RECURSIVE EMBEDDINGS OF
PARTIAL ORDERINGS

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Introduction. Let \( \mathcal{A} \) be a countable atomless Boolean algebra and let \( X \) be a countable partial ordering. We prove that there exists an embedding of \( X \) into \( \mathcal{A} \) which is recursive in \( X \), \( \mathcal{A} \) and which destroys all suprema and infima of \( X \) which can be destroyed. We show that the above theorem is false when we try to preserve all suprema and infima of \( X \) instead of destroying them. Finally we indicate that if \( \mathcal{A} \) and \( \mathcal{B} \) are countable Boolean algebras and \( \mathcal{B} \) is atomless then \( \mathcal{A} \) can be embedded into \( \mathcal{B} \) by a function which is recursive in \( \mathcal{A}, \mathcal{B} \). If \( \mathcal{A} \) is also atomless, then there is an isomorphism from \( \mathcal{A} \) into \( \mathcal{B} \) which is recursive in \( \mathcal{A}, \mathcal{B} \).

1. Preliminaries. Throughout the paper \( \omega \) denotes the set of natural numbers, and \( \emptyset \) the empty set. If \( X \) is a set and \( n \) a natural number then \( X^n \) denotes the set of all \( n \)-tuples of elements of \( X \). We say that \( X \) is a partial ordering on a set \( A \) (p.o. on \( A \)) if for some \( B \subset A \) \( X \subset B^2 \) and for all \( x, y, z \in B \)

1) \((x, x) \in X,

2) \((x, y) \in X \land (y, x) \in X) \Rightarrow x = y,

3) \((x, y) \in X \land (y, z) \in X) \Rightarrow (x, z) \in X.

If \((x, y) \in X\), we write \( x \leq X y \). If \((x, y) \in X \land x \neq y\), we write \( x < X y \).

If \((x, y) \notin X\) and \((y, x) \notin X\), we say that \( x \) and \( y \) are \( X \)-incomparable and we write \( x \parallel y \).

\( z \) is called the supremum of \( x \) and \( y \) in \( X(x \cup y = z)\), if

\[
x \leq X z \land y \leq X z \land \forall t[(x \leq X t \land y \leq X t) \Rightarrow z \leq X t],
\]

and \( z \) is called the infimum of \( x \) and \( y \) in \( X(x \cap y = z)\), if

\[
z \leq X x \land z \leq X y \land \forall t[(t \leq X x \land t \leq X y) \Rightarrow t \leq X z].
\]

By \( \text{Fid}(X) \) we denote the set \( \{x : (x, x) \in X\} \).

For the definition of a Boolean algebra we refer the reader to Sikorski [4].

If \( \mathcal{A} \) is a Boolean algebra then \( 0 \) denotes its smallest element and \( 1 \) the greatest one. If \( x \) and \( y \) are elements of \( \mathcal{A} \), then we write \( x \leq y \) if \( x \cup y = y \) and \( x < y \) if \( x \leq y \) and \( x \neq y \). We write \( x \parallel y \) if \( \neg (x \leq y) \) and \( \neg (y \leq x) \).

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We say that $\mathcal{A}$ is a Boolean algebra on a set $A$, if every element of $\mathcal{A}$ is an element of $A$.

In this paper we are interested in partial orderings on $\omega$ and Boolean algebras on $\omega$.

**Definition 1.** Let $X$ be a p.o. on a set $A$ and $\mathcal{A}$ a Boolean algebra. $f$ is called an embedding of $X$ into $\mathcal{A}$ if $f$ is an injective function from $\text{Fld}(X)$ into $\mathcal{A}$ such that for all $x, y \in \text{Fld}(X)$

$$x \preceq y \iff f(x) \preceq f(y).$$

We say that an embedding $f$ of $X$ into $\mathcal{A}$ preserves all suprema and infima of $X$ if

I) whenever $x \cup y = z$, then $f(x) \cup f(y) = f(z)$; and

II) whenever $x \cap y = z$, then $f(x) \cap f(y) = f(z)$.

We say that an embedding $f$ of $X$ into $\mathcal{A}$ destroys all suprema and infima of $X$ if

I) whenever $x \preceq y$ and $x \cup y = z$, then $f(x) \cup f(y) \neq f(z)$; and

II) whenever $x \preceq y$ and $x \cap y = z$, then $f(x) \cap f(y) \neq f(z)$.

Observe that if $x \preceq x \cup y$, then $x \cup y = y$ and $x \cap y = x$, so for any embedding $f$ of $X$ into $\mathcal{A}$, $f(x) \cup f(y) = f(x \cup y)$ and $f(x) \cap f(y) = f(x \cap y)$. Thus an embedding of $X$ into $\mathcal{A}$ cannot destroy suprema and infima of $X$-comparable elements.

All the notions from recursion theory we use can be found in Shoenfield [2]. In particular, $\text{Seq}(x)$ means that $x$ codes a finite sequence of natural numbers, and $\text{lh}(x)$ is the length of that sequence. If $\text{Seq}(x)$ then $x = \langle (x)_0, \ldots, (x)_{\text{lh}(x)-1} \rangle$. If $a = \langle a_1, \ldots, a_n \rangle$ and $b = \langle b_1, \ldots, b_n \rangle$, then $a \ast b = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$.

All the mentioned functions and relations are recursive.

If $A = \{a_1, \ldots, a_k\}$ then $x$ is called the code of $A$ (a = \langle A \rangle) if $x$ is the least number $z$ such that $\text{seq}(z) = k$ and $\{ (z)_i : i < \text{lh}(z) \} = A$. If $f(x_1, \ldots, x_n)$ is a function then $\text{graph}(f) = \{(x_1, \ldots, x_n, y) : f(x_1, \ldots, x_n) = y \}$.

**Definition 2.** Let $\mathcal{A} = \langle A, \cup, \cap, -, 0, 1 \rangle$ be a Boolean algebra on $\omega$. We say that $f$ is recursive in $\mathcal{A}$ if $f$ is recursive in $\{ A, \text{graph}(\cup), \text{graph}(\cap), \text{graph}(-) \}$.

Similarly we say that $f$ is recursive in $\mathcal{B}$ where $\mathcal{B}$ is another Boolean algebra on $\omega$ or that $f$ is recursive in $X, \mathcal{A}$ for a set $X$.

**Definition 3.** Let $\mathcal{A}$ be a Boolean algebra. Suppose that $A$ and $B$ are sets of elements of $\mathcal{A}$. Then

I) if $a \leq b$ for all $a \in A, b \in B$ we write $A \leq B$;

II) if $a < b$ for all $a \in A, b \in B$ we write $A < B$;

III) if $\forall (a \leq b)$ for all $a \in A, b \in B$ we write $A \nleq B$;

IV) if $a \nleq b$ for all $a \in A, b \in B$ we write $A \nleq B$.

Instead of $\{a\} < A$ we write $a < A$. Similarly with other relations. Observe that for every set $A, \phi < A, A < \phi, \phi \nleq A, A \nleq \phi$ and $\phi \nleq A$. 

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If \(A\) is a finite set of elements of \(\mathcal{A}\) then \(\sup A\) denotes the least element \(a\) of \(\mathcal{A}\) such that \(A \subseteq a\), and \(\inf A\) denotes the greatest element \(a\) of \(\mathcal{A}\) such that \(a \subseteq A\). Observe that \(\sup \phi = 0\) and \(\inf \phi = 1\). Recall that a Boolean algebra \(\mathcal{A}\) is atomless if \(0 < x\) implies for some \(y, 0 < y < x\).

2. Embeddings destroying suprema and infima. In this section we prove the following theorem:

**Theorem 1.** Let \(X\) be a partial ordering on \(\omega\) and let \(\mathcal{A}\) be an atomless Boolean algebra on \(\omega\). Then there exists an embedding \(f\) of \(X\) into \(\mathcal{A}\) such that

I) \(f\) destroys all suprema and infima of \(X\), and
II) \(f\) is recursive in \(X, \mathcal{A}\).

We first present an informal idea of the proof. Let \(\text{Fld}(X) = \{a_0, a_1, \ldots\}\) be a recursive in \(X\) enumeration of \(\text{Fld}(X)\). We want to build the required embedding by induction. Suppose that for \(i \leq n\) we already defined some elements \(b_i\) of \(\mathcal{A}\) such that

\[
a_i <_X a_j \Leftrightarrow b_i < b_j \quad \text{for } i, j \leq n.
\]

We want to define an element \(b_{n+1}\) of \(\mathcal{A}\) such that

\[
(\ast) \quad a_i <_X a_j \Leftrightarrow b_i < b_j \quad \text{for } i, j \leq n + 1.
\]

If we do not impose any conditions on \(b_i - s\) we can be stuck. For example, if \(a_0 < a_2, a_1 < a_2, a_0 < a_3, a_1 < a_3\) and \(a_2 < a_3\) (represented schematically by the following diagram)

![Diagram](https://doi.org/10.4153/CJM-1977-038-5)

and we choose \(b_0, b_1, b_2\) in such a way that \(b_0 \cup b_1 = b_2\) then there is no \(b_3\) such that \(b_2 < b_3, b_0 < b_3\) and \(b_1 < b_3\).

In order to prevent such situations we choose \(b_i - s\) in a more careful way. For example, the above difficulty would not occur if \(b_0 \cup b_1 < b_2\). Thus we assume that the elements \(b_0, \ldots, b_n\) satisfy an additional property, namely that the set \(\{b_0, \ldots, b_n\}\) is normal (see Definition 4).

Let

\[
A = \{b_i : a_i <_X a_{n+1}, i \leq n\},
\]

\[
B = \{b_i : a_{n+1} <_X a_i, i \leq n\},
\]

\[
C = \{b_i : a_{n+1} = a_i, i \leq n\}.
\]

Then \(A \cup B \cup C = \{b_0, \ldots, b_n\}\). Observe that \(A < B, C \not< A\) and \(B \not< C\).

Since \(A \cup B \cup C\) is a normal set we get from this that \(\sup A < \inf B, C \not<\)
sup $A$ and inf $B \not\equiv C$. We are looking for an element $b_{n+1}$ such that sup $A < b_{n+1} < \inf B$ and $b_{n+1} \parallel C$. Then (*) holds. The existence of such a $b_{n+1}$ is guaranteed by Lemma 1.

But we want also to preserve our additional condition, so we claim also that the set $A \cup B \cup C \cup \{b_{n+1}\}$ is normal. Lemma 2 shows that the required $b_{n+1}$ still can be found. Its proof uses Lemma 1, but in an appropriately modified way.

Thus the induction step works. The obtained embedding destroys all suprema and infima of $X$ which is an immediate consequence of the fact that for each $n$, the set $\{b_0, \ldots, b_n\}$ is normal.

Choosing at each time the smallest $b_{n+1}$ satisfying the above conditions (see the definition of the function $g$ in the proof of Theorem 1) we ensure that the above embedding is recursive in $X, \mathcal{A}$.

We present now the precise proof of the theorem. We first prove two lemmata.

**Lemma 1.** Let $\mathcal{A}$ be an atomless Boolean algebra. Suppose that $A \cup \{a, b\}$ is a finite set of elements of $\mathcal{A}$, such that:

1) $a < b$;
2) $A \not\equiv a$;
3) $b \not\equiv A$.

Then there exists an element $c$ of $\mathcal{A}$, such that $a < c < b$ and $c \parallel A$.

Obviously Conditions 2) and 3) have to be satisfied if we want to prove the claim. The lemma shows that 2) and 3) are also sufficient conditions.

**Proof.** At first we "modify" $A$ to a set $A'$ such that $a < A' < b$. We find then an element $c$ such that $a < c < b$ and $c \parallel A'$. It turns out that also $c \parallel A$.

Let $A' = \{b \cap d : d \in A$ and $a < b \cap d\} \cup \{a \cup d : d \in A$ and $a \cup d < b\}$.

Suppose that $x = b \cap d$ for some $d \in A$ such that $a < b \cap d$. Then $x \leq b$. If $x = b$ then $b \leq d$, which violates our assumptions. Thus $a < x < b$.

Suppose now that $x = a \cup d$ for some $d \in A$ such that $a \cup d < b$. Then $a \leq x$. If $a = x$ then $d \leq a$, which violates our assumptions. Thus $a < x < b$.

So $a < A' < b$.

We can treat the set $B = \{x : a \leq x \leq b\}$ as a Boolean algebra with the operations induced by $\mathcal{A}$.

\[
\begin{align*}
x \cup y &= x \cup y \\
x \cap y &= x \cap y \\
\hat{0} &= a \\
\hat{1} &= b \\
\hat{x} &= a \cup (b \cap \neg x)
\end{align*}
\]
Let $A' = \{a_1, \ldots, a_n\}$. We just proved that $\hat{0} < a_i$ and $\hat{0} < -a_i$ for all $i \leq n$. Let $C = \{b_1 \cap \ldots \cap b_n : \text{for all } i \leq n, b_i = a_i \text{ or } b_i = \neg a_i\}$. Then each $a_i \text{ or } \neg a_i$ is a sum of elements of $C$. For each $i \leq 2n$, pick an element $c_i$ from $C$ such that

$$\hat{0} < c_i \leq a_i \quad \text{and} \quad \hat{0} < c_{i+n} \leq -a_i \quad \text{for all } j \leq n.$$

$\mathcal{A}$ is atomless so there exist elements $d_i$ such that for $i \leq 2n, \hat{0} < d_i < c_i$. We can choose $d_i - s$ in such a way that $d_i = d_j$ if $c_i = c_j$.

Finally let $c = d_1 \cup \ldots \cup d_{2n}$. We claim that $c$ is the desired element. We prove at first that $c \upharpoonright A'$. Suppose that for some $i \leq n, c \leq a_i$. Then

$$\hat{0} < d_{i+n} \leq a_i \quad \text{and} \quad d_{i+n} < -a_i$$

which is clearly impossible. If for some $i \leq n, a_i \leq c$ then

$$c_i \cap \neg d_i \leq a_i \leq c.$$

Observe that for $x, y \in C$ either $x = y$ or $x \cap y = \hat{0}$. Hence for $k \leq n$ either $c_k = c_i$ or $c_k \cap c_i = \hat{0}$. In the first case $d_k = d_i$, in the second $d_k \cap (c_i \cap \neg d_i) = \hat{0}$. So in both cases we obtain $d_k \cap (c_i \cap \neg d_i) = \hat{0}$. Finally we obtain:

$$c_i \cap \neg d_i = c_i \cap \neg d_i \cap c = \bigcup_{k=1}^{2n} d_k \cap (c_i \cap \neg d_i) = \hat{0}$$

which contradicts the choice of $d_i$.

Observe that by construction $a < c < b$. We prove now that $c \upharpoonright A$. Suppose that $x \in A$. There are 3 possible cases:

I) $x < b$. There are two possible cases:

1) $a \cup x < b$. Then $a \cup x \in A'$. So $a \cup x \| c$. If $x \leq c$ then $a \cup x \leq c$, which is impossible; if $c \leq x$ then $c \leq a \cup x$, which is impossible. Thus $c \| x$.

2) $a \cup x = b$. If $x \leq c$, then $a \cup x \leq c$, so $b \leq c$ which is impossible; if $c \leq x$, then $a \leq x$, so $a \cup x = x$, i.e. $b = x$ which contradicts our assumptions. Thus $c \| x$.

II) $a < x$. There are two possible cases:

1) $a < b \cap x$. Then $b \cap x \in A'$, so $b \cap x \| c$. If $x \leq c$, then $b \cap x \leq c$, which is impossible. If $c \leq x$, then $c \leq b \cap x$, which is impossible. Thus $x \| c$.

2) $a = b \cap x$. If $x \leq c$, then $x \leq b$, so $b \cap x = x$, i.e. $a = x$ which contradicts our assumptions. If $c \leq x$, then $c \leq b \cap x$, i.e. $c \leq a$ which is impossible. Thus $c \| x$.

This concludes the proof of the lemma.

**Definition 4.** Let $\mathcal{A}$ be a Boolean algebra. A finite set $T$ of elements of $\mathcal{A}$ is called normal if for all $A$ and $B$ such that $A \cup B \subseteq T$ we have

$$A < B \quad \text{implies} \quad \sup A < \inf B,$$

$$A \nleq B \quad \text{implies} \quad \inf A \nleq \sup B.$$
Lemma 2. Let $\mathcal{A}$ be an atomless Boolean algebra. Suppose that for some finite sets $A$, $B$ and $C$ of elements of $\mathcal{A}$,

$$A < B, \ C \not\subseteq A, \ B \not\subseteq C, \text{ and } A \cup B \cup C \text{ is normal.}$$

Then there exists an element $a$ of $\mathcal{A}$ such that

$$\sup A < a < \inf B, \ a\not\subseteq C, \text{ and } A \cup B \cup C \cup \{a\} \text{ is normal.}$$

Proof. Let $S$ be a subalgebra of $\mathcal{A}$ generated by the set $A \cup B \cup C$. Let

$$T = \{x : x \in S \land \nexists (x \leq \sup A) \land \nexists (\inf B \leq x)\}.$$ 

The set $T$ is of course finite.

Since $A \cup B \cup C$ is normal we get from our assumptions and Lemma 1 that for some $a$ in $A$, $\sup A < a < \inf B$ and $a \not\subseteq T$. We claim that $a$ is the required element. If $c \in C$, then $c \not\subseteq A$ and $B \not\subseteq c$. Since $A \cup B \cup C$ is normal, $c \not\subseteq \sup A$ and $\inf B \not\subseteq c$. Thus $C \subseteq T$, i.e. $a \subseteq C$.

It remains to prove that $A \cup B \cup C \cup \{a\}$ is normal. Let $K \cup L \subseteq A \cup B \cup C$. We have to consider the following four possible cases:

1) $K < L$ and $a < L$. We prove then that $\sup (K \cup \{a\}) < \inf L$. We always have $\sup (K \cup \{a\}) \leq \inf L$ so suppose that $\sup (K \cup \{a\}) = \inf L$. Then $\sup K \cup a = \inf L$, so $\inf L \cap - \sup K \leq a$, which indicates that $\inf L \cap - \sup K \not\subseteq T$. There are two possibilities:

1) $\inf B \subseteq \inf L \cap \sup K$. Then $\inf B \leq a$ which contradicts the choice of $a$.

II) $\inf L \cap - \sup K \leq \sup A$. Then $\inf L \leq \sup A \cup \sup K$, i.e. $\inf L \leq \sup (A \cup K)$. The assumption $a < L$ implies, by the choice of $a$, that $L \subseteq B$. Thus $A < L$ since $A \not\subseteq B$. So $A \cup K < L$. But $A \cup B \cup C$ is normal, so we get that $\sup (A \cup K) < \inf L$, which contradicts our previous statement.

2) $K < L$ and $K < a$. We prove that $\sup K < \inf (L \cup \{a\})$. We always have $\sup K \leq \inf (L \cup \{a\})$, so suppose that $\sup K = \inf (L \cup \{a\})$. Then $\sup K = \inf L \cap a$, so $a \leq \sup K \cup - \inf L$. This indicates that $\sup K \cup - \inf L \not\subseteq T$. There are two possibilities:

1) $\sup K \cup - \inf L \leq \sup A$. Then $a \leq \sup A$ which is impossible.

II) $\sup K \cup - \inf L \not\subseteq \sup A$, i.e. $\inf (B \cup L) \leq \sup K$. But $K < a$, so $K \subseteq A$, i.e. $K \not\subseteq B$. Thus $K < L \cup B$. Since $A \cup B \cup C$ is normal we get $\sup K < \inf (B \cup L)$, which contradicts the former statement.

3) $K \not\subseteq L$ and $a \not\subseteq L$. We prove that $\inf (K \cup \{a\}) \not\subseteq L$. Suppose that $\inf (K \cup \{a\}) \leq \sup L$, i.e. $\inf K \cap a \leq \sup L$. Then $a \leq \sup L \cup - \inf K$, so $L \cup - \inf K \not\subseteq T$. There are two possibilities:

1) $\sup L \cup - \inf K \leq \sup A$. Then $a \leq \sup A$, which contradicts the choice of $a$.

II) $\inf B \leq \sup L \cup - \inf K$. Then $\inf B \cup \inf K \leq \sup L$, i.e. $\inf (B \cup K) \leq \sup L$. But $a \not\subseteq L$, so by the choice of $a$, $B \not\subseteq L$, i.e. $B \cup K \not\subseteq L$. Since $A \cup B \cup C$ is normal we get that $\inf (B \cup K) \not\subseteq L$, which contradicts the former statement.

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4) $K \not\leq L$, $K \not\leq a$. We prove that $\inf K \not\leq \sup (L \cup \{a\})$. Suppose that $\inf K \leq \sup (L \cup \{a\})$, i.e. $\inf K \leq \sup L \cup a$. Then $\inf K \cap - \sup L \leq a$, so $\inf K \cap - \sup L \notin T$. There are two possibilities:

I) $\inf K \cap - \sup L \leq \sup A$. Then $\inf K \leq \sup A \cup \sup L$, i.e. $\inf K \leq \sup (A \cup L)$. On the other hand, $K \not\leq a$, so by the choice of $a$, $K \not\leq A$, i.e. $K \not\leq A \cup L$. Now, $A \cup B \cup C$ is normal, so $\inf K \not\leq \sup (A \cup L)$, which gives the contradiction.

II) $\inf B \leq \inf K \cap - \sup L$. Then $\inf B \leq a$, which contradicts the choice of $a$.

This completes the proof that $A \cup B \cup C \cup \{a\}$ is normal, so the proof of the lemma is concluded.

**Proof of Theorem 1.** Observe that the relation $P(x) \iff x$ is a code of a finite set is recursive. It is easy to see that the relation $T(x) \iff x$ is a code of a normal set of elements of $\mathcal{A}$ is recursive to $\mathcal{A}$. Define a function $g$ as follows:

$$g(x, y, z) = \begin{cases} 
\mu a \ (a \text{ satisfies the claim of Lemma 2}) \text{ if } x, y \text{ and } z \text{ are respectively codes of the sets } A, B \text{ and } C \text{ satisfying the conditions of Lemma 2}, \\
0 \quad \text{otherwise}
\end{cases}$$

Then $g$ is a total function recursive in $\mathcal{A}$. $F_{\text{ld}}(X)$ is recursive in a set $X$, so for some total injective function $a(x)$, which is recursive in $X$,

$$F_{\text{ld}}(X) = \{a(0), a(1), \ldots\}.$$ 

For any total function $h(x)$ and $n \geq 0$, let

$$A(h, n) = \{h(k) : a(k) < x a(n + 1), k \leq n\},$$

$$B(h, n) = \{h(k) : a(n + 1) < x a(k), k \leq n\},$$

$$C(h, n) = \{h(k) : a(k) \not\leq x a(n + 1), k \leq n\}.$$ 

Let $b$ be an arbitrary element of $\mathcal{A}$ such that $0 < b < 1$. Define a function $h$ as follows:

$$h(0) = b$$

$$h(n + 1) = g(A(h, n), B(h, n), C(h, n)).$$

$h$ is a well defined total function. It is easy to see that $h$ is recursive in $X$, $\mathcal{A}$. Finally define

$$f(a(n)) = h(n) \quad \text{for } n \geq 0.$$ 

We claim that $f$ is the required function. Observe that

$$f(x) = y \iff \exists n (x = a(n) \land y = h(n)),$$
so $f$ is recursive in $X$, $\mathcal{A}$. By induction on $k$, we prove that for all $k$,

I) $a(i) <_X a(j)$ if and only if $f(a(i)) < f(a(j))$ for all $i, j \leq k$, and

II) the set $\{f(a(i)) : i \leq k\}$ is normal.

Observe that the set $\{f(a(0))\}$ is normal, so I) and II) are true for $k = 0$.

Suppose that I) and II) are true for $k$. Then I) implies that

$$A(h, k) < B(h, k), C(h, k) \nsubseteq A(h, k), \text{ and } B(h, k) \nsubseteq C(h, k).$$

Also $A(h, k) \cup B(h, k) \cup C(h, k) = \{f(a(i)) : i \leq k\}$ so it is a normal set.

Thus the sets $A = A(h, k), B = B(h, k), C = C(h, k)$ satisfy the claim of Lemma 2.

Now, $g(\langle A(h, k), \langle B(h, k), \langle C(h, k) \rangle \rangle) = f(a(k + 1))$, so by the definition of the function $g$,

$$\sup A(h, k) < f(a(k + 1)) < \inf B(h, k),$$

$f(a(k + 1)) \parallel C(h, k)$ and $A(h, k) \cup B(h, k) \cup C(h, k) \cup \{f((a(k + 1))\}$ is a normal set. Observe now that for $i < k + 1$,

$$a(i) <_X a(k + 1) \iff f(a(i)) \in A(h, k) \iff f(a(i)) < f(a(k + 1))$$

$$a(k + 1) <_X a(i) \iff f(a(i)) \in B(h, k) \iff f(a(k + 1)) < f(a(i))$$

$$a(i) \parallel_X a(k + 1) \iff f(a(i)) \in C(h, k) \iff f(a(i)) \parallel f(a(k + 1)).$$

Thus I) and II) are true for $k + 1$. Hence by induction for all $i$ and $j$,

$$a(i) <_X a(j) \iff f(a(i)) < f(a(j)).$$

Since $f$ is also injective it is an embedding of $X$ into $\mathcal{A}$.

It remains to show that $f$ destroys all suprema and infima. Suppose that for some $i, j, k, a(i) \parallel_X a(j)$ and $a(i) \cup a(j) = a(k)$. Then $a(i) <_X a(k)$ and $a(j) <_X a(k)$, so $f(a(i)) < f(a(k))$ and $f(a(j)) < f(a(k))$. The set $\{f(a(n)) : n \leq \max (i, j, k)\}$ is normal, thus

$$f(a(i)) \cup f(a(j)) < f(a(k)),$$

i.e. $f$ destroys the supremum $a(i) \cup a(j)$. The same argument applies in the case of infimum of $X$-incomparable elements. This concludes the proof of the theorem.

3. Embedding preserving suprema and infima. Let

$$A = \{x : \text{Seq } (x) \land \forall i (i < \text{lh } (x) \Rightarrow ((x), i = 0 \lor (x), i = 1))\}. $$

Thus $A$ is the set of codes of all finite sequences of zeroes and ones.

Let $\cup$ and $\cap$ be some operations on $A$ satisfying the following property:

If $\langle k_1, \ldots, k_n \rangle \in A,$
then
\[
\langle k_1, \ldots, k_n, 0 \rangle \cup \langle k_1, \ldots, k_n, 1 \rangle = \langle k_1, \ldots, k_n \rangle
\]
\[
\langle k_1, \ldots, k_n, 0 \rangle \cap \langle k_1, \ldots, k_n, 1 \rangle = \langle 0 \rangle.
\]

Let \( \mathcal{M} \) be the Boolean algebra generated by \( \mathcal{A} \) and by operations \( \cup \) and \( \cap \) satisfying the above property. It is well known that \( \mathcal{M} \) is (isomorphic to) the Boolean algebra of all clopen subsets of the Cantor Space. The elements of \( \mathcal{M} \) are just all the finite joins and meets of \( \mathcal{A} \).

It is easy to see that \( \mathcal{M} \) is recursive, that is to say
\[
\mathcal{M} = \langle A, \cup, \cap, -, 0, 1 \rangle,
\]
where \( A \) is a recursive set and the graphs of partial functions \( \cup, \cap \) and \( - \) are recursive. \( \mathcal{M} \) is an atomless Boolean algebra.

We prove the following theorem.

**Theorem 2.** There exists a recursive partial ordering \( X \) on \( \omega \), such that

1) there is an embedding of \( X \) into \( \mathcal{M} \) which preserves all suprema and infima of \( X \), and

2) no such embeddings are recursive.

**Proof.** Let \( P(x) \) be a \( \Sigma_2^0 \) \( \rightarrow \Pi_2^0 \) relation. For some recursive \( R \),
\[
P(x) \iff \exists y \forall z R(x, y, z).
\]
Define a partial function \( g \) as follows:
\[
g(x, y) \simeq \langle x, y, \mu z \forall R(x, y, z) \rangle
\]
Observe that graph \( (g) \) is recursive. Define
\[
h(x, y) \simeq \langle g(x, 0), \ldots, g(x, y - 1) \rangle \quad \text{where} \ y - 1 = \max (y - 1, 0).
\]
Clearly \( h \) is a partial recursive function. Observe that
1) \( (h(x, y) \) is defined and \( z < y) \Rightarrow (h(x, z) \) is defined)
2) For all \( x \) \( \forall y h(x, y) \) is total \( \iff \forall y g(x, y) \) is total
3) \( \text{graph}(h)(x, y, z) \Rightarrow \text{Seq}(z) \land \text{lh}(z) = y \land \forall i(i < y \Rightarrow \text{graph}(g)(x, i, (z)i)).
\)

Our ordering \( X \) looks as follows:

```
(\langle 0 \rangle)   \quad (\langle 1 \rangle)   \quad (\langle 2 \rangle)   \ldots
```

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  \bullet

  \bullet

  h(0, 1)   \quad h(1, 1)

  h(0, 0)   \quad h(1, 0)   \ldots
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More formally,
\[ X = \{ (\langle x \rangle, \langle x \rangle) : x \geq 0 \} \cup \{ (h(x, m), h(x, n)) : x \geq 0, n \geq m \geq 0 \} \]
\[ \cup \{ \langle h(x, m), \langle x \rangle \rangle : m \geq 0, x \geq 0 \} \]
\[ \cup \{ (h(x, m), \langle x + 1 \rangle) : x \geq 0, m \geq 0 \} \} . \]

\( X \) is clearly a recursive set. Now let \( T \) be the following relation:
\[
T(x) \iff \langle \langle x \rangle \rangle \cap \langle \langle x + 1 \rangle \rangle \text{ exists.}
\]

Then
\[
T(x) \iff \lambda y h(x, y) \text{ is not total,}
\]
\[
\iff \lambda y g(x, y) \text{ is not total,}
\]
\[
\iff \exists y (g(x, y) \text{ is not defined,})
\]
\[
\iff \exists y (\forall z R(x, y, z)),
\]
\[
\iff P(x).
\]

Hence \( T \) is a \( \Sigma^0_2 - \Pi^0_2 \) relation.

It is easy to see that there is an embedding of \( X \) into \( \mathcal{M} \) which preserves all suprema and infima of \( X \). Let \( f \) be such an embedding. Then
\[
T(x) \iff \exists z (z \in \text{Fld}(X) \land (f(\langle \langle x \rangle \rangle) \cap f(\langle \langle x + 1 \rangle \rangle) = f(z)).
\]

Thus, if \( f \) was recursive then \( T \) would be a \( \Sigma^0_1 \) set, which is not the case. Hence no such embeddings are recursive, completing the proof.

The above theorem shows that Theorem 1 is not true when \( I \) is changed for \( I' \) \( f \) preserves all suprema and infima of \( X \).

We pass now to the problem of recursive embeddings of Boolean algebras into Boolean algebras. Abian in [1] proves the following lemma.

**Lemma 3.** (Abian). Let \( \mathcal{A} \) and \( \mathcal{B} \) be countable Boolean algebras and let \( \mathcal{B} \) be atomless. Let \( f \) be an isomorphism from a finite subalgebra \( \mathcal{A}_1 \) of \( \mathcal{A} \) onto a finite subalgebra \( \mathcal{B}_1 \) of \( \mathcal{B} \). Then for every \( a \in \mathcal{A} - \mathcal{A}_1 \) there exists \( b \in \mathcal{B} - \mathcal{B}_1 \) such that the assignment \( f(a) = b \) extends the isomorphism \( f \) from the subalgebra of \( \mathcal{A} \) generated by \( \mathcal{A}_1 \cup \{a\} \) onto the subalgebra of \( \mathcal{B} \) generated by \( \mathcal{B}_1 \cup \{b\} \).

Using this lemma, Abian gives an algebraic proof of the well-known theorem that two countable atomless Boolean algebras are isomorphic. In fact this isomorphism is recursive in the considered algebras. More precisely, we have the following theorem.

**Theorem 3.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be countable Boolean algebras on \( \omega \) and let \( \mathcal{B} \) be atomless. Then

1) there exists an embedding of \( \mathcal{A} \) into \( \mathcal{B} \) (as Boolean algebras) which is recursive in \( \mathcal{A} \), \( \mathcal{B} \), and

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II) if $\mathcal{A}$ is atomless, then there exists an isomorphism of $\mathcal{A}$ and $\mathcal{B}$ which is recursive in $\mathcal{A}, \mathcal{B}$.

Proof. I) follows by the repeated use of Lemma 3. II) follows using Lemma 3 repeatedly back and forth. It is clear that in both cases the constructed embedding $f$ is recursive in $\mathcal{A}, \mathcal{B}$.

Remark. This paper is closely related with the Van Emde Boas [2] paper. Van Emde Boas proves there that every recursive partial ordering can be recursively embedded into the Boolean algebra $\mathcal{M}$ defined earlier. We obtained Theorem 1 independently of his paper.

References