NOTE ON A THEOREM ON SINGULAR MATRICES

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J. A. Erdös proved recently [1] that every singular matrix over a field \( F \) is a product of idempotent matrices. He gave two proofs, one valid for matrices which are similar to triangular matrices and the other valid in general. We shall give a simple geometric proof of the above result. Instead of matrices we use linear operators. Moreover we get an explicit factorization in terms of projectors (idempotent operators).

Let \( A \) be a singular linear operator in \( n \)-dimensional vector space \( V \) over \( F \). By a well known decomposition theorem ([2], p. 189) \( V \) decomposes into a direct sum

\[
V = V_1 \oplus \ldots \oplus V_k
\]

such that each subspace \( V_i \) is \( A \)-cyclic and the minimal polynomial of the restriction of \( A \) to \( V_i \) is a power of a prime polynomial over \( F \). Since \( V_i \) is \( A \)-cyclic it has a basis \( e_i^j (j = 1, \ldots, m_i) \) such that

\[
e_i^1 A \in V_i \quad \text{and} \quad e_i^j A = e_i^{j-1} \quad \text{for} \quad j = 2, \ldots, m_i.
\]

\( A \) being singular, we can assume that \( e_i^1 A = 0 \). The vectors \( e_i^j (i = 1, \ldots, k; j = 1, \ldots, m_i) \) form a basic set of \( V \). If \( e \) is any of these vectors let \( V(e) \) be the \((n - 1)\)-dimensional subspace of \( V \) spanned by all basic vectors \( e_i^j \) except \( e \). If \( x \in V(e) \), we define \( P(e, x) \) to be the operator which maps \( e \) onto \( x \) and leaves \( V(e) \) pointwise fixed. It is obvious that \( P(e, x) \) is a projector of nullity 1. If

\[
P_0 = P(e_2^1, e_1^1)P(e_3^1, e_2^1)\ldots P(e_{m_1}^1, e_{m_1-1}^1),
\]

(1)

\[
P_i = P(e_i^1, e_{i-1}^1)P(e_{i+1}^1, e_i^1)P(e_{i+2}^1, e_{i+1}^1)\ldots P(e_{m_i}^1, e_{m_i-1}^1)P(e_{i+1}^1, e_{i+1}^1 A)
\]

for \( i = 2, \ldots, k \), then we claim that
(2) \[ A = P(e_1^j, 0)P_2P_3\ldots P_kP_0. \]

This is easy to verify since both sides in (2) have the same effect when applied to basic vectors \( e_j^i \).

REFERENCES


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