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Strong shift equivalence of 2×2 matrices of non-negative integers

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Abstract. The concept of strong shift equivalence of square non-negative integral matrices has been used by R. F. Williams to characterize topological isomorphism of the associated topological Markov chains. However, not much has been known about sufficient conditions for strong shift equivalence even for 2×2 matrices (other than those of unit determinant). The main theorem of this paper is: If A and B are positive 2×2 integral matrices of non-negative determinant and are similar over the integers, then A and B are strongly shift equivalent.

1. *Introduction*

For non-negative square integral matrices A, B possibly of different sizes, write $A \approx_1 B$ if there exist non-negative integral matrices R, S with $A = RS$, $B = SR$. The transitive closure of the relation $\approx_1$ is called *strong shift equivalence*, here denoted by $\approx$; thus $A \approx B$ if and only if there exist $k \geq 0$ and non-negative square integral matrices $C_0, C_1, \ldots, C_k$ with

$$A = C_0 \approx_1 C_1 \approx_1 \cdots \approx_1 C_k = B.$$

The importance of this relation lies in its use by R. F. Williams [**27**], [**28**] in topological dynamics to characterize isomorphisms of topological Markov chains. A *topological dynamical system* (X, σ) is a compact metric space X together with a homeomorphism σ of X onto X [**8**]. With each non-negative $n \times n$ integral matrix A is associated a *topological Markov chain* (X_A, σ_A), a particular kind of topological dynamical system with X_A 0-dimensional: A can be regarded as the adjacency matrix of a directed graph G with a set V of n vertices ('symbols' or 'states'), a set E of edges, and A_{ij} edges from the ith vertex to the jth vertex [**1**]. (Thus multiple edges and loops are permitted.) By an 'infinite walk' in G let us mean a doubly infinite sequence of edges $(\ldots, e_k, \ldots) = \mathbf{e}$, where for each k the terminal vertex of e_k is the initial vertex of e_{k+1}(cf. [**1**]). Then X_A is defined to be the set of infinite walks as a subspace of the compact metrizable sequence space $E^{\mathbf{Z}}$; σ_A is the left shift, $\sigma_A(\mathbf{e})_k = e_{k+1}$ (cf. [**7**]). (If A is a 0–1 matrix, i.e. G has no multiple edges, then a walk is determined by a sequence of vertices, so that X_A can be realized more economically inside $V^{\mathbf{Z}}$, again with σ_A the left shift [**5**], [**8**], [**28**]. A then represents allowed transitions between states.) Williams' theorem ([**28**]; see also [**21**]) is that (X_A, σ_A) and (X_B, σ_B) are isomorphic as topological dynamical systems if and only if A and B are strongly shift equivalent. (Here an isomorphism is a homeomorphism φ of X_A with X_B such that $\varphi\sigma_A = \sigma_B\varphi$.)

The importance of topological Markov chains, in turn, lies in their fundamental role in the Smale–Bowen study of Axiom A diffeomorphisms f of a compact manifold M by means of the action of f on the non-wandering set Ω_f (the set of points of M lacking a neighbourhood whose iterates under f are disjoint) [**25**], [**3**], [**4**], [**8**]: Ω_f is partitioned into finitely many invariant clopen 'basic' sets, each of which, as a dynamical system, is the quotient of a topological Markov chain, or is even isomorphic to a topological Markov chain if Ω_f is 0-dimensional [**3**].

The nature of strong shift equivalence, however, has not been well understood even for 2×2 matrices. For the particular case $A, B \in \mathrm{GL}(2, \mathbb{Z})$, Cuntz and Krieger [**7**], [**17**] used Effros' and Shen's theory of dimension groups [**11**], [**9**] to show $A \approx B$ if A and B are similar over $\mathbb{Z}$; in this case, 2×2 intermediate matrices C_i suffice. For more general 2×2 matrices, however, there have been many examples of positive 2×2 integral A, B similar over $\mathbb{Z}$ (enough to satisfy all known necessary conditions) for which strong shift equivalence has not been proved. The main result of this paper resolves all such examples with non-negative determinant in the affirmative, helps to clarify the nature of the 2×2 case, and provides a perspective for study of the $n\times n$ case:

1.1. THEOREM. *If A and B are positive 2×2 integral matrices of non-negative determinant and are similar over $\mathbb{Z}$, then $A\approx B$.*

The proof is in two steps. In § 2, it is shown that a similarity of A and B (via a unimodular matrix, later seen to be no restriction) can be replaced by a sequence of similarities via 'unit shears'; in § 3, it is shown that similarity via a unit shear gives strong shift equivalence using 3×3 intermediate matrices. The method is algorithmic.

Let $f_A(x)$ denote the characteristic polynomial of A. If A and B are of the same size, an easy necessary condition for $A\approx B$ is that $f_A(x) = f_B(x)$. The theorem of Latimer, MacDuffee & Taussky [**18**], [**26**], [**20**] for monic irreducible polynomials $f(x)$ with integer coefficients states that the similarity classes over $\mathbb{Z}$ of square matrices A with $f_A(x) = f(x)$ correspond one-to-one to ideal classes in the ring $\mathbb{Z}[\lambda]$, where λ is a root of $f(x)$. An immediate consequence of this fact and theorem 1.1. is:

1.2. COROLLARY. *Let $\mathcal{M}$ be a set of positive 2×2 integral matrices of positive determinant all having a common irrational eigenvalue λ. Then the number of strong-shift-equivalence classes of members of $\mathcal{M}$ is bounded by the number of ideal classes of $\mathbb{Z}[\lambda]$.*

For example, let p be any prime with $p<223$, $p\neq 79$. If $p\equiv 1 \pmod 4$, choose any odd integer $t>\sqrt{p}$ and let $d=(t^2-p)/4$; if $p\not\equiv 1 \pmod 4$, choose any even integer $t>2\sqrt{p}$ and let $d=(t/2)^2-p$. Then all 2×2 integral matrices $A\geq 0$ with trace t and determinant d are strongly shift equivalent. Indeed $\mathbb{Z}[\lambda]$ equals $\mathbb{Z}[(1+\sqrt{p})/2]$ or $\mathbb{Z}[\sqrt{p}]$ in respective cases; then $\mathbb{Z}[\lambda]$ is known to be a maximal order of class number 1, [**2**, tables 1, 2].

A sharper necessary condition for strong shift equivalence was observed by Williams [27], [28]: non-negative square integral matrices A, B are said to be *shift equivalent*, here written $A \sim B$, if there exist $l \geq 0$ and integral matrices R, $S \geq 0$ with $RS = A^l$, $SR = B^l$, $AR = RB$, $SA = BS$. Then $A \approx B$ implies $A \sim B$. 'Williams' problem' [28, errata] [10] is to determine whether, conversely, $A \sim B$ implies $A \approx B$. A positive resolution of Williams' problem would be important, as shift equivalence is more readily computed than strong shift equivalence [28], [16]. Moreover, shift equivalence is less closely tied to order. In fact, in its definition, the condition R, $S \geq 0$ can be omitted if A, $B > 0$ [23], [22]. In particular, if $B = Q^{-1}AQ$ for $Q \in \mathrm{GL}(2, \mathbb{Z})$, then $A = RS$ and $B = SR$ for $R = AQ$, $S = Q^{-1}$, so that $A \sim B$. Theorem 1.1 can therefore be interpreted as a positive answer to Williams' problem in the somewhat restricted case of positive 2×2 integral matrices in a fixed similarity class over $\mathbb{Z}$ of non-negative determinant. Strongly shift equivalent matrices, however, can be in different similarity classes. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

factor as RS and SR for

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix},$$

but are not similar mod 2 and hence are not similar over $\mathbb{Z}$.

In the remainder of this paper, we shall work with matrices $A = [A_{ij}]$ variously over $\mathbb{R}$ and $\mathbb{Z}$. $M_n(S)$ is the set of all $n \times n$ matrices with entries in S. The notation $A \geq 0$ means that A has non-negative entries; $A > 0$ means that all entries of A are strictly positive. A^{tr} is the transpose of A. Matrices are at times regarded as linear transformations on column vectors: $A(x) = Ax$. Elements of $\mathbb{R}^2$ may nevertheless be written in row form. A and B are said to be similar via Q if $B = Q^{-1}AQ$. C_+ denotes the standard positive cone of $\mathbb{R}^2$ (the closed first quadrant). A 'cone' C in general is a closed convex cone in $\mathbb{R}^2$. $\mathrm{int}_0\, C$ is the interior of C together with the origin.

General references are [12] and [20] for matrices, [15] and [24] for number theory, [1] for graph theory, [8] for topological dynamical systems, and [8], [19], [21], [22], [23], [28] for perspective on the use of symbol spaces (symbolic dynamics). Topological Markov chains are also known as 'two-sided subshifts of finite type.' Krieger [17] presents an alternate realization of X_A. Handelman [13], [14] has developed further deep properties of shift equivalence.

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2. *Farey approximations, shears, and positivity*

Let us call each of the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a *unit shear*. The purpose of this section is to reduce the proof of theorem 1.1. to the case where A and B are similar via a unit shear, by proving this fact:

2.1 LEMMA. *Let A and $B \in M_2(\mathbb{R})$ be positive, of positive determinant, and similar via* SL $(2, \mathbb{Z})$. *Then there exists a sequence $A = A_0, A_1, \ldots, A_k = B$ in $M_2(\mathbb{R})$ such that*

(a) $A_i > 0$ *for each i, and*

(b) *for each $i = 1, \ldots, k$, $A_i = H_i^{-1} A_{i-1} H_i$, where H_i is a unit shear or the inverse of a unit shear.*

The proof, given as 2.6 below, depends on the relationship between positivity, similarity, and rational approximation of Perron eigenvectors, as expressed in the lemmas to follow.

2.2 *Definitions.* Let us say that the cone C in $\mathbb{R}^2$ *straddles* the basis $\mathbf{v}$, $\mathbf{w}$ of $\mathbb{R}^2$ if $\mathbf{v} \in \operatorname{int} C$ but $\mathbf{w} \notin C$, $-\mathbf{w} \notin C$. In other words, in the coordinate system determined by $\mathbf{v}$, $\mathbf{w}$, C contains the 'positive x-axis' in its interior and lies in the 'open right half plane'. By a *Perron basis* for the 2×2 matrix $A > 0$ let us mean a basis $\mathbf{v}$, $\mathbf{w}$ of $\mathbb{R}^2$ in which $\mathbf{v}$ is a Perron eigenvector of A and $\mathbf{w}$ is a non-Perron eigenvector.

Observe that C_+ always straddles a Perron basis [**12**, p. 53 and p. 63, remark 3]. If A has positive determinant, a more general statement is possible:

2.3. LEMMA. *Let $A \in M_2(\mathbb{R})$ with $A > 0$,* $\det A > 0$. *Then for any $P \in$* GL $(2, \mathbb{R})$, *$P^{-1}AP > 0$ if and only if $P(C_+)$ or $-P(C_+)$ straddles a Perron basis for A.*

Proof. Let λ_1 be the Perron eigenvalue of A and λ_2 the other eigenvalue. Since $\det A > 0$, $\lambda_1 > \lambda_2 > 0$. Let D be the diagonal matrix with diagonal entries λ_1, λ_2, Observe that for a cone C, $D(C) \subseteq \operatorname{int}_0 C$ if and only if C or $-C$ straddles the standard basis $(1, 0)$, $(0, 1)$ of $\mathbb{R}^2$; here the fact $\lambda_1 > \lambda_2 > 0$ is used. By a change of coordinates, it follows that for a cone C, $A(C) \subseteq \operatorname{int}_0 C$ if and only if C or $-C$ straddles a Perron basis for A. In particular, $P(C_+)$ or $-P(C_+)$ straddles a Perron basis for A if and only if $A(P(C_+)) \subseteq \operatorname{int}_0 P(C_+)$, or equivalently, $P^{-1}AP(C_+) \subseteq \operatorname{int}_0 (C_+)$, or equivalently, $P^{-1}AP > 0$. □

Now let us examine the case where P is a matrix $Q \in$ SL $(2, \mathbb{Z})$. Of special significance is the case where $Q \geq 0$. Indeed, if $Q \geq 0$, then $Q(C_+)$ straddles a Perron basis $\mathbf{v}$, $\mathbf{w}$ if and only if the entries of Q give a (strict) Farey approximation of the slope v_2/v_1 of $\mathbf{v}$:

$$Q_{21}/Q_{11} < v_2/v_1 < Q_{22}/Q_{12} \quad \text{with} \quad Q_{11}Q_{22} - Q_{12}Q_{21} = 1$$

[**24**], [**15**]. (Most presentations of Farey approximation assume $Q_{22}/Q_{12} \leq 1$, but the theory works as well without this restriction; indeed even the case where Q_{22}/Q_{12} is the formal fraction $1/0$ has been used historically [**6**] and can be justified by a matrix formulation.)

2.4 LEMMA. *Let A, $B \in M_2(\mathbb{R})$ be positive of positive determinant. Suppose $B = Q^{-1}AQ$, where $Q \in$* SL $(2, \mathbb{Z})$. *Then one of the matrices Q, $-Q$, Q^{-1}, $-Q^{-1}$ is non-negative.*

Proof. By lemma 2.3, either $Q(C_+)$ or $-Q(C_+)$ straddles a Perron basis $\mathbf{v}$, $\mathbf{w}$ for A. If the latter, we may without loss of generality replace Q by $-Q$ to obtain the former. Thus $Q(C_+)$ is contained in the 'right half plane' with respect to the Perron basis. Since $\mathbf{w}$ lies in the interior of the 2nd or 4th standard quadrant of $\mathbb{R}^2$ by [**12**,

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p. 63, remark 3] or by lemma 2.3 with $P = I$, $Q(C_+)$ is contained in the interior of the union of the 1st, 2nd, and 4th standard quadrants. Now observe that the columns of a member of SL $(2, \mathbb{Z})$ cannot lie in the interiors of adjacent quadrants. Since the columns of Q are in $Q(C_+)$, they must either both lie in the closed first quadrant, in which case $Q \geq 0$, or lie in the closed 2nd and 4th quadrants, in which case $Q(C_+) \supseteq C_+$, $Q^{-1}(C_+) \subseteq C_+$, so $Q^{-1} \geq 0$. □

2.5 Lemma. *Suppose C_+ straddles a basis* $\mathbf{v}$, $\mathbf{w}$ *of* $\mathbb{R}^2$ *and $Q(C_+)$ also straddles* $\mathbf{v}$, $\mathbf{w}$, *where $Q \in$ SL $(2, \mathbb{Z})$, $Q \geq 0$. Then there is a sequence $I = Q_0, Q_1, \ldots, Q_k = Q$ of matrices Q_i such that for each $i = 1, \ldots, k$,*

(a) $Q_i \in$ SL $(2, \mathbb{Z})$,

(b) $Q_i \geq 0$,

(c) $Q_i(C_+)$ *straddles* $\mathbf{v}, \mathbf{w}$,

(d) $Q_i = Q_{i-1}H_i$ *for a unit shear* H_i.

Proof. This is a matrix formulation of the Farey process [**24**] for approximating $\mu = v_2/v_1$. The process normally starts with $0/1 < \mu < 1/1$, but in matrix terms works as well if $1/1$ is replaced by the formal fraction $1/0$: In other words, the 0th approximation corresponds to I. The ith approximation Q_i is $Q_{i-1}H_i$ where H_i is one of the two unit shears. The lemma is a statement in matrix terms of the familiar fact that all Farey approximations of μ actually occur at some point in the process. A direct proof is easy: observe that if $Q \in$ SL $(2, \mathbb{Z})$ and $Q \geq 0$, $Q \neq I$, then one column of C dominates the other, so that $Q = Q'H$ for a unit shear H and $Q' \geq 0$ with smaller sum of entries than Q. Inductively, we obtain in reverse a sequence $I = Q_0, Q_1, \ldots, Q_k = Q$ with at least properties (a), (b), (d). Since

$$C_+ \supseteq Q_i(C_+) \supseteq Q(C_+)$$

for all i, property (c) holds as well. □

2.6 *Proof of lemma* 2.1. Given A, B, write $B = Q^{-1}AQ$, $Q \in \mathrm{SL}(2, \mathbb{Z})$. By lemma 2.4, one of Q, $-Q$, Q^{-1}, $-Q^{-1}$ is non-negative. Because the conclusion of lemma 2.1 is unaltered if Q is replaced by $-Q$ and/or A and B are interchanged, we may assume that $Q \geq 0$. Moreover, by lemma 2.3, one of $Q(C_+)$, $-Q(C_+)$, here obviously $Q(C_+)$, straddles a Perron basis for A. Then lemma 2.5 gives a sequence $I = Q_0, \ldots, Q_k = Q$. Let $A_i = Q_i^{-1}AQ_i$. By lemma 2.3 again and property (c) from lemma 2.5, $A_i > 0$. Finally,

$$A_i = (Q_{i-1}H_i)^{-1}A(Q_{i-1}H_i) = H_i^{-1}A_{i-1}H_i. \qquad \square$$

2.7 *Remarks.* The methods of this section fail for the case of negative determinant. For example, the matrices

$$\begin{bmatrix} 0 & 13 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

are similar over $\mathbb{Z}$, but not via a sequence of 2×2 shears with positive intermediate matrices. Thus they are shift equivalent, but it is not known whether they are strongly shift equivalent. The use of Farey approximations is related to the continued fractions of [**11**], [**7**]. Periodicity, however, is not used.

3. *Conclusion of the proof of theorem* 1.1

3.1. LEMMA. *Suppose the positive matrices A, $B \in M_2(\mathbb{Z})$ are similar via a unit shear. Then A and B are strongly shift equivalent.*

Proof. Consider the case $B = H^{-1}AH$,

$$H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(The case of the other unit shear H^{tr} is obtained from the present case simply by interchanging coordinates in $\mathbb{Z}^2$.)

Observe that $B_{11} = A_{11} - A_{21}$, so that $A_{11} > A_{21}$. If also $A_{12} \geq A_{22}$, then $H^{-1}A \geq 0$, yielding immediately

$$A = H(H^{-1}A) \approx_1 (H^{-1}A)H = B.$$

Let us therefore assume from now on that $A_{22} > A_{12}$. Consider the matrices

$$R = \begin{bmatrix} 1 & A_{21} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix}, \quad S = \begin{bmatrix} B_{11} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that $RS = A$, since $B_{11} = A_{11} - A_{21}$. Let $n = A_{22} - A_{12}$ (≥ 1 by our assumption). We make the following claims:

(a) $A_{22} \leq B_{11} + A_{12} - 1$;

(b) $B_{11} > 1$;

(c) $P^{-1}SR \geq 0$;

(d) $P^{-(n+1)}SH \geq 0$;

(e) $H^{-1}RP^n \geq 0$;

(f) $P^{-(k+1)}SRP^k \geq 0$ for $k = 0, 1, \ldots, n-1$;

(g) $P^{-k}SRP^k \approx_1 P^{-(k+1)}SRP^{k+1}$ for $k = 0, 1, \ldots, n-1$;

(h) $P^{-n}SRP^n \approx_1 H^{-1}AH = B$.

If these claims are granted, then

$$A = RS \approx_1 SR \approx_1 P^{-1}SRP \approx_1 P^{-2}SRP^{-2} \approx_1 \cdots \approx_1 P^{-n}SRP^n \approx_1 B$$

by (g) and (h), and we are done.

Proofs of claims. (The fact that entries of A, B are ≥ 1 is used repeatedly without comment.) For (a):

$$B_{11} + A_{12} - A_{22} = A_{11} - A_{21} + A_{12} - A_{22} = (H^{-1}AH)_{12} = B_{12} \geq 1.$$

For (b): $A_{22} > A_{12}$ by assumption; then by (a),

$$B_{11} + A_{12} - 1 \geq A_{22} > A_{12},$$

so $B_{11} > 1$. For (c): All entries of $P^{-1}SR$ are entries of A and B or products ≥ 0, except that

$$(P^{-1}SR)_{13} = B_{11}A_{12} - A_{22}.$$

By (a) and (b),

$$B_{11}A_{12} - A_{22} \geq B_{11}A_{12} - (B_{11} + A_{12} - 1) = (B_{11} - 1)(A_{12} - 1) \geq 0.$$

For (d):

$$(P^{-(n+1)}SH)_{11} = B_{11} > 0;$$

$$(P^{-(n+1)}SH)_{12} = B_{11} - n - 1 = A_{11} - A_{21} + A_{12} - A_{22} - 1 = B_{12} - 1 \geq 0;$$

the other entries are 0 or 1. For (e):

$$(H^{-1}RP^n)_{13} = A_{12} + n - A_{22} = 0;$$

other entries are among 0, 1, A_{21}, A_{22}. For (f):

$$(P^{-(k+1)}S)_{12} = -k - 1 \quad \text{and} \quad (RP^k)_{13} = A_{12} + k;$$

other entries do not depend on k. Then in $P^{-(k+1)}SRP^k$, all entries are linear in k, so it suffices to check non-negativity for $k=0$ and $k=n$. The case $k=0$ is (c); in the case $k=n$,

$$P^{-(n+1)}SRP^n = (P^{-(n+1)}SH)(H^{-1}RP^n) \geq 0$$

by (d) and (e). For (g):

$$P^{-k}SRP^k = P(P^{-(k+1)}SRP^k) \approx_1 (P^{-(k+1)}SRP^k)P = P^{-(k+1)}SRP^{k+1}$$

by (f). For (h):

$$P^{-n}SH = PP^{-(n+1)}SH \geq 0$$

by (d); using (e) as well we have

$$P^{-n}SRP^n = (P^{-n}SH)(H^{-1}RP^n) \approx_1 (H^{-1}RP^n)(P^{-n}SH) = H^{-1}RSH = H^{-1}AH = B.$$

□

3.2 *Proof of theorem* 1.1. If $A \geq 0$ has determinant 0, then $A = u^{tr}v$ for some integer vectors u, $v \geq 0$. Then $A \approx v^{tr}u = \text{trace}(A)$ as a 1×1 matrix. If $B \geq 0$ is similar to A over $\mathbb{Z}$, then $B \approx \text{trace}(B) = \text{trace}(A) \approx A$. Suppose now that A, $B > 0$ are of positive determinant and similar over $\mathbb{Z}$ via $Q \in \text{SL}(2, \mathbb{Z})$. Then lemma 2.1 reduces the proof to the case where Q is a unit shear or its inverse, and this case is verified by lemma 3.1 (with A and B interchanged in the inverse case). If $B = Q^{-1}AQ$ where $\det Q = -1$, let

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then B is similar to TAT via $TQ \in \text{SL}(2, \mathbb{Z})$, so from the earlier case $B \approx TAT$; but $TAT \approx ATT = A$. □

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