Nagoya Math. J. **208** (2012), 97–144
DOI 10.1215/00277630-1815222

ON THE KOTTWITZ-SHELSTAD NORMALIZATION OF TRANSFER FACTORS FOR AUTOMORPHIC INDUCTION FOR GL_n

KAORU HIRAGA AND ATSUSHI ICHINO

To the memory of Professor Hiroshi Saito

Abstract. Automorphic induction for GL_n is a case of endoscopic transfer, and its character identity was established by Henniart and Herb, up to a constant of proportionality. We determine this constant in terms of the Kottwitz-Shelstad normalization of transfer factors, which involves certain ε-factors.

CONTENTS

Introduction

Let F be a nonarchimedean local field of characteristic zero. Let $\mathbf{G} = \mathrm{GL}_n$, and let

$$\mathbf{H} = \mathrm{Res}_{E/F}\,\mathrm{GL}_{m_1} \times \cdots \times \mathrm{Res}_{E/F}\,\mathrm{GL}_{m_r},$$

Received March 17, 2011. Revised September 8, 2011. Accepted May 2, 2012.
2010 Mathematics Subject Classification. Primary 22E50.
Hiraga's work was partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (C) 22540018. Ichino's work was partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B) 22740021.

where E is a cyclic extension of F of degree d and $(m_1+\cdots+m_r)d=n$. Then $\mathbf{H}$ is a twisted endoscopic group of $\mathbf{G}$. Let $G=\mathbf{G}(F)$, and let $H=\mathbf{H}(F)$. By a result of Waldspurger [20, corollaire 1.7], we have a map

$$\mathrm{Tran}_H^G : \{\text{virtual characters of } H\} \longrightarrow \{\omega\text{-twisted virtual characters of } G\}$$

determined by matching orbital integrals. Here ω is a character of $F^\times$ of order d associated to E/F by class field theory.

Automorphic induction for GL_n asserts that for each irreducible tempered admissible representation π_H of H, there exist an irreducible tempered admissible representation π of G and a constant $c\in\mathbb{C}^\times$ such that $\pi\otimes\omega\cong\pi$ and

$$\mathrm{Tran}_H^G\big(\mathrm{trace}(\pi_H)\big)=c\cdot\mathrm{trace}(\pi\circ\mathcal{A}_\omega). \tag{0.1}$$

Here $\mathcal{A}_\omega:\pi\otimes\omega\to\pi$ is an intertwining operator. This was established by Henniart and Herb [7] when $r=1$, and it is given by the composition of the parabolic induction from H to $\mathrm{GL}_{m_1+\cdots+m_r}(E)$ with the automorphic induction from $\mathrm{GL}_{m_1+\cdots+m_r}(E)$ to G in general. Henniart and Lemaire [8], [9], [10] extended the fundamental lemma and the automorphic induction to the case of positive characteristic. Also, the automorphic induction has been extended to the case of inner forms of GL_n in [11].

The purpose of this paper is to determine the constant c for automorphic induction for GL_n in the case of characteristic zero. To be precise, Tran_H^G depends on a transfer factor Δ which is well defined up to a scalar. Also, $\mathcal{A}_\omega$ is well defined up to a scalar. In particular, we first need to normalize Δ and $\mathcal{A}_\omega$ to determine the constant c. Following Kottwitz and Shelstad [13], we can take Δ and $\mathcal{A}_\omega$ which depend only on an F-splitting $\mathbf{spl}_G$ of $\mathbf{G}$ and a nontrivial character ψ of F since $\mathbf{G}$ is quasi-split. Hence, c is well defined.

For $n=2$, Labesse and Langlands [14] proved that $c=1$. For arbitrary n, Henniart and Lemaire [9] proved that c does not depend on the representations and that $c=1$ if E/F is unramified. In this paper, we will prove that

$$c=1$$

in general (see Theorem 1.4). We remark that the transfer factor Δ involves a certain ε-factor which depends on the twisted endoscopic group $\mathbf{H}$.

This paper is organized as follows. In Section 1, we recall the definition of Tran_H^G and state the main theorem. We prove standard reductions in Sections 2–4 and recall a result of Henniart and Lemaire [9] on the product

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

formula for the constant c in Section 5. These imply that we may assume that E/F is tamely ramified, $\mathbf{H} = \mathrm{Res}_{E/F}\mathbb{G}_m$, and $\pi_H = \mathbb{1}$. To prove that $c = 1$ in this case, we will compute the two sides of (0.1) explicitly. In Section 6, we construct the intertwining operator $\mathcal{A}_\omega$ by using a result of Shahidi [17]. In Section 7, we compute the twisted character $\mathrm{trace}(\pi(f^G) \circ \mathcal{A}_\omega)$ for a certain $f^G \in C_c^\infty(G)$. Since $\pi_H = \mathbb{1}$, it remains to compute the twisted orbital integral for f^G and the transfer factor Δ. Following Waldspurger [19], we compute the twisted orbital integral and prove descent to topological unipotent elements in Section 8. In Section 9, we compute the transfer factor Δ explicitly. Using these and the fundamental lemma [19], we finish the proof of the main theorem in Section 10.

NOTATION. Let F be a nonarchimedean local field of characteristic zero. We fix an algebraic closure $\bar{F}$ of F. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group of F, and let W_F be the Weil group of F. Let $\mathfrak{o} = \mathfrak{o}_F$ be the maximal compact subring of F, let $\mathfrak{p} = \mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}$, let $\varpi = \varpi_F$ be a generator of $\mathfrak{p}$, and let $q = q_F$ be the cardinality of $\mathfrak{o}/\mathfrak{p}$. The valuation ord_F and the absolute value $|\cdot|$ on F are normalized so that $\mathrm{ord}_F\,\varpi = 1$ and $|\varpi| = q^{-1}$. We extend $|\cdot|$ to the absolute value on $\bar{F}$.

If $\mathbf{G}$ is a connected reductive algebraic group defined over F, let $G = \mathbf{G}(F)$ be the group of rational points of $\mathbf{G}$, let $\mathbf{G}_{\mathrm{SC}}$ be the simply connected cover of the derived group of $\mathbf{G}$, and let $\hat{G}$ be the Langlands dual group of $\mathbf{G}$. If $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, let $\mathbf{T}_{\mathrm{sc}}$ be the preimage of $\mathbf{T}$ in $\mathbf{G}_{\mathrm{SC}}$, and let

$$D_G(\gamma) = \prod_{\alpha \in R(\mathbf{G},\mathbf{T})} |\alpha(\gamma) - 1|^{1/2}$$

for $\gamma \in T$, where $R(\mathbf{G},\mathbf{T})$ is the set of roots of $\mathbf{T}$ in $\mathbf{G}$.

§1. Statement of the main theorem

Let $\mathbf{G} = \mathrm{GL}_n$. Let $\mathbf{spl}_G = (\mathbf{B}_0, \mathbf{T}_0, \{X_\alpha\})$ be an F-splitting of $\mathbf{G}$, and let ψ be a nontrivial character of F. Then $\mathbf{spl}_G$ and ψ determine a character χ of U_0, where $\mathbf{U}_0$ is the unipotent radical of $\mathbf{B}_0$ and

$$\chi\big(\exp(xX_\alpha)\big) = \psi(x)$$

for $x \in F$ and a simple root α of $\mathbf{T}_0$ in $\mathbf{B}_0$. We call $(\mathbf{B}_0, \chi)$ a *Whittaker datum* for $\mathbf{G}$.

Let $\mathbf{a} \in \mathrm{H}^1(W_F, Z(\hat{G}))$, where $Z(\hat{G})$ is the center of $\hat{G}$. Let $(\mathbf{H}, \mathcal{H}, s, \xi)$ be an endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$ (see [13, Section 2.1]).

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Since $\mathbf{G}$ is quasi-split over F, we have the absolute transfer factor Δ_0 (see [15, Section 3.7], [13, Section 5.3]). Recall that Δ_0 is a function on

$$\{G\text{-regular semisimple elements in } H\}$$
$$\times \{\text{regular semisimple elements in } G\}$$

given by the product of five terms:

$$\Delta_0 = \Delta_{\mathrm{I}} \cdot \Delta_{\mathrm{II}} \cdot \Delta_{\mathrm{III}_1} \cdot \Delta_{\mathrm{III}_2} \cdot \Delta_{\mathrm{IV}}.$$

To be precise, we implicitly fix an F-splitting $\mathbf{spl}_H = (\mathbf{B}_{H,0}, \mathbf{T}_{H,0}, \{Y_\alpha\})$ of $\mathbf{H}$ and a Γ-splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ (resp., $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$) of $\hat{G}$ (resp., $\hat{H}$). Moreover, by replacing $(\mathbf{H}, \mathcal{H}, s, \xi)$ by an endoscopic datum $(\mathbf{H}, \mathcal{H}, \mathrm{Int}(g)(s), \mathrm{Int}(g) \circ \xi)$ with some $g \in \hat{G}$ if necessary, we assume that $\xi(\mathcal{B}_H) \subset \mathcal{B}$ and that $\xi(\mathcal{T}_H) = \mathcal{T}$. However, Δ_0 depends only on $\mathbf{spl}_G$. Following [13, Section 5.3], we define a normalized transfer factor $\Delta = \Delta_{G,H}$ by

$$\Delta = \varepsilon(V, \psi) \cdot \Delta_0,$$

where

$$V = X^*(\mathbf{T}_0) \otimes \mathbb{C} - X^*(\mathbf{T}_{H,0}) \otimes \mathbb{C} \tag{1.1}$$

is a virtual representation of Γ of dimension zero and $\varepsilon(V, \psi)$ is the local constant as in [18, (3.6)]. Then Δ depends only on the Whittaker datum $(\mathbf{B}_0, \chi)$ associated to $\mathbf{spl}_G$ and ψ.

Remark 1.1. By [11, Proposition A.2], the transfer factor given by Henniart and Herb [7] agrees with Δ if E/F is unramified, and it agrees with Δ up to a scalar in general.

Let ω be the character of $F^\times$ associated to $\mathbf{a}$. We write $\omega(g) = \omega(\det g)$ for $g \in G$. Fix Haar measures dg and dh on G and H, respectively. For a regular semisimple element $\gamma \in G$ such that $G_\gamma \subset \ker \omega$ and $f^G \in C_c^\infty(G)$, put

$$J^\omega(\gamma, f^G) = \int_{G_\gamma \backslash G} \omega(g) f^G(g^{-1}\gamma g) \, \frac{dg}{dt},$$

where $\mathbf{G}_\gamma$ is the centralizer of γ in $\mathbf{G}$ and dt is a Haar measure on G_γ. Similarly, for a G-regular semisimple element $\gamma_H \in H$ and $f^H \in C_c^\infty(H)$, put

$$J(\gamma_H, f^H) = \int_{H_{\gamma_H} \backslash H} f^H(h^{-1}\gamma_H h) \, \frac{dh}{du},$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

where $\mathbf{H}_{\gamma_H}$ is the centralizer of γ_H in $\mathbf{H}$ and du is a Haar measure on H_{γ_H}. If γ_H is a norm of γ, then we have $H_{\gamma_H} \cong G_\gamma \subset \ker\omega$, and we take compatible measures dt and du.

By a result of Waldspurger [20, corollaire 1.7], for each $f^G \in C_c^\infty(G)$ there exists $f^H \in C_c^\infty(H)$ such that

$$J(\gamma_H, f^H) = \sum_\gamma \Delta(\gamma_H, \gamma) J^\omega(\gamma, f^G)$$

for all G-regular semisimple elements $\gamma_H \in H$. Here the sum is taken over a set of representatives for the conjugacy classes of $\gamma \in G$ whose norm is γ_H. We say that f^G and f^H have *matching orbital integrals*. For a virtual character θ of H, we define an ω-invariant distribution $\mathrm{Tran}_H^G(\theta)$ on G by

$$\mathrm{Tran}_H^G(\theta)(f^G) = \theta(f^H)$$

for $f^G \in C_c^\infty(G)$. Here we say that a distribution D on G is *ω-invariant* if

$$D\big(\mathrm{Ad}(g)(f^G)\big) = \omega(g)^{-1} D(f^G)$$

for all $g \in G$ and $f^G \in C_c^\infty(G)$.

On the other hand, by a result of Henniart and Herb [7, Theorem 1.3], for each irreducible tempered admissible representation π_H of H there exist an irreducible tempered admissible representation π of G on a space V_π and a constant $c \in \mathbb{C}^\times$ such that $\pi \otimes \omega \cong \pi$ and such that

$$\mathrm{Tran}_H^G\big(J(\pi_H)\big) = c \cdot J^\omega(\pi). \tag{1.2}$$

Here $J(\pi_H, f^H) = \mathrm{trace}(\pi_H(f^H))$ for $f^H \in C_c^\infty(H)$, and $J^\omega(\pi, f^G) = \mathrm{trace}(\pi(f^G) \circ \mathcal{A}_\omega)$ for $f^G \in C_c^\infty(G)$, where $\mathcal{A}_\omega : V_\pi \to V_\pi$ is an isomorphism such that

$$\mathcal{A}_\omega \circ (\pi \otimes \omega)(g) = \pi(g) \circ \mathcal{A}_\omega$$

for all $g \in G$. We remark that $J^\omega(\pi)$ and $J(\pi_H)$ depend on dg and dh, but the constant c does not depend on the choice of Haar measures. Also, $J^\omega(\pi)$ depends on the choice of $\mathcal{A}_\omega$. Since π is generic, we can normalize $\mathcal{A}_\omega$ so that

$$\lambda \circ \mathcal{A}_\omega = \lambda,$$

where $\lambda : V_\pi \to \mathbb{C}$ is a Whittaker functional with respect to χ.

We now recall a result of Henniart and Lemaire [9].

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

THEOREM 1.2 [9, théorème 4.11]. *The constant c does not depend on π_H. Moreover, $c=1$ if E/F is unramified.*

REMARK 1.3. Henniart [6, théorème 2] proved an analogous result for $F=\mathbb{R}$.

Our main result is as follows.

THEOREM 1.4. *We have*

$$c=1.$$

REMARK 1.5. Arthur and Clozel [1] proved an analogous result for base change for GL_n. Also, assuming certain properties of tempered L-packets, Waldspurger [21, lemme 4.8] proved an analogous result for special orthogonal groups.

§2. Reduction to the standard L-group structures

We retain the notation of Section 1. Let $(\mathbf{H},\mathcal{H},s,\xi)$ be an endoscopic datum for $(\mathbf{G},1,\mathbf{a})$.

Let $\gamma\in G$ be a regular semisimple element. Let $\gamma_H\in H$ be a G-regular semisimple element. Assume that γ_H is the image of γ. Let $\mathbf{T}$ (resp., $\mathbf{T}_H$) be the centralizer of γ (resp., γ_H) in $\mathbf{G}$ (resp., $\mathbf{H}$). Let $i_0:\mathbf{T}_{H,0}\to\mathbf{T}_0$ be the dual of $\xi^{-1}:\mathcal{T}\to\mathcal{T}_H$. We choose an admissible embedding

$$\begin{array}{ccc} \mathbf{T}_H & \xrightarrow{\ i\ } & \mathbf{T} \\ {\scriptstyle \mathrm{Int}(h^{-1})}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(g^{-1})} \\ \mathbf{T}_{H,0} & \xrightarrow{\ i_0\ } & \mathbf{T}_0 \end{array}$$

such that $i(\gamma_H)=\gamma$, where $h\in\mathbf{H}_{\mathrm{SC}}(\bar{F})$ and $g\in\mathbf{G}_{\mathrm{SC}}(\bar{F})$.

Let Δ be the transfer factor associated to $(\mathbf{H},\mathcal{H},s,\xi)$.

LEMMA 2.1. *If Δ' is the transfer factor associated to an endoscopic datum $(\mathbf{H},\mathcal{H},s',\xi)$ with $s'\in\hat{G}$, then*

$$\Delta'=\Delta.$$

Proof. By [11, Section 10], we have

$$s'\in s\cdot\left(Z(\hat{H})^{\Gamma}\right)^0.$$

Hence, the image of s' in $\pi_0(\hat{T}_{\mathrm{ad}}^{\Gamma})$ is equal to that of s. This completes the proof. □

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Recall that $\mathcal{H}$ is a split extension

$$1 \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1.$$

Let ${}^L H = \hat{H} \rtimes W_F$ denote the L-group of $\mathbf{H}$ with an action of Γ on $\hat{H}$ determined by $\mathbf{spl}_{\hat{H}}$. Since the derived group of $\mathbf{H}$ is simply connected, the identity map $\hat{H} \to \hat{H}$ gives rise to an L-isomorphism $\mathcal{H} \to {}^L H$ (see [13, Lemma 2.2.A]). This isomorphism is well defined up to a 1-cocycle of W_F in $Z(\hat{H})$.

We now take $\mathcal{H} = {}^L H$. If b is a 1-cocycle of W_F in $Z(\hat{H})$, let ω_b be the character of H associated to b. Let $\xi_b : {}^L H \hookrightarrow {}^L G$ be the L-embedding given by

$$\xi_b(h \times w) = \xi\big(b(w)h \times w\big)$$

for $h \in \hat{H}$ and $w \in W_F$. Let Δ_b be the transfer factor associated to an endoscopic datum $(\mathbf{H}, {}^L H, s, \xi_b)$. Let c_b be the constant for $(\mathbf{H}, {}^L H, s, \xi_b)$ given by (1.2).

LEMMA 2.2. *We have*

$$\Delta_b(\gamma_H, \gamma) = \omega_b(\gamma_H) \cdot \Delta(\gamma_H, \gamma).$$

Proof. It is easy to see that Δ_{I}, Δ_{II}, Δ_{III_1}, Δ_{IV}, and $\varepsilon(V, \psi)$ do not depend on b. Let $\xi_T : {}^L T \to {}^L G$, and let $\xi_{T_H} : {}^L T_H \to {}^L H$ be as in [15, Section 2.6]. We use the same χ-data to compute the transfer factor Δ_b. In particular, ξ_T and ξ_{T_H} do not depend on b. Let $\xi \circ \xi_{T_H} = a \cdot \xi_T$, where a is a 1-cocycle of W_F in $\mathcal{T}$. Then we have

$$\xi_b \circ \xi_{T_H} = (\xi \circ b) \cdot a \cdot \xi_T.$$

Thus, we obtain

$$\Delta_{\mathrm{III}_2, b}(\gamma_H, \gamma) = \omega_b(\gamma_H) \cdot \Delta_{\mathrm{III}_2}(\gamma_H, \gamma).$$

This completes the proof. □

LEMMA 2.3. *The constant c_b does not depend on b.*

Proof. Let ϕ_H be a tempered Langlands parameter for $\mathbf{H}$. Put $\phi_{H,b} = b^{-1} \cdot \phi_H$. Let π_H (resp., $\pi_{H,b}$) be the irreducible tempered admissible representation of H associated to ϕ_H (resp., $\phi_{H,b}$). By [3, Lemma VII.2.1], we have

$$J(\pi_{H,b}, \gamma_H) = \omega_b(\gamma_H)^{-1} \cdot J(\pi_H, \gamma_H).$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Hence, we have

$$\mathrm{Tran}^G_{H,b}\big(J(\pi_{H,b})\big) = \mathrm{Tran}^G_H\big(J(\pi_H)\big)$$

by Lemma 2.2. Here $\mathrm{Tran}^G_{H,b}$ is the transfer for $(\mathbf{H}, {}^LH, s, \xi_b)$. Since $\xi_b \circ \phi_{H,b} = \xi \circ \phi_H$, the assertion follows. □

Let $\mathbf{m} = (m_1, \dots, m_r)$ be a partition of n/d, and let $(\mathbf{H}_\mathbf{m}, {}^LH_\mathbf{m}, s_\mathbf{m}, \xi_\mathbf{m})$ be an endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$ defined in [11, Section 10]. To prove Theorem 1.4, we may assume that $(\mathbf{H}, \mathcal{H}, s, \xi) = (\mathbf{H}_\mathbf{m}, {}^LH_\mathbf{m}, s_\mathbf{m}, \xi_\mathbf{m})$ by Lemmas 2.1 and 2.3.

For convenience, we recall the definition of $(\mathbf{H}_\mathbf{m}, {}^LH_\mathbf{m}, s_\mathbf{m}, \xi_\mathbf{m})$. Let $\zeta = e^{2\pi\sqrt{-1}/n}$. Let a be the unique 1-cocycle of W_F in $Z(\hat{G})$ representing $\mathbf{a}$. Let E be the cyclic extension of F of degree d associated to $\ker a$ by class field theory. Let σ be the generator of $\mathrm{Gal}(E/F)$ such that $a(\sigma) = \zeta^{n/d}$. We fix an element $w_\sigma \in W_F$ in the preimage of σ.

For each integer m, let $\mathbf{H}_m = \mathrm{Res}_{E/F}\,\mathrm{GL}_m$, let $\hat{H}_m = \mathrm{GL}_m(\mathbb{C})^d$, and let ${}^LH_m = \hat{H}_m \rtimes W_F$. Here W_E acts on $\hat{H}_m$ trivially, and σ acts on $\hat{H}_m$ by

$$(g_1, \dots, g_d) \longmapsto (g_d, g_1, \dots, g_{d-1}).$$

We take the standard splittings $\mathbf{spl}_{H_m}$ and $\mathbf{spl}_{\hat{H}_m}$ of $\mathbf{H}_m$ and $\hat{H}_m$, respectively. For $x \in \mathbb{C}^\times$, put

$$s_m(x) = \mathrm{diag}(x \cdot \mathbf{1}_m, \zeta^{n/d}x \cdot \mathbf{1}_m, \dots, \zeta^{(d-1)n/d}x \cdot \mathbf{1}_m) \in \mathrm{GL}_{md}(\mathbb{C}).$$

We define a homomorphism $\xi_{m,0} : {}^LH_m \to \mathrm{GL}_{md}(\mathbb{C})$ by

$$\xi_{m,0}\big((g_1, \dots, g_d) \times w\big) = \mathrm{diag}(g_1, \dots, g_d)$$

for $(g_1, \dots, g_d) \in \hat{H}_m$ and $w \in W_E$, and

$$\xi_{m,0}(1 \times w_\sigma) = \begin{pmatrix} & & & \mathbf{1}_m \\ \mathbf{1}_m & & & \\ & \ddots & & \\ & & \mathbf{1}_m & \end{pmatrix}.$$

Let $\mathbf{H}_\mathbf{m} = \mathbf{H}_{m_1} \times \cdots \times \mathbf{H}_{m_r}$, let $\hat{H}_\mathbf{m} = \hat{H}_{m_1} \times \cdots \times \hat{H}_{m_r}$, and let ${}^LH_\mathbf{m} = \hat{H}_\mathbf{m} \rtimes W_F$. Put

$$s_\mathbf{m} = \mathrm{diag}\big(s_{m_1}(x_1), \dots, s_{m_r}(x_r)\big) \in \mathrm{GL}_n(\mathbb{C}),$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

where we choose $x_1, \dots, x_r \in \mathbb{C}^\times$ such that

$$\{x_i, \zeta^{n/d}x_i, \dots, \zeta^{(d-1)n/d}x_i\} \cap \{x_j, \zeta^{n/d}x_j, \dots, \zeta^{(d-1)n/d}x_j\} = \varnothing$$

if $i \neq j$. We define a homomorphism $\xi_{\mathbf{m}} : {}^L H_{\mathbf{m}} \to {}^L G$ by

$$\xi_{\mathbf{m}}\big((h_1, \dots, h_r) \times w\big) = \operatorname{diag}\big(\xi_{m_1,0}(h_1 \times w), \dots, \xi_{m_r,0}(h_r \times w)\big) \times w$$

for $(h_1, \dots, h_r) \in \hat{H}_{\mathbf{m}}$ and $w \in W_F$. Note that the equivalence class of $(\mathbf{H_m}, {}^L H_{\mathbf{m}}, s_{\mathbf{m}}, \xi_{\mathbf{m}})$ does not depend on the choice of $x_1, \dots, x_r$.

§3. Reduction to the standard Whittaker data

We retain the notation of Section 1. Let $(\mathbf{B}_0, \chi)$ be the Whittaker datum for $\mathbf{G}$ associated to $\mathbf{spl}_G$ and ψ, where $\mathbf{spl}_G$ is an F-splitting of $\mathbf{G}$ and ψ is a nontrivial character of F. Let $\mathbf{m}$ be a partition of n/d, and let $(\mathbf{H}, \mathcal{H}, s, \xi) = (\mathbf{H_m}, {}^L H_{\mathbf{m}}, s_{\mathbf{m}}, \xi_{\mathbf{m}})$ be the endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$ defined in Section 2.

Let $\tilde{t} \in T_0$, and let $\mathbf{spl}'_G = \mathrm{Int}(\tilde{t})(\mathbf{spl}_G)$. Let Δ' be the transfer factor with respect to $\mathbf{spl}'_G$ and ψ.

LEMMA 3.1. *We have*

$$\Delta' = \omega(\det \tilde{t}) \cdot \Delta.$$

Proof. By replacing $\mathbf{spl}_G$ by its G_{SC}-conjugate if necessary, we may assume that $\mathbf{spl}_G$ is the standard splitting of $\mathbf{G}$. Let $\mathbf{T}_H$ be a maximal torus of $\mathbf{H_m}$ which is maximally split. We choose an admissible embedding $\mathbf{T}_H \to \mathbf{G}$, and let $\mathbf{T}$ be its image. We identify $\mathbf{T}_H$ with $\mathbf{T}$. Then we have an exact sequence

$$1 \longrightarrow \mathbf{T}_{\mathrm{sc}} \longrightarrow \mathbf{T} \xrightarrow{\det} \mathbb{G}_m \longrightarrow 1$$

and its dual

$$1 \longleftarrow \hat{T}_{\mathrm{ad}} \longleftarrow \hat{T} \longleftarrow \mathbb{C}^\times \longleftarrow 1.$$

These induce long exact sequences

$$\cdots \longrightarrow \mathrm{H}^0(F, \mathbf{T}) \longrightarrow \mathrm{H}^0(F, \mathbb{G}_m) \xrightarrow{\delta} \mathrm{H}^1(F, \mathbf{T}_{\mathrm{sc}}) \longrightarrow \mathrm{H}^1(F, \mathbf{T}) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow \mathrm{H}^1(W_F, \hat{T}) \longleftarrow \mathrm{H}^1(W_F, \mathbb{C}^\times) \xleftarrow{\hat{\delta}} \mathrm{H}^0(W_F, \hat{T}_{\mathrm{ad}})$$
$$\longleftarrow \mathrm{H}^0(W_F, \hat{T}) \longleftarrow \cdots.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

It is easy to see that

$$\hat{\delta}(\text{the image of } s_{\mathbf{m}} \text{ in } \hat{T}_{\mathrm{ad}}) = \mathbf{a}.$$

Hence, we have

$$\langle \delta(x), s_{\mathbf{m}} \rangle = \omega(x) \tag{3.1}$$

for $x \in F^{\times}$, where

$$\langle \cdot, \cdot \rangle : \mathrm{H}^1(F, \mathbf{T}_{\mathrm{sc}}) \times \pi_0(\hat{T}_{\mathrm{ad}}^{\Gamma}) \longrightarrow \mathbb{C}^{\times}$$

is the Tate-Nakayama pairing.

For $\tilde{g} \in G$, let $g = z \cdot \tilde{g} \in \mathbf{G}_{\mathrm{SC}}(\bar{F})$, where we choose $z \in \bar{F}^{\times}$ such that $\det \tilde{g} = z^{-n}$. Then we have $\mathrm{Int}(g) = \mathrm{Int}(\tilde{g})$ and $z^{-n} \in F^{\times}$. Let $\mathbf{z}$ be the element in $\mathrm{H}^1(F, Z(\mathbf{G}_{\mathrm{SC}}))$ induced by

$$\tau(g)^{-1} g = \tau(z)^{-1} z \cdot \mathbf{1}_n$$

for $\tau \in \Gamma$. We remark that the 1-cocycle of Γ as in [13, p. 33] is given by $\tau \mapsto g^{-1}\tau(g)$, which is the inverse of ours. We can define $\langle \mathbf{z}, s_{\mathbf{m}} \rangle$ by using the natural homomorphism $\mathrm{H}^1(F, Z(\mathbf{G}_{\mathrm{SC}})) \to \mathrm{H}^1(F, \mathbf{T}_{\mathrm{sc}})$. Since $\delta(z^n)$ is equal to the image of $\mathbf{z}$ in $\mathrm{H}^1(F, \mathbf{T}_{\mathrm{sc}})$, (3.1) implies that

$$\langle \mathbf{z}, s_{\mathbf{m}} \rangle = \omega(z^n) = \omega(\det \tilde{g})^{-1}. \tag{3.2}$$

Hence, the lemma follows from [13, Section 5.3] and (3.2). □

LEMMA 3.2. *The constant c given by (1.2) does not depend on $(\mathbf{B}_0, \chi)$.*

Proof. Let $(\mathbf{B}_0', \chi')$ be another Whittaker datum for $\mathbf{G}$ associated to $\mathbf{spl}_G'$ and ψ', where $\mathbf{spl}_G'$ is an F-splitting of $\mathbf{G}$ and ψ' is a nontrivial character of F. By [13, Section 5.3], Δ' does not depend on the choice of $\mathbf{spl}_G'$ and ψ' giving rise to the same $(\mathbf{B}_0', \chi')$. By replacing $\mathbf{spl}_G'$ and ψ' if necessary, we may assume that $\psi' = \psi$. Moreover, by replacing $\mathbf{spl}_G'$ by its G_{SC}-conjugate if necessary, we may assume that $\mathbf{spl}_G' = \mathrm{Int}(\tilde{t})(\mathbf{spl}_G)$ for some $\tilde{t} \in T_0$. By Lemma 3.1, we have

$$\mathrm{Tran}'^{G}_{H}\big(J(\pi_H)\big) = \omega(\det \tilde{t}) \cdot \mathrm{Tran}^{G}_{H}\big(J(\pi_H)\big)$$

for an irreducible tempered admissible representation π_H of H, where Tran'^{G}_{H} is the transfer with respect to $(\mathbf{B}_0', \chi')$.

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

On the other hand, let π be the irreducible tempered admissible representation of G such that $\pi \otimes \omega \cong \pi$. Let $\lambda : V_\pi \to \mathbb{C}$ be a Whittaker functional with respect to χ. Since $\chi'(u) = \chi(\mathrm{Int}(\tilde{t}^{-1})u)$ for $u \in U_0$, the map $\lambda' : V_\pi \to \mathbb{C}$ given by $\lambda' = \lambda \circ \pi(\tilde{t}^{-1})$ is a Whittaker functional with respect to χ'. Thus, we obtain

$$\mathcal{A}'_\omega = \omega(\det \tilde{t}) \cdot \mathcal{A}_\omega,$$

where $\mathcal{A}'_\omega : V_\pi \to V_\pi$ is the intertwining operator with respect to λ'. This yields the lemma. □

§4. Reduction to Levi subgroups

Let $\mathbf{G} = \mathrm{GL}_n$, and let $\mathbf{a} \in \mathrm{H}^1(W_F, Z(\hat{G}))$. Let ψ be a nontrivial character of F. Let $(\mathbf{H}, {}^L H, s, \xi)$ be an endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$. We fix an F-splitting $\mathbf{spl}_G = (\mathbf{B}_0, \mathbf{T}_0, \{X_\alpha\})$ (resp., $\mathbf{spl}_H = (\mathbf{B}_{H,0}, \mathbf{T}_{H,0}, \{Y_\alpha\})$) of $\mathbf{G}$ (resp., $\mathbf{H}$) and a Γ-splitting $\mathbf{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ (resp., $\mathbf{spl}_{\hat{H}} = (\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$) of $\hat{G}$ (resp., $\hat{H}$). We may assume that the image of W_F in $\hat{G}$ under ξ is bounded. Moreover, we may assume that $\xi(\mathcal{B}_H) \subset \mathcal{B}$ and that $\xi(\mathcal{T}_H) = \mathcal{T}$.

We first prove descent to Levi subgroups of $\mathbf{H}$. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{H}$, and let ${}^L L$ be the Levi subgroup of ${}^L H$ associated to $\mathbf{L}$. Let $(\mathbf{L}, {}^L L, s_L, \xi_L)$ be the endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$, where $s_L = s \cdot z$ with some $z \in \xi(Z(\hat{L})^0)$ such that the connected centralizer of s_L in $\hat{G}$ is $\xi(\hat{L})$, and $\xi_L = \xi|_{{}^L L}$. We take the F-splitting $\mathbf{spl}_L$ of $\mathbf{L}$ and the Γ-splitting $\mathbf{spl}_{\hat{L}}$ of $\hat{L}$ given by $\mathbf{spl}_L = (\mathbf{B}_{H,0} \cap \mathbf{L}, \mathbf{T}_{H,0}, \{Y_\alpha\}_{\alpha \in S(\mathbf{L})})$ and $\mathbf{spl}_{\hat{L}} = (\mathcal{B}_H \cap \hat{L}, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\}_{\alpha \in S(\mathbf{L})})$, respectively. Here $S(\mathbf{L})$ is the set of simple roots of $\mathbf{T}_{H,0}$ in $\mathbf{B}_{H,0} \cap \mathbf{L}$. Let $\Delta_{G,H}$ and $\Delta_{G,L}$ be the transfer factors with respect to $\mathbf{spl}_G$ and ψ as in Section 1.

LEMMA 4.1. *Let $\gamma \in G$ be a regular semisimple element. Let $\gamma_H \in L$ be a G-regular semisimple element. If γ_H is the image of γ with respect to the endoscopic datum $(\mathbf{L}, {}^L L, s_L, \xi_L)$ for $(\mathbf{G}, 1, \mathbf{a})$, then we have*

$$\Delta_{G,L}(\gamma_H, \gamma) = \Delta_{G,H}(\gamma_H, \gamma) \cdot \frac{D_H(\gamma_H)}{D_L(\gamma_H)}.$$

Proof. Assume that γ_H is the image of γ with respect to $\mathbf{L}$. Let $\mathbf{T}$ (resp., $\mathbf{T}_H$) be the centralizer of γ (resp., γ_H) in $\mathbf{G}$ (resp., $\mathbf{L}$). Let i_0 :

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

$\mathbf{T}_{H,0} \to \mathbf{T}_0$ be the dual of $\xi^{-1} : \mathcal{T} \to \mathcal{T}_H$. We choose an admissible embedding

$$\begin{array}{ccc} \mathbf{T}_H & \xrightarrow{i} & \mathbf{T} \\ {\scriptstyle \mathrm{Int}(h^{-1})}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(g^{-1})} \\ \mathbf{T}_{H,0} & \xrightarrow{i_0} & \mathbf{T}_0 \end{array}$$

with respect to $\mathbf{L}$ such that $i(\gamma_H) = \gamma$, where $h \in \mathbf{L}_{\mathrm{SC}}(\bar{F})$ and $g \in \mathbf{G}_{\mathrm{SC}}(\bar{F})$. We identify h with its image in $\mathbf{H}_{\mathrm{SC}}(\bar{F})$ and regard i as an admissible embedding with respect to $\mathbf{H}$. We can take a-data $\{a_\alpha\}$ and χ-data $\{\chi_\alpha\}$ so that $a_\alpha = 1$ and $\chi_\alpha = \mathbb{1}$ for $\alpha \in R(\mathbf{H}, \mathbf{T}_H) - R(\mathbf{L}, \mathbf{T}_H)$ since such an α is asymmetric (see [15, Section 2.1]).

By definition, we have $\Delta_{G,L,\mathrm{I}}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{I}}(\gamma_H, \gamma)$ and $\Delta_{G,L,\mathrm{III}_1}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{III}_1}(\gamma_H, \gamma) = 1$. Since $\chi_\alpha = \mathbb{1}$ for $\alpha \in R(\mathbf{H}, \mathbf{T}_H) - R(\mathbf{L}, \mathbf{T}_H)$, we have $\Delta_{G,L,\mathrm{II}}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{II}}(\gamma_H, \gamma)$.

Let $\bullet = \mathbf{L}$ or $\mathbf{H}$. Let $\xi_T^\bullet$ and $\xi_{T_H}^\bullet$ be as in [15, Section 2.6] with respect to $\bullet$. By definition, we have $\xi_T^L = \xi_T^H$. Let $n(\omega_{T_H}^\bullet(\sigma))$ be as in [15, Section 2.6] with respect to $\bullet$. Then we have $\omega_{T_H}^\bullet(\sigma) \in \Omega(\mathbf{L}, \mathbf{T}_{H,0}) = \Omega(\hat{L}, \mathcal{T}_H)$, so that $n(\omega_{T_H}^L(\sigma)) = n(\omega_{T_H}^H(\sigma))$ by the definition of $\mathbf{spl}_{\hat{L}}$. Let $r_{p_0}^\bullet$ and $s_{p/p_0}^\bullet$ be as in [15, Section 2.6] with respect to $\bullet$, where p_0 and p are gauges as in [15, p. 235] and [15, p. 238], respectively. To be precise, we identify $R(\bullet, \mathbf{T}_H)$ with $R(\bullet, \mathbf{T}_{H,0}) = \check{R}(\hat{\bullet}, \mathcal{T}_H)$, where $\check{R}(\hat{\bullet}, \mathcal{T}_H)$ is the set of coroots of $\mathcal{T}_H$ in $\hat{\bullet}$, and we take gauges $p_0^\bullet$ on $R(\bullet, \mathbf{T}_H)$ so that the restriction of p_0^H to $R(\mathbf{L}, \mathbf{T}_H)$ is equal to p_0^L. If $\alpha \in R(\mathbf{H}, \mathbf{T}_H) - R(\mathbf{L}, \mathbf{T}_H)$, then α does not contribute to $r_{p_0}^H$ since $\chi_\alpha = \mathbb{1}$, and α does not contribute to s_{p/p_0}^H since α is asymmetric. Indeed, we have $p_0(\tau^{-1}(\alpha)) = p_0(\alpha)$ and $p(\tau^{-1}(\alpha)) = p(\alpha)$ for $\tau \in \Gamma$ and $\alpha \in R(\mathbf{H}, \mathbf{T}_H) - R(\mathbf{L}, \mathbf{T}_H)$. Hence, we have $r_{p_0}^L = r_{p_0}^H$ and $s_{p/p_0}^L = s_{p/p_0}^H$. Thus, we obtain $\xi_{T_H}^L = \xi_{T_H}^H$, and hence $\Delta_{G,L,\mathrm{III}_2}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{III}_2}(\gamma_H, \gamma)$. This completes the proof. □

Let ϕ_L be a tempered Langlands parameter for $\mathbf{L}$. Let ϕ_H be the composition of ϕ_L with the embedding ${}^L L \subset {}^L H$. Put $\phi_G = \xi_L \circ \phi_L$. Let π_L, π_H, π_G be the irreducible tempered admissible representations of L, H, G associated to ϕ_L, ϕ_H, ϕ_G, respectively. As in (1.2), we define c_L and c_H by $\mathrm{Tran}_L^G(J(\pi_L)) = c_L \cdot J^\omega(\pi_G)$ and $\mathrm{Tran}_H^G(J(\pi_H)) = c_H \cdot J^\omega(\pi_G)$, respectively.

LEMMA 4.2. *We have*

$$c_L = c_H.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. By [11, Lemma 8.8] and Lemma 4.1, we have $\mathrm{Tran}_L^G(J(\pi_L)) = \mathrm{Tran}_H^G(J(\pi_H))$. This completes the proof. □

We next prove descent to Levi subgroups of $\mathbf{G}$. Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$, and let ${}^LM$ be the Levi subgroup of ${}^LG$ associated to $\mathbf{M}$. Assume that $\xi({}^LH) \subset {}^LM$. We may regard $(\mathbf{H}, {}^LH, s, \xi)$ as an endoscopic datum for $(\mathbf{M}, 1, \mathbf{a})$. We take the F-splitting $\mathbf{spl}_M$ of $\mathbf{M}$ and the Γ-splitting $\mathbf{spl}_{\hat{M}}$ of $\hat{M}$ given by $\mathbf{spl}_M = (\mathbf{B}_0 \cap \mathbf{M}, \mathbf{T}_0, \{X_\alpha\}_{\alpha \in S(\mathbf{M})})$ and $\mathbf{spl}_{\hat{M}} = (\mathcal{B} \cap \hat{M}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\}_{\alpha \in S(\mathbf{M})})$, respectively. Here $S(\mathbf{M})$ is the set of simple roots of $\mathbf{T}_0$ in $\mathbf{B}_0 \cap \mathbf{M}$. Let $\Delta_{G,H}$ (resp., $\Delta_{M,H}$) be the transfer factor with respect to $\mathbf{spl}_G$ and ψ (resp., $\mathbf{spl}_M$ and ψ) as in Section 1.

LEMMA 4.3. *Let $\gamma \in M$ be a G-regular semisimple element. Let $\gamma_H \in H$ be a G-regular semisimple element. If γ_H is the image of γ with respect to the endoscopic datum $(\mathbf{H}, {}^LH, s, \xi)$ for $(\mathbf{M}, 1, \mathbf{a})$, then we have*

$$\Delta_{M,H}(\gamma_H, \gamma) = \Delta_{G,H}(\gamma_H, \gamma) \cdot \frac{D_M(\gamma)}{D_G(\gamma)}.$$

Proof. Assume that γ_H is the image of γ with respect to $\mathbf{M}$. Let $\mathbf{T}$ (resp., $\mathbf{T}_H$) be the centralizer of γ (resp., γ_H) in $\mathbf{M}$ (resp., $\mathbf{H}$). Let $i_0 : \mathbf{T}_{H,0} \to \mathbf{T}_0$ be the dual of $\xi^{-1} : \mathcal{T} \to \mathcal{T}_H$. We choose an admissible embedding

$$\begin{array}{ccc} \mathbf{T}_H & \xrightarrow{i} & \mathbf{T} \\ {\scriptstyle \mathrm{Int}(h^{-1})}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(g_M^{-1})} \\ \mathbf{T}_{H,0} & \xrightarrow{i_0} & \mathbf{T}_0 \end{array}$$

with respect to $\mathbf{M}$ such that $i(\gamma_H) = \gamma$, where $h \in \mathbf{H}_{\mathrm{SC}}(\bar{F})$ and $g_M \in \mathbf{M}_{\mathrm{SC}}(\bar{F})$. We regard i as an admissible embedding with respect to $\mathbf{G}$ by using the image g_G of g_M in $\mathbf{G}_{\mathrm{SC}}(\bar{F})$. We can take a-data $\{a_\alpha\}$ and χ-data $\{\chi_\alpha\}$ so that $a_\alpha = 1$ and $\chi_\alpha = \mathbb{1}$ for $\alpha \in R(\mathbf{G}, \mathbf{T}) - R(\mathbf{M}, \mathbf{T})$ since such an α is asymmetric (see [15, Section 2.1]).

By definition, we have $\Delta_{M,H,\mathrm{III}_1}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{III}_1}(\gamma_H, \gamma) = 1$. Since $\chi_\alpha = \mathbb{1}$ for $\alpha \in R(\mathbf{G}, \mathbf{T}) - R(\mathbf{M}, \mathbf{T})$, we have $\Delta_{M,H,\mathrm{II}}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{II}}(\gamma_H, \gamma)$.

Let $\bullet = \mathbf{M}$ or $\mathbf{G}$. Let $\mathbf{T}^\bullet_{\mathrm{sc}}$ (resp., $\mathbf{T}^\bullet_{0,\mathrm{sc}}$) be the preimage of $\mathbf{T}$ (resp., $\mathbf{T}_0$) in $\bullet_{\mathrm{SC}}$. Let $m(\sigma_{T^\bullet_{\mathrm{sc}}}) = x(\sigma_{T^\bullet_{\mathrm{sc}}}) n(\omega_{T^\bullet_{\mathrm{sc}}}(\sigma))$ and $\lambda(\mathbf{T}^\bullet_{\mathrm{sc}}) \in \mathrm{H}^1(F, \mathbf{T}^\bullet_{\mathrm{sc}})$ be as in [15, Section 2.3]. Then the image of $\omega_{T^M_{\mathrm{sc}}}(\sigma) = \mathrm{Int}(g_M^{-1}\sigma(g_M)) \in \Omega(\mathbf{M}_{\mathrm{SC}}, \mathbf{T}^M_{0,\mathrm{sc}})$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

in $\Omega(\mathbf{G}_{\mathrm{SC}}, \mathbf{T}^G_{0,\mathrm{sc}})$ is equal to $\omega_{T^G_{\mathrm{sc}}}(\sigma) = \mathrm{Int}(g_G^{-1}\sigma(g_G))$, so that the image of $n(\omega_{T^M_{\mathrm{sc}}}(\sigma))$ in $\mathbf{G}_{\mathrm{SC}}(\bar{F})$ is equal to $n(\omega_{T^G_{\mathrm{sc}}}(\sigma))$ by the definition of $\mathbf{spl}_M$. Since $a_\alpha = 1$ for $\alpha \in R(\mathbf{G}, \mathbf{T}) - R(\mathbf{M}, \mathbf{T})$, the image of $x(\sigma_{T^M_{\mathrm{sc}}})$ in $\mathbf{T}^G_{\mathrm{sc}}(\bar{F})$ is equal to $x(\sigma_{T^G_{\mathrm{sc}}})$. Hence, the image of $m(\sigma_{T^M_{\mathrm{sc}}})$ in $\mathbf{G}_{\mathrm{SC}}(\bar{F})$ is equal to $m(\sigma_{T^G_{\mathrm{sc}}})$, so that the image of $\lambda(\mathbf{T}^M_{\mathrm{sc}})$ in $\mathrm{H}^1(F, \mathbf{T}^G_{\mathrm{sc}})$ is equal to $\lambda(\mathbf{T}^G_{\mathrm{sc}})$. Let $\mathbf{s}^\bullet_T \in \pi_0((\hat{T}/Z(\hat{\bullet}))^\Gamma)$ be the image of $s \in \mathcal{T}$ under

$$\begin{array}{ccccccc} \mathcal{T} = \hat{T}_0 & \longrightarrow & \hat{T} & \longrightarrow & \hat{T}/Z(\hat{G}) & \longrightarrow & \hat{T}/Z(\hat{M}) \\ & & & & \downarrow & & \downarrow \\ & & & & \pi_0\big((\hat{T}/Z(\hat{G}))^\Gamma\big) & \longrightarrow & \pi_0\big((\hat{T}/Z(\hat{M}))^\Gamma\big) \end{array}$$

In particular, the image of $\mathbf{s}^G_T$ in $\pi_0((\hat{T}/Z(\hat{M}))^\Gamma)$ is equal to $\mathbf{s}^M_T$. Thus, we obtain

$$\Delta_{M,H,\mathrm{I}}(\gamma_H, \gamma) = \langle \lambda(T^M_{\mathrm{sc}}), \mathbf{s}^M_T \rangle = \langle \lambda(T^G_{\mathrm{sc}}), \mathbf{s}^G_T \rangle = \Delta_{G,H,\mathrm{I}}(\gamma_H, \gamma).$$

Let $\xi^\bullet_T$ and $\xi^\bullet_{T_H}$ be as in [15, Section 2.6] with respect to $\bullet$. By definition, we have $\xi^M_{T_H} = \xi^G_{T_H}$. Let $n(\omega^\bullet_T(\sigma))$ be as in [15, Section 2.6] with respect to $\bullet$. Then the image of $\omega^M_T(\sigma) \in \Omega(\mathbf{M}, \mathbf{T}_0) = \Omega(\hat{M}, \mathcal{T})$ in $\Omega(\hat{G}, \mathcal{T})$ is equal to $\omega^G_T(\sigma)$, so that $n(\omega^M_T(\sigma)) = n(\omega^G_T(\sigma))$ by the definition of $\mathbf{spl}_{\hat{M}}$. Let $r^\bullet_{p_0}$ and $s^\bullet_{p/p_0}$ be as in [15, Section 2.6] with respect to $\bullet$. To be precise, we take gauges $p^\bullet_0$ on $R(\bullet, \mathbf{T})$ so that the restriction of p^G_0 to $R(\mathbf{M}, \mathbf{T})$ is equal to p^M_0. If $\alpha \in R(\mathbf{G}, \mathbf{T}) - R(\mathbf{M}, \mathbf{T})$, then α does not contribute to $r^G_{p_0}$ since $\chi_\alpha = \mathbb{1}$, and α does not contribute to s^G_{p/p_0} since α is asymmetric. Hence, we have $r^M_{p_0} = r^G_{p_0}$ and $s^M_{p/p_0} = s^G_{p/p_0}$. Thus, we obtain $\xi^M_T = \xi^G_T$ and $\Delta_{M,H,\mathrm{III}_2}(\gamma_H, \gamma) = \Delta_{G,H,\mathrm{III}_2}(\gamma_H, \gamma)$. This completes the proof. □

Let ϕ_H be a tempered Langlands parameter for $\mathbf{H}$. Put $\phi_G = \xi \circ \phi_H$. We may regard ϕ_G as a tempered Langlands parameter ϕ_M for $\mathbf{M}$. Let π_H, π_M, π_G be the irreducible tempered admissible representations of H, M, G associated to ϕ_H, ϕ_M, ϕ_G, respectively.

Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{B}_0$, and let χ be the generic character associated to $\mathbf{spl}_G$ and ψ. Put $\chi^M = \chi|_{U_0 \cap M}$. Then χ^M is the generic character associated to $\mathbf{spl}_M$ and ψ. Let $\mathcal{A}^M_\omega : V_{\pi_M} \to V_{\pi_M}$ be the intertwining

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

operator such that $\mathcal{A}_\omega^M \circ (\pi_M \otimes \omega) = \pi_M \circ \mathcal{A}_\omega^M$ and $\lambda^M \circ \mathcal{A}_\omega^M = \lambda^M$, where $\lambda^M : V_{\pi_M} \to \mathbb{C}$ is a Whittaker functional with respect to χ^M.

We now construct the intertwining operator $\mathcal{A}_\omega^G : V_{\pi_G} \to V_{\pi_G}$ explicitly. We may assume that

$$\mathbf{M} = \{\mathrm{diag}(g_1, \dots, g_r) \mid g_1 \in \mathrm{GL}_{m_1}, \dots, g_r \in \mathrm{GL}_{m_r}\},$$

where $m_1 + \cdots + m_r = n$. We can take the standard splittings $\mathbf{spl}_G$ and $\mathbf{spl}_M$ of $\mathbf{G}$ and $\mathbf{M}$, respectively. Then we have $\pi_G = \mathrm{Ind}_P^G(\pi_M)$, where $\mathbf{P} = \mathbf{MN}$ is the standard parabolic subgroup of $\mathbf{G}$ and the space V_{π_G} consists of locally constant V_{π_M}-valued functions Φ on G such that

$$\Phi(mng) = \delta_P^{1/2}(m)\pi_M(m)\Phi(g)$$

for $m \in M$, $n \in N$, $g \in G$. Here δ_P is the modulus character of P. We define $\mathcal{A}_\omega^G$ by

$$(\mathcal{A}_\omega^G \Phi)(g) = \omega(g)\mathcal{A}_\omega^M \Phi(g)$$

for $\Phi \in V_{\pi_G}$ and $g \in G$. Then we have $\mathcal{A}_\omega^G \circ (\pi_G \otimes \omega) = \pi_G \circ \mathcal{A}_\omega^G$.

LEMMA 4.4. *We have*

$$\lambda^G \circ \mathcal{A}_\omega^G = \lambda^G,$$

where $\lambda^G : V_{\pi_G} \to \mathbb{C}$ *is a Whittaker functional with respect to* χ.

Proof. By [16, Proposition 3.1], we may assume that the Whittaker functional λ^G is given by

$$\lambda^G(\Phi) = \int_{U'} \lambda^M\big(\Phi(w_0 u')\big)\overline{\chi(u')}\, du'$$

for $\Phi \in V_{\pi_G}$, where

$$w_0 = \begin{pmatrix} & & \mathbf{1}_{m_1} \\ & \iddots & \\ \mathbf{1}_{m_r} & & \end{pmatrix}$$

and

$$\mathbf{U}' = \left\{ \begin{pmatrix} \mathbf{1}_{m_r} & * & * \\ & \ddots & * \\ & & \mathbf{1}_{m_1} \end{pmatrix} \right\}.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Since $\pi_M \otimes \omega \cong \pi_M$, the order of ω divides $m_1, \dots, m_r$, so that $\omega(w_0) = 1$. Hence, we have

$$\begin{aligned}
\lambda^G(\mathcal{A}^G_\omega \Phi) &= \int_{U'} \lambda^M\big(\mathcal{A}^G_\omega \Phi(w_0 u')\big)\overline{\chi(u')}\, du' \\
&= \int_{U'} \lambda^M\big(\mathcal{A}^M_\omega \omega(w_0 u')\Phi(w_0 u')\big)\overline{\chi(u')}\, du' \\
&= \int_{U'} \lambda^M\big(\mathcal{A}^M_\omega \Phi(w_0 u')\big)\overline{\chi(u')}\, du' \\
&= \int_{U'} \lambda^M\big(\Phi(w_0 u')\big)\overline{\chi(u')}\, du' \\
&= \lambda^G(\Phi). \qquad \square
\end{aligned}$$

As in (1.2), we define c_M and c_G by $\mathrm{Tran}^M_H(J(\pi_H)) = c_M \cdot J^\omega(\pi_M)$ and $\mathrm{Tran}^G_H(J(\pi_H)) = c_G \cdot J^\omega(\pi_G)$, respectively.

LEMMA 4.5. *We have*

$$c_M = c_G.$$

Proof. By [11, Lemma 8.6] and Lemmas 4.3 and 4.4, $\mathrm{Tran}^M_H(J(\pi_H)) = c_M \cdot J^\omega(\pi_M)$ implies that

$$\mathrm{Tran}^G_H\big(J(\pi_H)\big) = c_M \cdot J^\omega(\pi_G).$$

This completes the proof. $\square$

§5. Reduction to tamely ramified cyclic extensions

Let E be a cyclic extension of F of degree d. We first refine a result of Henniart [4, lemme 3.6], [5, Section 3].

LEMMA 5.1. *There exist a cyclic extension $\mathbb{E}$ of a number field $\mathbb{F}$ of degree d and a place v_0 of $\mathbb{F}$ such that*

- *v_0 is inert in $\mathbb{E}$, and $\mathbb{E}_{v_0}/\mathbb{F}_{v_0}$ is isomorphic to E/F;*
- *$\mathbb{E}_w/\mathbb{F}_v$ is tamely ramified if $v \neq v_0$ is a nonarchimedean place of $\mathbb{F}$ and w is a place of $\mathbb{E}$ lying above v; and*
- *$\mathbb{F}_v \cong \mathbb{C}$ if v is an archimedean place of $\mathbb{F}$.*

Proof. We first recall the Grunwald-Wang theorem, from which the lemma follows. Let $\mathbb{F}$ be a number field. For each positive integer r, fix a primitive 2^rth root of unity ξ_r such that $\xi_{r+1}^2 = \xi_r$. Put $\eta_r = \xi_r + \xi_r^{-1}$. Note that $\eta_2 = 0$.

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Let $s \geq 2$ be the integer such that $\eta_s \in \mathbb{F}$ but $\eta_{s+1} \notin \mathbb{F}$. Let S_0 be the set of places v of $\mathbb{F}$ lying above 2 such that -1, $2+\eta_s$, $-(2+\eta_s)$ are not square in $\mathbb{F}_v$. Let S be a finite set of places of $\mathbb{F}$, and let d be a positive integer. We say that we are in the special case if

- -1, $2+\eta_s$, $-(2+\eta_s)$ are not square in $\mathbb{F}$,
- $2^{s+1} \mid d$,
- $S_0 \subset S$.

For $v \in S$, let ω_v be a character of $\mathbb{F}_v^\times$ of order dividing d. Assume that

$$\prod_{v \in S_0} \omega_v(2+\eta_s)^{d/2} = 1 \tag{5.1}$$

if we are in the special case. Then, by the Grunwald-Wang theorem, there exists a character $\boldsymbol{\omega}$ of $\mathbb{A}_{\mathbb{F}}^\times/\mathbb{F}^\times$ of order dividing d such that $\boldsymbol{\omega}_v = \omega_v$ for $v \in S$ (see [2, Chapter X, Section 2, Theorem 5]).

Let p be the residual characteristic of F. Fix a character ω of $F^\times$ of order d associated to E/F by class field theory.

We first assume that $p \neq 2$. By the weak approximation theorem and Krasner's lemma, we can take a number field $\mathbb{F}$ and a place v_0 of $\mathbb{F}$ so that $\mathbb{F}_{v_0} \cong F$ and $\mathbb{F}_v \cong \mathbb{C}$ if v is an archimedean place of $\mathbb{F}$. Let S_∞ be the set of archimedean places of $\mathbb{F}$, and let S' the set of nonarchimedean places v of $\mathbb{F}$ such that the residual characteristic of $\mathbb{F}_v$ divides $2d$. Put $S = S_\infty \cup S' \cup \{v_0\}$. Then we have $S_0 \subset S$ and $v_0 \notin S_0$. We take

$$\omega_v = \begin{cases} \omega & \text{if } v = v_0, \\ \mathbb{1} & \text{if } v \neq v_0. \end{cases}$$

Then (5.1) holds, and hence there exists a character $\boldsymbol{\omega}$ of $\mathbb{A}_{\mathbb{F}}^\times/\mathbb{F}^\times$ of order d such that $\boldsymbol{\omega}_v = \omega_v$ for $v \in S$. Let $\mathbb{E}$ be the cyclic extension of $\mathbb{F}$ of degree d associated to $\boldsymbol{\omega}$ by class field theory. By construction, v_0 is inert in $\mathbb{E}$, $\mathbb{E}_{v_0}/\mathbb{F}_{v_0}$ is isomorphic to E/F, and $\mathbb{E}_w \cong \mathbb{F}_v$ for $v \in S - \{v_0\}$. Here w is a place of $\mathbb{E}$ lying above v. If $v \notin S$, then the residual characteristic of $\mathbb{F}_v$ does not divide d, so that $\mathbb{E}_w/\mathbb{F}_v$ is tamely ramified.

We next assume that $p = 2$. We modify the argument above as follows.

Let $\breve{F}$ be the unramified extension of $\mathbb{Q}_2$ of degree $[F:\mathbb{Q}_2]$. Since $\mathbb{Q}_2(\xi_r)$ is a totally ramified extension of $\mathbb{Q}_2$ of degree 2^{r-1}, we have $\eta_r \notin \breve{F}$ if $r \geq 3$. Also, -1, 2, -2 are not square in $\breve{F}$. Recall that 2 is split in $\mathbb{Q}(\sqrt{-7})$. By the weak approximation theorem and Krasner's lemma, we can take an extension $\mathbb{F}$ of $\mathbb{Q}(\sqrt{-7})$ and two places v_0, v_1 of $\mathbb{F}$ so that $\mathbb{F}_{v_0} \cong F$, $\mathbb{F}_{v_1} \cong \breve{F}$,

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

and $\mathbb{F}_v \cong \mathbb{C}$ if v is an archimedean place of $\mathbb{F}$. In particular, $s = 2$ and $v_1 \in S_0$. Put $S = S_\infty \cup S' \cup \{v_0\}$. Then we have $S_0 \subset S$. Moreover, we are in the special case if and only if $8 \mid d$.

For $v \in S$, we choose a character ω_v of $\mathbb{F}_v^\times$ as follows. Fix an unramified character $\breve{\omega}$ of $\breve{F}^\times$ of order d. Note that $\breve{\omega}(2)^{d/2} = -1$ if d is even. If we are in the special case, $v_0 \in S_0$, and $\omega(2)^{d/2} = -1$, we take

$$\omega_v = \begin{cases} \omega & \text{if } v = v_0, \\ \breve{\omega} & \text{if } v = v_1, \\ \mathbb{1} & \text{if } v \notin \{v_0, v_1\}. \end{cases}$$

Otherwise, we take

$$\omega_v = \begin{cases} \omega & \text{if } v = v_0, \\ \mathbb{1} & \text{if } v \neq v_0. \end{cases}$$

Then (5.1) holds if we are in the special case. Indeed, if we are in the special case and $v_0 \in S_0$, then we have

$$\prod_{v \in S_0} \omega_v(2 + \eta_s)^{d/2} = \omega(2)^{d/2} \cdot \omega_{v_1}(2)^{d/2} = 1.$$

Hence, there exists a character $\boldsymbol{\omega}$ of $\mathbb{A}_\mathbb{F}^\times / \mathbb{F}^\times$ of order d such that $\boldsymbol{\omega}_v = \omega_v$ for $v \in S$. Let $\mathbb{E}$ be the cyclic extension of $\mathbb{F}$ of degree d associated to $\boldsymbol{\omega}$ by class field theory. By construction, if we are in the special case, $v_0 \in S_0$, and $\omega(2)^{d/2} = -1$, then v_1 is inert in $\mathbb{E}$ and $\mathbb{E}_{v_1}/\mathbb{F}_{v_1}$ is unramified. This completes the proof. □

Let ω be a character of $F^\times$ of order d associated to E/F by class field theory. Fix a character $\boldsymbol{\omega}$ of $\mathbb{A}_\mathbb{F}^\times / \mathbb{F}^\times$ of order d associated to $\mathbb{E}/\mathbb{F}$ by class field theory such that $\boldsymbol{\omega}_{v_0} = \omega$. Let $\mathbf{G} = \mathrm{GL}_d$, and let $\mathbf{H} = \mathrm{Res}_{\mathbb{E}/\mathbb{F}} \mathbb{G}_m$. We fix an $\mathbb{F}$-splitting $\mathbf{spl}_G$ of $\mathbf{G}$ and a nontrivial character $\boldsymbol{\psi}$ of $\mathbb{A}_\mathbb{F}/\mathbb{F}$. For each place v of $\mathbb{F}$, $\mathbf{spl}_G$ and $\boldsymbol{\psi}$ induce an $\mathbb{F}_v$-splitting $\mathbf{spl}_G$ of GL_d and a nontrivial character of $\boldsymbol{\psi}_v$ of $\mathbb{F}_v$, respectively.

We now recall a special case of a result of Henniart and Lemaire [9] on the product formula for the constant.

Proposition 5.2 [9, section 4.6]. *Let π_H be a unitary character of $E^\times$. Then there exists a unitary character $\Pi_H = \bigotimes_v \Pi_{H,v}$ of $\mathbb{A}_\mathbb{E}^\times / \mathbb{E}^\times$ such that $\Pi_{H,v_0} = \pi_H$ and*

$$\prod_{v \in S} c_v = 1$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

for a sufficiently large finite set S of places of $\mathbb{F}$ which contains v_0. Here $c_v \in \mathbb{C}^\times$ is the constant for $\Pi_{H,v}$ given by (1.2).

§6. Intertwining operators

To prove Theorem 1.4, we may assume that

- E is a tamely ramified cyclic extension of F of degree d,
- $\mathbf{G} = \mathrm{GL}_d$,
- $(\mathbf{H}, \mathcal{H}, s, \xi) = (\mathrm{Res}_{E/F}\mathbb{G}_m, {}^L H_{\mathbf{m}}, s_{\mathbf{m}}, \xi_{\mathbf{m}})$ with $\mathbf{m} = (1)$,
- $\mathbf{spl}_G$ is standard,
- ψ is of order zero, and
- $\pi_H = \mathbb{1}$

by Theorem 1.2, Proposition 5.2, and Lemmas 2.1, 2.3, 3.2, 4.2, 4.5, and 5.1. The rest of this paper is devoted to the proof of Theorem 1.4 under this assumption. In this section, we construct the intertwining operator $\mathcal{A}_\omega$ explicitly.

Let e and f be the ramification index and the residual degree of E/F, respectively. Note that $d = ef$. Fix a character ω of $F^\times$ of order d associated to E/F by class field theory. Then ω is trivial on $1 + \mathfrak{p}$ and the restriction of ω to $\mathfrak{o}^\times$ is of order e.

Recall that $\mathbf{B}_0 = \mathbf{T}_0\mathbf{U}_0$ is the standard Borel subgroup of $\mathbf{G}$, where $\mathbf{T}_0$ is the subgroup of diagonal matrices and

$$\mathbf{U}_0 = \{u = (u_{i,j}) \in \mathbf{G} \mid u_{i,i} = 1,\ u_{i,j} = 0 \text{ if } i > j\}.$$

The generic character χ of U_0 associated to $\mathbf{spl}_G$ and ψ is given by

$$\chi(u) = \psi(u_{1,2} + \cdots + u_{d-1,d})$$

for $u \in U_0$.

Let μ be the character of T_0 defined by

$$\mu = \mathbb{1} \otimes \omega^e \otimes \cdots \otimes \omega^{e(f-1)} \otimes \omega \otimes \omega^{e+1} \otimes \cdots \otimes \omega^{e(f-1)+1} \otimes \cdots \otimes \omega^{e-1} \otimes \omega^{2e-1} \otimes \cdots \otimes \omega^{ef-1}.$$

For $s = (s_1, \ldots, s_d) \in \mathbb{C}^d$, we put

$$\mu_s = \mu|\cdot|^s,$$

where $|t|^s = |t_1|^{s_1} \cdots |t_d|^{s_d}$ for $t = \mathrm{diag}(t_1, \ldots, t_d) \in T_0$. Let

$$\pi_s = \mathrm{Ind}_{B_0}^G(\mu_s)$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

be the principal series representation of G on the space V_s of locally constant functions Φ_s such that

$$\Phi_s(tug) = \delta_{B_0}^{1/2}(t)\mu_s(t)\Phi_s(g)$$

for $t \in T_0$, $u \in U_0$, $g \in G$. Here δ_{B_0} is the modulus character of B_0, and G acts on V_s by right translation. Let

$$\lambda_s : V_s \longrightarrow \mathbb{C}$$

be the Whittaker functional given by

$$\lambda_s(\Phi_s) = \int_{U_0} \Phi_s(w_0 u)\overline{\chi(u)}\, du,$$

where

$$w_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

and du is the Haar measure on U_0 such that $\mathrm{vol}(\mathbf{U}_0(\mathfrak{o})) = 1$. We write

$$\pi = \pi_0, \qquad V = V_0, \qquad \lambda = \lambda_0.$$

Then π is the automorphic induction of the trivial representation of H.

Let μ' be the character of T_0 defined by

$$\mu' = \omega \otimes \omega^{e+1} \otimes \cdots \otimes \omega^{e(f-1)+1} \otimes \cdots \otimes \omega^{e-1} \otimes \omega^{2e-1} \otimes \cdots \otimes \omega^{ef-1} \otimes \omega^{e} \otimes \cdots \otimes \omega^{e(f-1)} \otimes \mathbb{1}.$$

Note that $\mu' = \mu\omega$. For $s \in \mathbb{C}^d$, we put $\mu'_s = \mu'|\cdot|^s$. We define a principal series representation $\pi'_s = \mathrm{Ind}_{B_0}^G(\mu'_s)$ of G on the space V'_s and a Whittaker functional

$$\lambda'_s : V'_s \longrightarrow \mathbb{C}$$

similarly. We write

$$\pi' = \pi'_0, \qquad V' = V'_0, \qquad \lambda' = \lambda'_0.$$

We define an isomorphism

$$\mathcal{A} : V \longrightarrow V'$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

as vector spaces by

$$\mathcal{A}(\Phi)(g) = \Phi(g)\omega(g)$$

for $\Phi \in V$ and $g \in G$. Here we write $\omega(g) = \omega(\det g)$ for $g \in G$.

LEMMA 6.1. *We have*

$$\mathcal{A} \circ (\pi \otimes \omega) = \pi' \circ \mathcal{A}.$$

Proof. Let $\Phi \in V$, and let $g, g' \in G$. Then we have

$$\begin{aligned}\mathcal{A}\big((\pi \otimes \omega)(g')\Phi\big)(g) &= \mathcal{A}\big(\pi(g')\Phi\big)(g)\omega(g') = \pi(g')\Phi(g)\omega(g)\omega(g') \\ &= \Phi(gg')\omega(gg')\end{aligned}$$

and

$$\pi'(g')\mathcal{A}(\Phi)(g) = \mathcal{A}(\Phi)(gg') = \Phi(gg')\omega(gg').$$

□

LEMMA 6.2. *We have*

$$\lambda' \circ \mathcal{A} = \omega(-1)^{d(d-1)/2} \cdot \lambda.$$

Proof. Let $\Phi \in V$. Then we have

$$\begin{aligned}\lambda'\big(\mathcal{A}(\Phi)\big) &= \int_{U_0} \mathcal{A}(\Phi)(w_0 u)\overline{\chi(u)}\, du \\ &= \int_{U_0} \Phi(w_0 u)\omega(w_0 u)\overline{\chi(u)}\, du \\ &= \omega(w_0)\int_{U_0} \Phi(w_0 u)\overline{\chi(u)}\, du \\ &= \omega(w_0)\cdot\lambda(\Phi).\end{aligned}$$

□

Let $\mathbf{P}$ be the standard parabolic subgroup of $\mathbf{G}$ with Levi factor

$$\mathbf{L} = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \middle| \, g_1 \in \mathrm{GL}_{d-f}, g_2 \in \mathrm{GL}_f \right\}.$$

Let

$$\begin{aligned}\mathbf{L}_1 &= \big\{\mathrm{diag}(g_1, \mathbf{1}_f) \in \mathbf{L} \mid g_1 \in \mathrm{GL}_{d-f}\big\}, \\ \mathbf{L}_2 &= \big\{\mathrm{diag}(\mathbf{1}_{d-f}, g_2) \in \mathbf{L} \mid g_2 \in \mathrm{GL}_f\big\}.\end{aligned}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

For $i = 1, 2$, put

$$\pi'_{s,i} = \mathrm{Ind}_{B_0 \cap L_i}^{L_i}(\mu'_{s,i}),$$

where $\mu'_{s,i} = \mu'_s|_{B_0 \cap L_i}$. We write $\pi'_i = \pi'_{0,i}$ and $\mu'_i = \mu'_{0,i}$. Note that

$$\pi'_s \cong \mathrm{Ind}_P^G(\pi'_{s,1} \otimes \pi'_{s,2})$$

and that

$$\mu'_1 = \omega \otimes \omega^{e+1} \otimes \cdots \otimes \omega^{e(f-1)+1} \otimes \cdots \otimes \omega^{e-1} \otimes \omega^{2e-1} \otimes \cdots \otimes \omega^{ef-1},$$
$$\mu'_2 = \omega^e \otimes \cdots \otimes \omega^{e(f-1)} \otimes \mathbb{1}.$$

For $s = (s_1, \ldots, s_d) \in \mathbb{C}^d$, put $s' = (s_{d-f+1}, \ldots, s_d, s_1, \ldots, s_{d-f})$. For $\Phi'_s \in V'_s$ and $g \in G$, put

$$\mathcal{M}_s(\Phi'_s)(g) = \int_{\mathrm{M}_{f,d-f}(F)} \Phi'_s\left(w \begin{pmatrix} \mathbf{1}_f & x \\ 0 & \mathbf{1}_{d-f} \end{pmatrix} g\right) dx,$$

where

$$w = \begin{pmatrix} 0 & \mathbf{1}_{d-f} \\ \mathbf{1}_f & 0 \end{pmatrix}$$

and dx is the Haar measure on $\mathrm{M}_{f,d-f}(F)$ such that $\mathrm{vol}(\mathrm{M}_{f,d-f}(\mathfrak{o})) = 1$. This integral is absolutely convergent for $\mathrm{Re}(s_1) \gg \mathrm{Re}(s_2) \gg \cdots \gg \mathrm{Re}(s_d) \gg 0$, has a meromorphic continuation to $\mathbb{C}^d$, and defines an intertwining operator

$$\mathcal{M}_s : V'_s \longrightarrow V_{s'}.$$

Then we have $\mathcal{M}_s \circ \pi'_s = \pi_{s'} \circ \mathcal{M}_s$. By [17, Theorem 5.1], we have

$$\lambda'_s = \gamma(0, \pi'_{s,1} \times \check{\pi}'_{s,2}, \psi) \cdot \lambda_{s'} \circ \mathcal{M}_s,$$

where $\check{\pi}'_{s,2}$ is the contragredient representation of $\pi'_{s,2}$. We remark that $\pi'_{s,2}$ is unramified.

We define a normalized intertwining operator

$$\mathcal{N} : V' \longrightarrow V$$

by

$$\mathcal{N} = \lim_{s \to 0} \gamma(0, \pi'_{s,1} \times \check{\pi}'_{s,2}, \psi) \mathcal{M}_s.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Then we have

$$\mathcal{N} \circ \pi' = \pi \circ \mathcal{N} \tag{6.1}$$

and

$$\lambda' = \lambda \circ \mathcal{N}. \tag{6.2}$$

We define an isomorphism

$$\mathcal{A}_\omega : V \longrightarrow V$$

as vector spaces by

$$\mathcal{A}_\omega = \omega(-1)^{d(d-1)/2} \cdot \mathcal{N} \circ \mathcal{A}.$$

By Lemma 6.1 and (6.1), we have

$$\mathcal{A}_\omega \circ (\pi \otimes \omega) = \pi \circ \mathcal{A}_\omega.$$

LEMMA 6.3. *We have*

$$\lambda \circ \mathcal{A}_\omega = \lambda.$$

Proof. By Lemma 6.2 and (6.2), we have

$$\lambda \circ \mathcal{A}_\omega = \omega(-1)^{d(d-1)/2} \cdot \lambda \circ \mathcal{N} \circ \mathcal{A} = \omega(-1)^{d(d-1)/2} \cdot \lambda' \circ \mathcal{A} = \lambda. \qquad \square$$

§7. Calculation of a twisted character

We retain the notation of Section 6. In this section, we compute the twisted character $J^\omega(\pi, f^G)$ for a certain $f^G \in C_c^\infty(G)$.

Let $K = \mathrm{GL}_d(\mathfrak{o})$. Let I be the parahoric subgroup of G given by

$$I = \{ g \in K \mid g_{i,j} \in \mathrm{M}_f(\mathfrak{p}) \text{ if } i > j \}.$$

Here we write $g = (g_{i,j})_{1 \le i,j \le e}$ with $g_{i,j} \in \mathrm{M}_f(F)$. We take a Haar measure dg on G so that $\mathrm{vol}(K) = 1$.

Put

$$\kappa(g) = \omega(g_{2,2})\omega^2(g_{3,3}) \cdots \omega^{e-1}(g_{e,e})$$

for $g = (g_{i,j})_{1 \le i,j \le e} \in I$ with $g_{i,j} \in \mathrm{M}_f(F)$. Here we write $\omega(g_{i,i}) = \omega(\det g_{i,i})$ for $g_{i,i} \in \mathrm{GL}_f(F)$. Since ω is trivial on $1 + \mathfrak{p}$, κ is a character of I. Note that $\kappa(g) = 1$ if $g \in I$ is topologically unipotent. We define $f^G \in C_c^\infty(G)$ so that

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

- $\operatorname{supp} f^G = \eta^{-1} I$,
- $f^G(\eta^{-1} g) = \kappa^{-1}\omega^{-1}(g) \cdot \operatorname{vol}(I)^{-1}$ for $g \in I$,

where

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_{d-f} \\ \varpi \cdot \mathbf{1}_f & 0 \end{pmatrix}.$$

Note that f^G does not depend on the choice of ϖ.

LEMMA 7.1. *We have*

$$f^G(g_1 g g_2) = \kappa^{-1}(g_1)\kappa^{-1}\omega^{-1}(g_2) f^G(g)$$

for $g \in G$ *and* $g_1, g_2 \in I$.

Proof. Since $\eta I \eta^{-1} = I$, it suffices to show that

$$\kappa^{-1}\omega^{-1}(\eta g \eta^{-1}) = \kappa^{-1}(g)$$

for $g \in I$. We write $\eta g \eta^{-1} = (g'_{i,j})_{1 \le i,j \le e} \in I$ for $g = (g_{i,j})_{1 \le i,j \le e} \in I$. Then we have

$$g'_{i,i} = \begin{cases} g_{i+1,i+1} & \text{if } 1 \le i < e, \\ g_{1,1} & \text{if } i = e. \end{cases}$$

Hence, we have

$$\begin{aligned} \kappa^{-1}\omega^{-1}(\eta g \eta^{-1}) &= \omega^{-1}(g_{3,3}) \cdots \omega^{-e+2}(g_{e,e})\omega^{-e+1}(g_{1,1}) \\ &\quad \cdot \omega^{-1}(g_{2,2}) \cdots \omega^{-1}(g_{e,e})\omega^{-1}(g_{1,1}) \\ &= \omega^{-1}(g_{2,2})\omega^{-2}(g_{3,3}) \cdots \omega^{-e+1}(g_{e,e}) \\ &= \kappa^{-1}(g). \end{aligned}$$

∎

Let

$$V_s^\kappa = \{\Phi_s \in V_s \mid \Phi_s(gg') = \Phi_s(g)\kappa(g') \text{ for all } g \in G,\ g' \in I\}.$$

We define $V_s^{\kappa\omega}$ and $V_s'^{,\kappa\omega}$ similarly. We write $V^\kappa = V_0^\kappa$, $V^{\kappa\omega} = V_0^{\kappa\omega}$, and $V'^{,\kappa\omega} = V_0'^{,\kappa\omega}$.

LEMMA 7.2. *We have*

$$\dim_{\mathbb{C}} V_s^\kappa = \dim_{\mathbb{C}} V_s^{\kappa\omega} = \dim_{\mathbb{C}} V_s'^{,\kappa\omega} = 1.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. We only consider V_s^κ. We have an Iwasawa decomposition $G = B_0K$. Let $\mathbf{P}'$ be the standard parabolic subgroup of $\mathbf{G}$ with Levi factor

$$\mathbf{L}' = \{\mathrm{diag}(g_{1,1}, \dots, g_{e,e}) \mid g_{i,i} \in \mathrm{GL}_f\}.$$

Let Ω and Ω' be the Weyl groups of $\mathbf{G}$ and $\mathbf{L}'$, respectively. We take permutation matrices in G and L' as representatives of elements in Ω and Ω', respectively. Then we have a Bruhat decomposition

$$K = \coprod_{w' \in \Omega/\Omega'} (B_0 \cap K) w' I.$$

Let $\Phi_s \in V_s$ such that $\mathrm{supp}\, \Phi_s \subset B_0 w' I$. Since ω^e is unramified, the restriction of μ_s to $T_0 \cap K$ is given by

$$\underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{f} \otimes \underbrace{\omega \otimes \cdots \otimes \omega}_{f} \otimes \cdots \otimes \underbrace{\omega^{e-1} \otimes \cdots \otimes \omega^{e-1}}_{f}.$$

Hence, we have

$$\Phi_s(w't) = \Phi_s(w'tw'^{-1} \cdot w') = \prod_{i=0}^{e-1} \prod_{j=1}^{f} \omega^i(t_{w'(if+j)}) \cdot \Phi_s(w')$$

for $t = \mathrm{diag}(t_1, \dots, t_d) \in T_0 \cap K$. If $\Phi_s \in V_s^\kappa$, then we have

$$\Phi_s(w't) = \Phi_s(w') \cdot \prod_{i=0}^{e-1} \prod_{j=1}^{f} \omega^i(t_{if+j})$$

for $t = \mathrm{diag}(t_1, \dots, t_d) \in T_0 \cap K$. Since the restriction of ω to $\mathfrak{o}^\times$ is of order e, $\Phi_s(w') = 0$ unless $w' \in \Omega'$. ∎

We define

$$\Phi_s^\kappa \in V_s^\kappa, \qquad \Phi_s^{\kappa\omega} \in V_s^{\kappa\omega}, \qquad \Phi_s'^{,\kappa\omega} \in V_s'^{,\kappa\omega}$$

so that

$$\mathrm{supp}\, \Phi_s^\kappa = B_0 I, \qquad \Phi_s^\kappa(1) = 1,$$
$$\mathrm{supp}\, \Phi_s^{\kappa\omega} = B_0 w^{-1} I, \qquad \Phi_s^{\kappa\omega}(w^{-1}) = 1,$$
$$\mathrm{supp}\, \Phi_s'^{,\kappa\omega} = B_0 I, \qquad \Phi_s'^{,\kappa\omega}(1) = 1.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Here

$$w = \begin{pmatrix} 0 & \mathbf{1}_{d-f} \\ \mathbf{1}_f & 0 \end{pmatrix}.$$

We write $\Phi^\kappa = \Phi_0^\kappa$, $\Phi^{\kappa\omega} = \Phi_0^{\kappa\omega}$, $\Phi'^{,\kappa\omega} = \Phi_0'^{,\kappa\omega}$.

LEMMA 7.3. *We have*

$$\mathcal{A}(\Phi^\kappa) = \Phi'^{,\kappa\omega}.$$

Proof. We have $\mathcal{A}(\Phi^\kappa) \in V'^{,\kappa\omega}$ and

$$\mathcal{A}(\Phi^\kappa)(1) = \Phi^\kappa(1)\omega(1) = 1.$$ □

LEMMA 7.4. *We have*

$$\mathcal{M}_s(\Phi_s'^{,\kappa\omega}) = q^{-(e-1)f^2} \cdot \Phi_{s'}^{\kappa\omega}.$$

Proof. We have $\mathcal{M}_s(\Phi_s'^{,\kappa\omega}) \in V_{s'}^{\kappa\omega}$ and

$$\begin{aligned} \mathcal{M}_s(\Phi_s'^{,\kappa\omega})(w^{-1}) &= \int_{\mathrm{M}_{f,d-f}(F)} \Phi_s'^{,\kappa\omega}\left(w \begin{pmatrix} \mathbf{1}_f & x \\ 0 & \mathbf{1}_{d-f} \end{pmatrix} w^{-1}\right) dx \\ &= \int_{\mathrm{M}_{f,d-f}(F)} \Phi_s'^{,\kappa\omega}\left(\begin{pmatrix} \mathbf{1}_{d-f} & 0 \\ x & \mathbf{1}_f \end{pmatrix}\right) dx. \end{aligned}$$

Since

$$\begin{pmatrix} \mathbf{1}_{d-f} & 0 \\ x & \mathbf{1}_f \end{pmatrix} \in B_0 I$$

if and only if $x \in \mathrm{M}_{f,d-f}(\mathfrak{p})$, this integral is equal to $\mathrm{vol}(\mathrm{M}_{f,d-f}(\mathfrak{p})) = q^{-f(d-f)}$. □

Let

$$\lambda(E/F, \psi) = \prod_{i=0}^{f-1} \prod_{j=1}^{e-1} \varepsilon(1/2, \omega^{ie+j}, \psi)$$

denote the Langlands λ-factor.

LEMMA 7.5. *We have*

$$\lambda(E/F, \psi)^f = \omega(-1)^{((e-1)e/2)[(f+1)/2]} \lambda(E/F, \psi)^{-1}.$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. Note that $\omega^e(-1)=1$. If f is even, then we have

$$\begin{aligned}\lambda(E/F,\psi) &= \prod_{i=0}^{f/2-1}\prod_{j=1}^{e-1}\varepsilon(1/2,\omega^{ie+j},\psi)\varepsilon(1/2,\omega^{(f-i)e-j},\psi)\\ &= \prod_{i=0}^{f/2-1}\prod_{j=1}^{e-1}\omega^{ie+j}(-1)\\ &= \prod_{j=1}^{e-1}\omega^{j}(-1)^{f/2}\\ &= \omega(-1)^{(e-1)ef/4}.\end{aligned}$$

In particular, $\lambda(E/F,\psi)=\pm 1$. If f is odd, then we have

$$\begin{aligned}\lambda(E/F,\psi)^2 &= \prod_{i=0}^{f-1}\prod_{j=1}^{e-1}\varepsilon(1/2,\omega^{ie+j},\psi)\varepsilon(1/2,\omega^{(f-i)e-j},\psi)\\ &= \prod_{i=0}^{f-1}\prod_{j=1}^{e-1}\omega^{ie+j}(-1)\\ &= \prod_{j=1}^{e-1}\omega^{j}(-1)^{f}\\ &= \prod_{j=1}^{e-1}\omega^{j}(-1)\\ &= \omega(-1)^{(e-1)e/2}.\end{aligned}$$

∎

LEMMA 7.6. *We have*

$$\mathcal{N}(\Phi'^{,\kappa\omega}) = q^{-(e-1)f^2/2}\omega(-1)^{((e-1)e/2)[(f+1)/2]}\lambda(E/F,\psi)^{-1}\cdot\Phi^{\kappa\omega}.$$

Proof. We have

$$\gamma(0,\pi_1'\times\check{\pi}_2',\psi) = \prod_{i=0}^{f-1}\prod_{j=1}^{e-1}\prod_{k=0}^{f-1}\gamma(0,\omega^{ie+j}\cdot\omega^{-ke},\psi).$$

Since ω is of order ef, we have

$$\prod_{i=0}^{f-1}\gamma(0,\omega^{ie+j}\cdot\omega^{-ke},\psi) = \prod_{i=0}^{f-1}\gamma(0,\omega^{ie+j},\psi)$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

for fixed j, k. Hence, we have

$$\gamma(0, \pi_1' \times \check{\pi}_2', \psi) = \prod_{i=0}^{f-1} \prod_{j=1}^{e-1} \gamma(0, \omega^{ie+j}, \psi)^f.$$

Let $0 \le i \le f-1$, and let $1 \le j \le e-1$. Since ψ is of order zero and ω^{ie+j} is ramified with conductor 1, we have

$$\gamma(0, \omega^{ie+j}, \psi) = \varepsilon(0, \omega^{ie+j}, \psi) = q^{1/2} \cdot \varepsilon(1/2, \omega^{ie+j}, \psi).$$

Thus, we obtain

$$\gamma(0, \pi_1' \times \check{\pi}_2', \psi) = q^{(e-1)f^2/2} \prod_{i=0}^{f-1} \prod_{j=1}^{e-1} \varepsilon(1/2, \omega^{ie+j}, \psi)^f = q^{(e-1)f^2/2} \lambda(E/F, \psi)^f.$$

Hence, the lemma follows from Lemmas 7.4 and 7.5. □

LEMMA 7.7. *We have*

$$\pi(f^G)\Phi^{\kappa\omega} = q^{(e-1)f^2/2} \omega(\varpi)^{-d(f-1)/2} \cdot \Phi^{\kappa}.$$

Proof. By Lemma 7.1, we have $\pi(f^G)\Phi^{\kappa\omega} \in V^{\kappa}$ and

$$\begin{aligned} \pi(f^G)\Phi^{\kappa\omega}(1) &= \int_G f^G(g)\Phi^{\kappa\omega}(g)\, dg \\ &= \int_I f^G(\eta^{-1}g)\Phi^{\kappa\omega}(\eta^{-1}g)\, dg \\ &= f^G(\eta^{-1})\Phi^{\kappa\omega}(\eta^{-1}) \cdot \mathrm{vol}(I). \end{aligned}$$

We have $f^G(\eta^{-1}) = \mathrm{vol}(I)^{-1}$ and

$$\Phi^{\kappa\omega}(\eta^{-1}) = \Phi^{\kappa\omega}\left(\begin{pmatrix} \varpi^{-1} \cdot \mathbf{1}_f & 0 \\ 0 & \mathbf{1}_{d-f} \end{pmatrix} w^{-1}\right) = q^{(e-1)f^2/2} \omega(\varpi)^{-ef(f-1)/2}.$$

□

PROPOSITION 7.8. *We have*

$$J^{\omega}(\pi, f^G) = \lambda(E/F, \psi)^{-1} \times \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,} \\ \omega(\varpi)^{d/2} & \text{if } e \text{ is odd and } f \text{ is even,} \\ \omega(-1)^{e(f-1)/4} & \text{if } e \text{ is even and } f \text{ is odd,} \\ \omega(-1)^{ef/4}\omega(\varpi)^{d/2} & \text{if } e \text{ and } f \text{ are even.} \end{cases}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. By Lemma 7.1, we have $\mathrm{Im}(\pi(f^G)\circ\mathcal{A}_\omega)\subset V^\kappa$. Since $\dim_{\mathbb{C}} V^\kappa = 1$ by Lemma 7.2, we have

$$J^\omega(\pi, f^G) = C,$$

where $C\in\mathbb{C}$ is a constant such that $\pi(f^G)\mathcal{A}_\omega(\Phi^\kappa) = C\cdot\Phi^\kappa$. It remains to compute $C = \pi(f^G)\mathcal{A}_\omega(\Phi^\kappa)(1)$. By Lemmas 7.3, 7.6, and 7.7, we have

$$\begin{aligned}
&\pi(f^G)\mathcal{A}_\omega(\Phi^\kappa)(1)\\
&\quad = \omega(-1)^{d(d-1)/2}\pi(f^G)\mathcal{N}(\mathcal{A}(\Phi^\kappa))(1)\\
&\quad = \omega(-1)^{d(d-1)/2}\pi(f^G)\mathcal{N}(\Phi'^{,\kappa\omega})(1)\\
&\quad = q^{-(e-1)f^2/2}\omega(-1)^{d(d-1)/2+((e-1)e/2)[(f+1)/2]}\lambda(E/F,\psi)^{-1}\pi(f^G)\Phi^{\kappa\omega}(1)\\
&\quad = \omega(-1)^{d(d-1)/2+((e-1)e/2)[(f+1)/2]}\omega(\varpi)^{-d(f-1)/2}\lambda(E/F,\psi)^{-1}\Phi^\kappa(1)\\
&\quad = \omega(-1)^{d(d-1)/2+((e-1)e/2)[(f+1)/2]}\omega(\varpi)^{-d(f-1)/2}\lambda(E/F,\psi)^{-1}.
\end{aligned}$$

Since $\omega^e(-1) = 1$, we have

$$\omega(-1)^{d(d-1)/2} = \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,}\\ 1 & \text{if } e \text{ is odd and } f \text{ is even,}\\ \omega(-1)^{e/2} & \text{if } e \text{ is even and } f \text{ is odd,}\\ 1 & \text{if } e \text{ and } f \text{ are even} \end{cases}$$

and

$$\omega(-1)^{(e(e-1)/2)[(f+1)/2]} = \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,}\\ 1 & \text{if } e \text{ is odd and } f \text{ is even,}\\ \omega(-1)^{e(f+1)/4} & \text{if } e \text{ is even and } f \text{ is odd,}\\ \omega(-1)^{ef/4} & \text{if } e \text{ and } f \text{ are even.} \end{cases}$$

Since $\omega^d(\varpi) = 1$, we have $\omega(\varpi)^{-d(f-1)/2} = 1$ if f is odd. If f is even, then $\omega(\varpi)^{d/2} = \pm 1$, so that $\omega(\varpi)^{-d(f-1)/2} = \omega(\varpi)^{d/2}$. This completes the proof. □

§8. Calculation of a twisted orbital integral

We retain the notation of Sections 6 and 7. In this section, we compute the twisted orbital integral $J^\omega(\gamma, f^G)$ for $f^G\in C_c^\infty(G)$ defined in Section 7.

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

For each $0 \le i < d$, let $\mathcal{L}_{i,G}$ be the lattice of F^d given by

$$\mathcal{L}_{i,G} = \underbrace{\mathfrak{o} \oplus \cdots \oplus \mathfrak{o}}_{d-i} \oplus \underbrace{\mathfrak{p} \oplus \cdots \oplus \mathfrak{p}}_{i}.$$

For each integer $i = jd + k$ with $0 \le k < d$, let $\mathcal{L}_{i,G} = \varpi^j \mathcal{L}_{k,G}$. Then we have

$$I = \{ g \in G \mid g\mathcal{L}_{if,G} \subset \mathcal{L}_{if,G} \text{ for all } i \}.$$

Fix an isomorphism $\phi : E \to F^d$ as vector spaces over F such that

$$\phi(\mathfrak{p}_E^i) = \mathcal{L}_{if,G}$$

for all i. Let $\iota : \mathbf{H} \to \mathbf{G}$ be the embedding induced by ϕ. Then we have

$$\iota(H) \cap I = \iota(\mathfrak{o}_E^\times).$$

We identify H with the image in G and suppress ι from the notation.

Let $\gamma \in G$ be a regular semisimple element, and let $\mathbf{T}$ be the centralizer of γ in $\mathbf{G}$. Then $J^\omega(\gamma, f^G) = 0$ unless γ is conjugate to an element in H. If $\gamma \in H$ is G-regular, then $\mathbf{T} = \mathbf{H}$.

LEMMA 8.1. *Let $\gamma \in H$ be a G-regular semisimple element. If $J^\omega(\gamma, f^G) \neq 0$, then we have*

$$\operatorname{ord}_E \gamma = -1.$$

Proof. Then, since $\operatorname{supp} f^G = \eta^{-1} I$, we have $J^\omega(\gamma, f^G) = 0$ unless $\operatorname{ord}_F \det \gamma = -f$. □

Let $\gamma \in H$ be a G-regular semisimple element such that $\operatorname{ord}_E \gamma = -1$. By [12] and [19, section VII.1], there exist $\delta, \gamma' \in E^\times$ such that

- $\gamma = \delta\gamma'$,
- $\gamma' \in 1 + \mathfrak{p}_E$,
- δ is $F(\delta)/F$-cuspidal (see [19, section VI.2]).

Put $F' = F(\delta)$. Note that F' does not depend on the choice of δ, γ'.

LEMMA 8.2. *We have $\operatorname{ord}_{F'} \delta = -1$. Moreover, E/F' is unramified.*

Proof. Let e' be the ramification index of E/F'. Then we have

$$\operatorname{ord}_E \gamma = \operatorname{ord}_E \delta = e' \operatorname{ord}_{F'} \delta.$$

Hence, $e' = 1$. □

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Note that the ramification index of F'/F is e. Let f' be the residual degree of F'/F. Put $d' = d/ef'$. Let $\mathbf{G}' = \mathrm{Res}_{F'/F}\mathrm{GL}_{d'}$, and let $K' = \mathrm{GL}_{d'}(\mathfrak{o}_{F'})$. We take a Haar measure dg' on G' so that $\mathrm{vol}(K') = 1$. Fix an isomorphism $\phi' : F'^{d'} \to F^d$ such that $\phi'(\mathcal{L}_{i,G'}) = \mathcal{L}_{if',G}$ for all i. Let $\iota : \mathbf{G}' \to \mathbf{G}$ be the embedding induced by ϕ'. Then we have

$$\iota(G') \cap I = \iota(K'), \qquad \iota(G') \cap \eta^{-1} I = \iota(\delta K').$$

For simplicity, we will suppress ι from the notation. We may also regard H as a subgroup of G' via ϕ and ϕ'.

LEMMA 8.3. *Let $g \in G$ such that $g^{-1}\gamma g \in \eta^{-1} I$. Then we have*

$$g \in G' I.$$

Proof. The lemma follows from [19, lemme VI.3]. □

LEMMA 8.4. *We have $\mathrm{supp}\, f^G \cap G' = \delta K'$. Thus, let $g \in G'$ such that $g^{-1}\gamma' g \in K'$. Then we have*

$$f^G(g^{-1}\gamma g) = f^G(\delta).$$

Proof. The first assertion is obvious. Since $\gamma' \in 1 + \mathfrak{p}_E$, $g^{-1}\gamma' g \in K'$ is topologically unipotent, and hence $\kappa^{-1}\omega^{-1}(g^{-1}\gamma' g) = 1$. Thus, we obtain

$$f^G(g^{-1}\gamma g) = f^G(\delta g^{-1}\gamma' g) = f^G(\delta)\kappa^{-1}\omega^{-1}(g^{-1}\gamma' g) = f^G(\delta). \qquad \square$$

Let $\omega' = \omega \circ \mathrm{N}_{F'/F}$ be an unramified character of $F'^{\times}$ of order d' associated to E/F' by class field theory. Let $\mathbf{1}_{K'} \in C_c^\infty(G')$ be the characteristic function of K'. Put

$$J^{\omega'}(\gamma', \mathbf{1}_{K'}) = \int_{T\backslash G'} \omega'(g')\mathbf{1}_{K'}(g'^{-1}\gamma' g')\, dg'.$$

Note that $\mathbf{T}$ is also the centralizer of γ' in $\mathbf{G}'$.

LEMMA 8.5. *Let $\gamma \in H$ be a G-regular semisimple element such that $\mathrm{ord}_E \gamma = -1$. Let $\delta, \gamma' \in H$ as above. Then we have*

$$J^{\omega}(\gamma, f^G) = \mathrm{vol}(I) f^G(\delta) \cdot J^{\omega'}(\gamma', \mathbf{1}_{K'}).$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. The lemma follows from [19, lemme VI.4] with slight modifications. We include the proof for the sake of completeness.

Note that

$$\omega(g)f^G(g^{-1}\gamma g) = \omega(g)\kappa(g)\kappa^{-1}\omega^{-1}(g)f^G(\gamma) = f^G(\gamma)$$

for $g \in I$. By Lemma 8.3, we have

$$J^\omega(\gamma, f^G) = \sum_{g\in\Gamma} \mathrm{vol}(T\backslash TgI)\omega\big(\iota(g)\big)f^G(g^{-1}\gamma g).$$

Here Γ is a set of representatives of equivalence classes of G' for the equivalence $g_1 \sim g_2$ if and only if $g_1 \in Tg_2I$. Then Γ is also a set of representatives of $T\backslash G'/K'$. As in the proof of [19, lemme VI.4], we have $\mathrm{vol}(T\backslash TgI) = \mathrm{vol}(I)\,\mathrm{vol}(T\backslash TgK')$ and $\omega(\iota(g)) = \omega'(g)$ for $g \in \Gamma$. By Lemma 8.4, $f^G(g^{-1}\gamma g) = 0$ for $g \in \Gamma$ unless $g^{-1}\gamma' g \in K'$, and hence

$$\begin{aligned} J^\omega(\gamma, f^G) &= \mathrm{vol}(I)\sum_{g\in\Gamma} \mathrm{vol}(T\backslash TgK')\omega'(g)f^G(g^{-1}\gamma g)\mathbf{1}_{K'}(g^{-1}\gamma' g) \\ &= \mathrm{vol}(I)f^G(\delta)\sum_{g\in\Gamma} \mathrm{vol}(T\backslash TgK')\omega'(g)\mathbf{1}_{K'}(g^{-1}\gamma' g) \\ &= \mathrm{vol}(I)f^G(\delta)\cdot J^{\omega'}(\gamma', \mathbf{1}_{K'}). \end{aligned}$$

∎

§9. Calculation of the transfer factor

Let $\mathbf{G} = \mathrm{GL}_d$, and let $\mathbf{H} = \mathrm{Res}_{E/F}\,\mathbb{G}_m$, where E is a tamely ramified cyclic extension of F of degree d. Let $(\mathbf{H}, \mathcal{H}, s, \xi) = (\mathrm{Res}_{E/F}\,\mathbb{G}_m, {}^LH_{\mathbf{m}}, s_{\mathbf{m}}, \xi_{\mathbf{m}})$ be the endoscopic datum for $(\mathbf{G}, 1, \mathbf{a})$ with $\mathbf{m} = (1)$ as in Section 2. In this section, we compute the transfer factor $\Delta(\gamma, \iota(\gamma))$ for G-regular semisimple elements $\gamma \in H$, where $\iota : \mathbf{H} \to \mathbf{G}$ is the embedding defined in Section 8.

Let e and f be the ramification index and the residual degree of E/F, respectively. Let ω be the character of $F^\times$ of order d associated to $\mathbf{a}$. Let σ be the generator of $\mathrm{Gal}(E/F)$ as in [11, Section 10]. Let η_E be the unramified quadratic character of $E^\times$. If d is even, let $\sigma_+ = \sigma^{d/2}$, and let E_+ be the subfield of E corresponding to the subgroup $\{1, \sigma_+\}$ of $\mathrm{Gal}(E/F)$. Let $\omega_+ = \omega \circ \mathrm{N}_{E_+/F}$ be the quadratic character of $E_+^\times$ corresponding to E/E_+ by class field theory. If e is odd and f is even, then E/E_+ is unramified and the restriction of η_E to $E_+^\times$ is ω_+.

We fix a primitive $(q^f - 1)$th root of unity ζ in E. We choose ϖ_E and ϖ_F so that $\varpi_E^e = \varpi_F\zeta^a$ with some integer a. If e is odd and f is even, then we

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

may assume that $\varpi_E \in E_+$ since the ramification index of E_+/F is e and E/E_+ is unramified. Then we have

$$\sigma_+(\varpi_E) = \begin{cases} \varpi_E & \text{if } e \text{ is odd and } f \text{ is even,} \\ -\varpi_E & \text{if } e \text{ is even.} \end{cases}$$

We take $\{\varpi_E^i \zeta^j \mid 0 \le i < e, 0 \le j < f\}$ as a basis of $\mathfrak{o}_E$ over $\mathfrak{o}_F$. Put

$$\vec{u} = (\zeta^{f-1}, \zeta^{f-2}, \dots, \zeta, 1) \in E^f$$

and

$$\vec{v} = (\varpi_E^{e-1} \cdot \vec{u}, \varpi_E^{e-2} \cdot \vec{u}, \dots, \varpi_E \cdot \vec{u}, \vec{u}) \in E^d.$$

We may assume that the embedding $\iota : \mathbf{H} \to \mathbf{G}$ is given by

$$x \cdot \vec{v} = \vec{v} \cdot \iota(x)$$

for $x \in H$. Put

$$g = \begin{pmatrix} \vec{v} \\ \sigma(\vec{v}) \\ \vdots \\ \sigma^{d-1}(\vec{v}) \end{pmatrix} \in \mathrm{GL}_d(E).$$

Then we have

$$g\iota(x)g^{-1} = \mathrm{diag}\big(x, \sigma(x), \sigma^2(x), \dots, \sigma^{d-1}(x)\big)$$

for $x \in H$.

We will compute $\det g$.

LEMMA 9.1. *Let* $\mathbf{x}_i = \mathrm{diag}(x_{(i-1)f+1}, x_{(i-1)f+2}, \dots, x_{if})$. *Then we have*

$$\det \begin{pmatrix} \mathbf{1}_f & \mathbf{1}_f & \cdots & \mathbf{1}_f \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_e \\ \mathbf{x}_1^2 & \mathbf{x}_2^2 & \cdots & \mathbf{x}_e^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_1^{e-1} & \mathbf{x}_2^{e-1} & \cdots & \mathbf{x}_e^{e-1} \end{pmatrix} = \prod_{k=1}^{f} \prod_{0 \le i < j \le e-1} (x_{jf+k} - x_{if+k}).$$

Proof. See the Vandermonde determinant. □

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Put

$$\mathbf{u} = \begin{pmatrix} \vec{u} \\ \sigma(\vec{u}) \\ \vdots \\ \sigma^{f-1}(\vec{u}) \end{pmatrix} \in \mathrm{GL}_f(E).$$

Since $\sigma^f(\vec{u}) = \vec{u}$, we have $g = g'g''$, where $g' = (g'_{i,j})_{1\le i,j\le e}$, with

$$g'_{i,j} = \mathrm{diag}\big(\sigma^{(i-1)f}(\varpi_E)^{e-j}, \sigma^{(i-1)f+1}(\varpi_E)^{e-j}, \dots, \sigma^{if-1}(\varpi_E)^{e-j}\big) \in \mathrm{M}_f(E),$$

and $g'' = \mathrm{diag}(\mathbf{u}, \mathbf{u}, \dots, \mathbf{u})$. We have

$$\begin{aligned} \det g' &= (-1)^{(e-1)ef/2} \prod_{k=1}^{f} \prod_{0\le i<j\le e-1} \big(\sigma^{jf+k-1}(\varpi_E) - \sigma^{if+k-1}(\varpi_E)\big) \\ &= \prod_{k=1}^{f} \prod_{0\le i<j\le e-1} \big(\sigma^{if+k-1}(\varpi_E) - \sigma^{jf+k-1}(\varpi_E)\big) \end{aligned}$$

and

$$\begin{aligned} \det g'' &= \Big((-1)^{(f-1)f/2} \prod_{1\le i<j\le f} \big(\sigma^{j-1}(\zeta) - \sigma^{i-1}(\zeta)\big)\Big)^e \\ &= \Big(\prod_{1\le i<j\le f} \big(\sigma^{i-1}(\zeta) - \sigma^{j-1}(\zeta)\big)\Big)^e. \end{aligned}$$

Hence, we have

$$\begin{aligned} \det g = &\prod_{k=1}^{f} \prod_{0\le i<j\le e-1} \big(\sigma^{if+k-1}(\varpi_E) - \sigma^{jf+k-1}(\varpi_E)\big) \\ &\times \Big(\prod_{1\le i<j\le f} \big(\sigma^{i-1}(\zeta) - \sigma^{j-1}(\zeta)\big)\Big)^e. \end{aligned}$$

We define $h \in \mathrm{GL}_d(E)$ so that

$$h^{-1} = \begin{pmatrix} b_1 \cdot \vec{v} \\ b_2 \cdot \sigma(\vec{v}) \\ \vdots \\ b_d \cdot \sigma^{d-1}(\vec{v}) \end{pmatrix}, \tag{9.1}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

where

$$b_{jf+k} = \prod_{0\le i<j} \bigl(\sigma^{if+k-1}(\varpi_E) - \sigma^{jf+k-1}(\varpi_E)\bigr)^{-1} \times \prod_{1\le i<k} \bigl(\sigma^{i-1}(\zeta) - \sigma^{k-1}(\zeta)\bigr)^{-1}$$

for $0 \le j \le e-1$, $1 \le k \le f$. Then we have $\det h = 1$; that is, $h \in \mathbf{G}_{\mathrm{SC}}(E)$ and

$$h^{-1}\iota(x)h = \operatorname{diag}\bigl(x, \sigma(x), \sigma^2(x), \dots, \sigma^{d-1}(x)\bigr)$$

for $x \in H$.

Let $\mathbf{T}_H = \mathbf{H}$, and let $\mathbf{T} = \iota(\mathbf{T}_H)$. We choose an admissible embedding $\mathbf{T}_H \to \mathbf{T}$ induced by ι. Then we have

$$\Delta_{\mathrm{III}_1}\bigl(\gamma, \iota(\gamma)\bigr) = 1. \tag{9.2}$$

Let $\alpha_{i,j}$ be the root of $\mathbf{T}$ in $\mathbf{G}$ corresponding to the coroot $\mathbf{t}_{i,i}\mathbf{t}_{j,j}^{-1}$ of $\mathcal{T}$ in $\hat{G}$. Here $\mathbf{t}_{i,i}$ is the cocharacter of $\mathcal{T}$ given by $\mathbf{t}_{i,i}(z) = \operatorname{diag}(1, \dots, 1, \underset{i}{z}, 1, \dots, 1)$ for $z \in \mathbb{C}^\times$. Then $\alpha_{i,j}$ is symmetric (resp., asymmetric) if and only if $|i-j| = d/2$ (resp., $|i-j| \ne d/2$). We choose a-data $\{a_\alpha\}$ and χ-data $\{\chi_\alpha\}$ as follows. For simplicity, we write $a_{i,j} = a_{\alpha_{i,j}}$ and $\chi_{i,j} = \chi_{\alpha_{i,j}}$.

We require that $\sigma(a_{i,j}) = a_{i+1,j+1}$ and $a_{j,i} = -a_{i,j}$. Put

$$a_{1,jf+k} = \begin{cases} \nu(\sigma^{jf}(\varpi_E) - \varpi_E) & \text{if } k = 1, \\ \sigma^{k-1}(\zeta) - \zeta & \text{if } k \ne 1, \end{cases}$$

where

$$\nu = \prod_{1\le i<f} \bigl(\sigma^i(\zeta) - \zeta\bigr)^{-1}.$$

Note that $\sigma^f(\nu) = \nu$.

LEMMA 9.2. *We have*

$$\sigma^{d+1-i}(a_{1,i}) = -a_{1,d+2-i}$$

for $1 < i \le d/2 + 1$. *In particular,* $\{a_{1,i}\}$ *determines* a*-data* $\{a_\alpha\}$ *uniquely.*

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. Note that $i \equiv 1 \bmod f$ if and only if $d+2-i \equiv 1 \bmod f$. Let $i = jf+k$. If $k=1$, then we have

$$\sigma^{d+1-i}(a_{1,i}) = \sigma^{d-jf}(a_{1,jf+1}) = \nu\big(\varpi_E - \sigma^{d-jf}(\varpi_E)\big)$$
$$= -a_{1,d-jf+1} = -a_{1,d+2-i}.$$

If $k \neq 1$, then we have $d+2-i = (d-jf-f)+(f+2-k)$ and

$$\sigma^{d+1-i}(a_{1,i}) = \sigma^{d-jf-k+1}(a_{1,jf+k}) = \zeta - \sigma^{f-k+1}(\zeta) = -a_{1,d+2-i}. \qquad \square$$

Put $\chi_{1,i} = \mathbb{1}$ for $1 < i < d/2+1$. If d is even, let $\chi_{1,d/2+1}$ be a character of $E^\times$ whose restriction to $E_+^\times$ is ω_+. Note that $\chi_{1,d/2+1} \circ \sigma_+ = \chi_{1,d/2+1}^{-1}$. If e is odd and f is even, then we can take $\chi_{1,d/2+1} = \eta_E$. Then $\{\chi_{1,i}\}$ determines χ-data $\{\chi_\alpha\}$ uniquely.

LEMMA 9.3. *If e and f are odd, then we have*

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big)\Delta_{\mathrm{III}_2}\big(\gamma, \iota(\gamma)\big) = 1.$$

If e is odd and f is even, then we have

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big)\Delta_{\mathrm{III}_2}\big(\gamma, \iota(\gamma)\big) = \eta_E\big(\gamma - \sigma_+(\gamma)\big).$$

If e is even and f is odd, then we have

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big)\Delta_{\mathrm{III}_2}\big(\gamma, \iota(\gamma)\big) = \omega(-1)^{e(f-1)/4} \cdot \omega_+\Big(\frac{\gamma - \sigma_+(\gamma)}{2\varpi_E^{-1}}\Big).$$

If e and f are even, then we have

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big)\Delta_{\mathrm{III}_2}\big(\gamma, \iota(\gamma)\big) = \omega_+\Big(\frac{\gamma - \sigma_+(\gamma)}{2\varpi_E^{-1}(\sigma^{f/2}(\zeta) - \zeta)}\Big).$$

Proof. We only consider the case when d is even. By [11, Appendix A], we have

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big) = \chi_{1,d/2+1}\Big(\frac{\alpha_{1,d/2+1}(\gamma) - 1}{a_{1,d/2+1}}\Big) = \chi_{1,d/2+1}\Big(\frac{\gamma\sigma_+(\gamma)^{-1} - 1}{a_{1,d/2+1}}\Big)$$

and

$$\Delta_{\mathrm{III}_2}\big(\gamma, \iota(\gamma)\big) = \chi_{1,d/2+1}(\gamma)^{-1}.$$

Since $\sigma_+(a_{1,d/2+1}) = -a_{1,d/2+1}$, we have

$$\Delta_{\mathrm{II}}\big(\gamma, \iota(\gamma)\big)\Delta_{\mathrm{III}_1}\big(\gamma, \iota(\gamma)\big) = \chi_{1,d/2+1}\Big(\frac{\sigma_+(\gamma)^{-1} - \gamma^{-1}}{a_{1,d/2+1}}\Big) = \omega_+\Big(\frac{\gamma - \sigma_+(\gamma)}{a_{1,d/2+1}}\Big).$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Note that $\omega_+(x\sigma_+(x)) = 1$ for $x \in E^\times$.

If e is odd, then f is even and

$$a_{1,d/2+1} = a_{1,(e-1)f/2+f/2+1} = \sigma^{f/2}(\zeta) - \zeta \in \mathfrak{o}_E^\times,$$

so that

$$\chi_{1,d/2+1}(a_{1,d/2+1}) = \eta_E\big(\sigma^{f/2}(\zeta) - \zeta\big) = 1.$$

Assume that e is even. Then we have

$$a_{1,d/2+1} = a_{1,ef/2+1} = \nu\big(\sigma^{ef/2}(\varpi_E) - \varpi_E\big) = -2\varpi_E \cdot \nu$$

and

$$\chi_{1,d/2+1}(-\varpi_E) = \chi_{1,d/2+1}\big(\sigma_+(\varpi_E)\big) = \chi_{1,d/2+1}(\varpi_E^{-1}).$$

Since $\sigma^i(\zeta) - \zeta \in E_+^\times$ and

$$\omega_+\big(\sigma(x)\big) = \omega\big(\mathrm{N}_{E_+/F}(\sigma(x))\big) = \omega\big(\mathrm{N}_{E_+/F}(x)\big) = \omega_+(x)$$

for $x \in E_+^\times$, we have

$$\begin{aligned}\omega_+\big(\sigma^i(\zeta) - \zeta\big) &= \omega_+\big(\sigma^{f-i}(\sigma^i(\zeta) - \zeta)\big) = \omega_+\big(\zeta - \sigma^{f-i}(\zeta)\big)\\ &= \omega(-1)^{d/2}\omega_+\big(\sigma^{f-i}(\zeta) - \zeta\big).\end{aligned}$$

If f is odd, then we have

$$\begin{aligned}\omega_+(\nu) &= \prod_{1\le i\le (f-1)/2} \omega_+\big(\sigma^i(\zeta) - \zeta\big)^{-1} \cdot \prod_{(f+1)/2\le i\le f-1} \omega_+\big(\sigma^i(\zeta) - \zeta\big)^{-1}\\ &= \omega(-1)^{d(f-1)/4} \prod_{1\le i\le (f-1)/2} \omega_+\big(\sigma^i(\zeta) - \zeta\big)^{-2}\\ &= \omega(-1)^{d(f-1)/4}\\ &= \omega(-1)^{e(f-1)/4}.\end{aligned}$$

If f is even, then we have

$$\begin{aligned}\omega_+(\nu) = &\prod_{1\le i\le f/2-1} \omega_+\big(\sigma^i(\zeta) - \zeta\big)^{-1} \cdot \omega_+\big(\sigma^{f/2}(\zeta) - \zeta\big)^{-1}\\ &\cdot \prod_{f/2+1\le i\le f-1} \omega_+\big(\sigma^i(\zeta) - \zeta\big)^{-1}\end{aligned}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

$$\begin{aligned}
&= \omega(-1)^{d(f-2)/4} \prod_{1\le i\le f/2-1} \omega_+\big(\sigma^i(\zeta)-\zeta\big)^{-2}\cdot \omega_+\big(\sigma^{f/2}(\zeta)-\zeta\big)^{-1}\\
&= \omega(-1)^{d(f-2)/4}\cdot \omega_+\big(\sigma^{f/2}(\zeta)-\zeta\big)\\
&= \omega_+\big(\sigma^{f/2}(\zeta)-\zeta\big).
\end{aligned}$$

This completes the proof. □

LEMMA 9.4. *We have*

$$\Delta_{\mathrm{I}}\big(\gamma,\iota(\gamma)\big)=1.$$

Proof. Let $h\in \mathbf{G}_{\mathrm{SC}}(E)$ be given by (9.1). Let $m(\sigma_T)=x(\sigma_T)n(\omega_T(\sigma))$ be as in [15, Section 2.3]. Then we have

$$x(\sigma_T)=\mathrm{diag}\Big(\prod_{1<i\le d} a_{1,i}, a_{1,2}^{-1}, a_{1,3}^{-1},\dots,a_{1,d}^{-1}\Big)$$

and

$$n(\omega_T(\sigma))=\begin{pmatrix} 0 & 1\\ -\mathbf{1}_{d-1} & 0\end{pmatrix}.$$

Hence, we have

$$m(\sigma_T)\sigma(h^{-1})=x(\sigma_T)\cdot\begin{pmatrix}\sigma(b_d)\cdot\vec{v}\\ -\sigma(b_1)\cdot\sigma(\vec{v})\\ \vdots\\ -\sigma(b_{d-1})\cdot\sigma^{d-1}(\vec{v})\end{pmatrix}=\begin{pmatrix}\prod_{1<i\le d}a_{1,i}\cdot\sigma(b_d)\cdot\vec{v}\\ -a_{1,2}^{-1}\sigma(b_1)\cdot\sigma(\vec{v})\\ \vdots\\ -a_{1,d}^{-1}\sigma(b_{d-1})\cdot\sigma^{d-1}(\vec{v})\end{pmatrix}.$$

Let $i=jf+k\ge 2$ with $0\le j\le e-1$, $1\le k\le f$. If $k=1$, then we have

$$\begin{aligned}
-a_{1,i}^{-1}\sigma(b_{i-1}) &= -a_{1,jf+1}^{-1}\sigma(b_{(j-1)f+f})\\
&= \nu^{-1}\big(\varpi_E-\sigma^{jf}(\varpi_E)\big)^{-1}\prod_{0\le i<j-1}\big(\sigma^{(i+1)f}(\varpi_E)-\sigma^{jf}(\varpi_E)\big)^{-1}\\
&\quad\times\prod_{1\le i<f}\big(\sigma^i(\zeta)-\zeta\big)^{-1}\\
&= \prod_{0\le i<j}\big(\sigma^{if}(\varpi_E)-\sigma^{jf}(\varpi_E)\big)^{-1}\\
&= b_{jf+1}\\
&= b_i.
\end{aligned}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

If $k \neq 1$, then we have

$$\begin{aligned}
-a_{1,i}^{-1}\sigma(b_{i-1}) =& -a_{1,jf+k}^{-1}\sigma(b_{jf+k-1}) \\
=& \big(\zeta - \sigma^{k-1}(\zeta)\big)^{-1} \prod_{0\le i<j} \big(\sigma^{if+k-1}(\varpi_E) - \sigma^{jf+k-1}(\varpi_E)\big)^{-1} \\
& \times \prod_{1\le i<k-1} \big(\sigma^{i}(\zeta) - \sigma^{k-1}(\zeta)\big)^{-1} \\
=& \prod_{0\le i<j} \big(\sigma^{if+k-1}(\varpi_E) - \sigma^{jf+k-1}(\varpi_E)\big)^{-1} \\
& \times \prod_{1\le i<k} \big(\sigma^{i-1}(\zeta) - \sigma^{k-1}(\zeta)\big)^{-1} \\
=& b_{jf+k} \\
=& b_i.
\end{aligned}$$

Since

$$\begin{aligned}
\prod_{2\le i\le d} a_{1,i} &= \nu^{e-1} \prod_{1\le i\le e-1} \big(\sigma^{if}(\varpi_E) - \varpi_E\big) \cdot \Big(\prod_{1<k\le f} \big(\sigma^{k-1}(\zeta) - \zeta\big)\Big)^{e} \\
&= \nu^{-1} \prod_{1\le i<e} \big(\sigma^{if}(\varpi_E) - \varpi_E\big)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(b_d) &= \prod_{0\le i<e-1} \big(\sigma^{(i+1)f}(\varpi_E) - \varpi_E\big)^{-1} \cdot \prod_{1\le i<f} \big(\sigma^{i}(\zeta) - \zeta\big)^{-1} \\
&= \nu \prod_{1\le i<e} \big(\sigma^{if}(\varpi_E) - \varpi_E\big)^{-1},
\end{aligned}$$

we have

$$\prod_{2\le i\le d} a_{1,i} \cdot \sigma(b_d) = 1 = b_1.$$

Thus, we obtain

$$m(\sigma_T)\sigma(h^{-1}) = h^{-1},$$

so that the 1-cocycle $\lambda(\mathbf{T}_{\mathrm{sc}})$ as in [15, Section 2.3] is trivial. This yields the lemma. □

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

PROPOSITION 9.5. *If e and f are odd, then we have*

$$\Delta\big(\gamma,\iota(\gamma)\big)=\lambda(E/F,\psi)^{-1}\cdot\Delta_{\mathrm{IV}}\big(\gamma,\iota(\gamma)\big).$$

If e is odd and f is even, then we have

$$\Delta\big(\gamma,\iota(\gamma)\big)=\lambda(E/F,\psi)^{-1}\cdot\eta_E\big(\gamma-\sigma_+(\gamma)\big)\cdot\Delta_{\mathrm{IV}}\big(\gamma,\iota(\gamma)\big).$$

If e is even and f is odd, then we have

$$\Delta\big(\gamma,\iota(\gamma)\big)=\lambda(E/F,\psi)^{-1}\cdot\omega(-1)^{e(f-1)/4}\cdot\omega_+\Big(\frac{\gamma-\sigma_+(\gamma)}{2\varpi_E^{-1}}\Big)\cdot\Delta_{\mathrm{IV}}\big(\gamma,\iota(\gamma)\big).$$

If e and f are even, then we have

$$\Delta\big(\gamma,\iota(\gamma)\big)=\lambda(E/F,\psi)^{-1}\cdot\omega_+\Big(\frac{\gamma-\sigma_+(\gamma)}{2\varpi_E^{-1}\big(\sigma^{f/2}(\zeta)-\zeta\big)}\Big)\cdot\Delta_{\mathrm{IV}}\big(\gamma,\iota(\gamma)\big).$$

Proof. Let V be the virtual representation of Γ given by (1.1). Then we have

$$\varepsilon(V,\psi)=\lambda(E/F,\psi)^{-1}.$$

Hence, the proposition follows from (9.2) and Lemmas 9.3 and 9.4. □

§10. Proof of Theorem 1.4

In this section, we finish the proof of Theorem 1.4. We may keep the assumption of Section 6.

We fix a primitive (q^f-1)th root of unity ζ in E and choose ϖ_E, ϖ_F as in Section 9. Let $\iota:\mathbf{H}\to\mathbf{G}$ be the embedding defined in Section 9.

Let $\gamma\in H$ be a G-regular semisimple element such that $\mathrm{ord}_E\,\gamma=-1$. As in Section 8, we write $\gamma=\delta\gamma'$ so that δ is $F(\delta)/F$-cuspidal and $\gamma'\in1+\mathfrak{p}_E$. Put $F'=F(\delta)$, and put $\delta'=\varpi_E\delta$. By Lemma 8.2, E/F' is unramified and $\delta'\in\mathfrak{o}_E^\times$. We may assume that δ' is a root of unity in E.

Let $\mathbf{G}'=\mathrm{Res}_{F'/F}\,\mathrm{GL}_{d'}$, where $d'=d/ef'$ and f' is the residual degree of F'/F. Let $\iota':\mathbf{H}\to\mathbf{G}'$ be the embedding as in Section 9. By Proposition 9.5, we have

$$\Delta_{G',H}\big(\gamma',\iota'(\gamma')\big)=\begin{cases}\Delta_{G',H,\mathrm{IV}}(\gamma',\iota'(\gamma'))\\ \quad\text{if } d' \text{ is odd},\\ \eta_E(\gamma'-\sigma_+(\gamma'))\cdot\Delta_{G',H,\mathrm{IV}}(\gamma',\iota'(\gamma'))\\ \quad\text{if } d' \text{ is even}.\end{cases}\tag{10.1}$$

Note that γ' is G'-regular.

Let $F''\subset E$ be the unramified extension of F of degree f. Then $\delta'\in F''$.

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

LEMMA 10.1. *Assume that e is odd and that f is even. Then d' is even if and only if $\delta' \in E_+$.*

Proof. Since E/F is cyclic, d' is even if and only if $F' \subset E_+$. Since $F' = F(\delta)$ and $\varpi_E \in E_+$, the assertion follows. □

LEMMA 10.2. *Assume that e is even. Then d' is odd and $\delta' \in E_+$.*

Proof. We have $F'' \subset E_+$ since e is even. Hence, E/E_+ is ramified, so that d' must be odd since E/F' is unramified. □

LEMMA 10.3. *We have*

$$\Delta_{G,H}\big(\gamma, \iota(\gamma)\big) = \lambda(E/F,\psi)^{-1} \cdot \Delta_{G',H}\big(\gamma', \iota(\gamma')\big) \times \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,} \\ \omega(\varpi_F)^{d/2} & \text{if } e \text{ is odd and } f \text{ is even,} \\ \omega(-1)^{e(f-1)/4} \cdot \omega_+(\delta') & \text{if } e \text{ is even and } f \text{ is odd,} \\ \omega_+(\sigma^{f/2}(\zeta) - \zeta) \cdot \omega_+(\delta') & \text{if } e \text{ and } f \text{ are even.} \end{cases}$$

Proof. By [19, lemme VII.3], we have

$$\Delta_{G,H,\mathrm{IV}}\big(\gamma, \iota(\gamma)\big) = \Delta_{G',H,\mathrm{IV}}\big(\gamma', \iota'(\gamma')\big).$$

By Proposition 9.5, (10.1), and Lemmas 10.1 and 10.2, it suffices to show that

$$\eta_E\big(\gamma - \sigma_+(\gamma)\big) = \omega(\varpi_F)^{d/2} \times \begin{cases} \eta_E(\gamma' - \sigma_+(\gamma')) & \text{if } e \text{ is odd, } f \text{ is even, and } \delta' \in E_+, \\ 1 & \text{if } e \text{ is odd, } f \text{ is even, and } \delta' \notin E_+ \end{cases}$$

and that

$$\omega_+\Big(\frac{\gamma - \sigma_+(\gamma)}{2\varpi_E^{-1}}\Big) = \omega_+(\delta')$$

if e is even.

We first assume that e is odd and f is even. Then we have

$$\eta_E\big(\gamma - \sigma_+(\gamma)\big) = \eta_E(\varpi_E^{-1}) \cdot \eta_E\big(\delta'\gamma' - \sigma_+(\delta'\gamma')\big).$$

Since $\eta_E^2 = \mathbb{1}$ and e is odd, we have

$$\eta_E(\varpi_E^{-1}) = \eta_E(\varpi_E)^e = \eta_E(\varpi_F \zeta^a).$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Since η_E is unramified, we have $\eta_E(\zeta^a) = 1$. Hence, we have

$$\eta_E(\varpi_E^{-1}) = \eta_E(\varpi_F) = \omega\big(\mathrm{N}_{E_+/F}(\varpi_F)\big) = \omega(\varpi_F^{d/2}).$$

If $\delta' \in E_+$, then we have

$$\eta_E\big(\delta'\gamma' - \sigma_+(\delta'\gamma')\big) = \eta_E(\delta') \cdot \eta_E\big(\gamma' - \sigma_+(\gamma')\big) = \eta_E\big(\gamma' - \sigma_+(\gamma')\big).$$

If $\delta' \notin E_+$, then $\sigma_+(\delta') \neq \delta'$, so that

$$\delta'\gamma' - \sigma_+(\delta'\gamma') \equiv \delta' - \sigma_+(\delta') \not\equiv 0 \mod \mathfrak{p}_E.$$

Hence, we have $\delta'\gamma' - \sigma_+(\delta'\gamma') \in \mathfrak{o}_E^\times$, so that $\eta_E(\delta'\gamma' - \sigma_+(\delta'\gamma')) = 1$.

We next assume that e is even. Then we have

$$\frac{\gamma - \sigma_+(\gamma)}{2\varpi_E^{-1}} = \frac{\varpi_E^{-1}\delta'(\gamma' + \sigma_+(\gamma'))}{2\varpi_E^{-1}} = \frac{\delta'(\gamma' + \sigma_+(\gamma'))}{2}.$$

Since E/F is tamely ramified and e is even, we have $2 \nmid q$, so that $(\gamma' + \sigma_+(\gamma'))/2 \in 1 + \mathfrak{p}_{E_+}$. Hence, we have

$$\omega_+\Big(\frac{\gamma' + \sigma_+(\gamma')}{2}\Big) = 1. \qquad \square$$

Let $f^G \in C_c^\infty(G)$ be as in Section 7. By [19] and [20], there exists $f^H \in C_c^\infty(H)$ such that f^G and f^H have matching orbital integrals. Then f^H is determined by

$$f^H(\gamma) = \Delta_{G,H}\big(\gamma, \iota(\gamma)\big) \cdot J^\omega(\gamma, f^G)$$

for G-regular semisimple elements $\gamma \in H$. By Lemma 8.1, $f^H(\gamma) = 0$ unless $\mathrm{ord}_E \gamma = -1$.

LEMMA 10.4. *Let $\gamma \in H$ be a G-regular semisimple element such that that $\mathrm{ord}_E \gamma = -1$. Then we have*

$$f^H(\gamma) = \lambda(E/F, \psi)^{-1} \times \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,} \\ \omega(\varpi_F)^{d/2} & \text{if } e \text{ is odd and } f \text{ is even,} \\ \omega(-1)^{e(f-1)/4} & \text{if } e \text{ is even and } f \text{ is odd,} \\ \omega_+(\sigma^{f/2}(\zeta) - \zeta) & \text{if } e \text{ and } f \text{ are even.} \end{cases}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Proof. By the definition of ι, we have

$$\iota(\varpi_E) = \eta \cdot \begin{pmatrix} g_{1,1} & 0 \\ 0 & \mathbf{1}_{d-f} \end{pmatrix}$$

with some $g_{1,1} \in \mathrm{GL}_f(\mathfrak{o}_F)$ and

$$\iota(\delta') = \mathrm{diag}(g', g', \dots, g')$$

with some $g' \in \mathrm{GL}_f(\mathfrak{o}_F)$ such that $\det g' = \mathrm{N}_{F''/F}(\delta')$. Hence, we have

$$f^G(\delta) = f^G(\varpi_E^{-1}\delta') = \mathrm{vol}(I)^{-1} \cdot \omega\big(\mathrm{N}_{F''/F}(\delta')\big)^{-e(e+1)/2}.$$

If e is odd, then we have

$$\omega\big(\mathrm{N}_{F''/F}(\delta')\big)^{-e(e+1)/2} = \omega^e\big(\mathrm{N}_{F''/F}(\delta')\big)^{-(e+1)/2} = 1.$$

If e is even, then we have

$$\omega\big(\mathrm{N}_{F''/F}(\delta')\big)^{-e(e+1)/2} = \omega\big(\mathrm{N}_{E_+/F}(\delta')\big)^{-(e+1)} = \omega_+(\delta')^{-(e+1)} = \omega_+(\delta').$$

By Lemmas 8.5 and 10.3, we have

$$\begin{aligned} &\Delta_{G,H}\big(\gamma, \iota(\gamma)\big) \cdot J^{\omega}(\gamma, f^G) \\ &\quad = \lambda(E/F, \psi)^{-1} \cdot \Delta_{G',H}\big(\gamma', \iota'(\gamma')\big) \cdot J^{\omega'}(\gamma', \mathbf{1}_{K'}) \\ &\qquad \times \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,} \\ \omega(\varpi_F)^{d/2} & \text{if } e \text{ is odd and } f \text{ is even,} \\ \omega(-1)^{e(f-1)/4} & \text{if } e \text{ is even and } f \text{ is odd,} \\ \omega_+(\sigma^{f/2}(\zeta) - \zeta) & \text{if } e \text{ and } f \text{ are even.} \end{cases} \end{aligned}$$

By the fundamental lemma [19, théorème VIII.5], we have

$$\Delta_{G',H}\big(\gamma', \iota'(\gamma')\big) \cdot J^{\omega'}(\gamma', \mathbf{1}_{K'}) = 1.$$

This completes the proof. □

Thus, we obtain

$$J(\pi_H, f^H) = \lambda(E/F, \psi)^{-1} \times \begin{cases} 1 & \text{if } e \text{ and } f \text{ are odd,} \\ \omega(\varpi_F)^{d/2} & \text{if } e \text{ is odd and } f \text{ is even,} \\ \omega(-1)^{e(f-1)/4} & \text{if } e \text{ is even and } f \text{ is odd,} \\ \omega_+(\sigma^{f/2}(\zeta) - \zeta) & \text{if } e \text{ and } f \text{ are even,} \end{cases}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

where $\pi_H = \mathbb{1}$. By Proposition 7.8 and Lemma 10.5 below, we have

$$J^\omega(\pi, f^G) = J(\pi_H, f^H).$$

Hence, we have $c = 1$.

To finish the proof of Theorem 1.4, it remains to prove the following lemma.

LEMMA 10.5. *Assume that e and f are even. Then we have*

$$\omega(-1)^{ef/4} \cdot \omega(\varpi_F)^{d/2} = \omega_+\big(\sigma^{f/2}(\zeta) - \zeta\big).$$

Proof. Note that $2 \nmid q$ since E/F is tamely ramified and e is even. By Lemma 10.6 below, we have $\varpi_E^e = \varpi_F \zeta^a$ with an odd integer a.

We first show that

$$\omega(\varpi_F)^{d/2} = -1.$$

The quadratic character $\omega_+ = \omega \circ \mathrm{N}_{E_+/F}$ of $E_+^\times$ associated to E/E_+ satisfies the following:

- $\omega_+(\zeta) = -1$,
- $\omega_+(-\varpi_E^2) = 1$,
- ω_+ is trivial on $1 + \mathfrak{p}_{E_+}$.

Note that $F'' \subset E_+$ and $\sigma_+(\varpi_E) = -\varpi_E$. Hence, we have

$$\begin{aligned}
\omega(\varpi_F)^{d/2} &= \omega_+(\varpi_F) \\
&= \omega_+(\varpi_E^e \zeta^{-a}) \\
&= \omega_+\big((-1)^{e/2} \cdot (-\varpi_E^2)^{e/2} \cdot \zeta^{-a}\big) \\
&= \omega_+(-1)^{e/2} \cdot \omega_+(\zeta)^{-a} \\
&= \omega(-1)^{ed/4} \cdot (-1)^{-a} \\
&= -1.
\end{aligned}$$

We next show that

$$\omega(-1)^{ef/4} = \begin{cases} 1 & \text{if } q \equiv 1 \bmod 4, \\ 1 & \text{if } q \equiv 3 \bmod 4 \text{ and } f \equiv 0 \bmod 4, \\ -1 & \text{if } q \equiv 3 \bmod 4 \text{ and } f \equiv 2 \bmod 4. \end{cases}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

Note that, if $q \equiv 3 \bmod 4$, then $e \equiv 2 \bmod 4$ since $e \mid (q-1)$. Assume that $q \equiv 1 \bmod 4$. Then a primitive fourth root of unity $\zeta^{(q^f-1)/4}$ belongs to F. Since $(-1)^{ef/4} = (\zeta^{(q^f-1)/4})^{d/2}$, we have

$$\omega(-1)^{ef/4} = \omega_+(\zeta^{(q^f-1)/4}) = (-1)^{(q^f-1)/4}.$$

Since f is even, $(q^f-1)/4$ is even, so that $\omega(-1)^{ef/4} = 1$. Assume that $q \equiv 3 \bmod 4$. It suffices to show that $\omega(-1) = -1$. Let $\zeta' = \zeta^{(q^f-1)/(q-1)}$ be a primitive $(q-1)$th root of unity in F. Then $\omega(\zeta')$ is a primitive eth root of unity in $\mathbb{C}^\times$ since the restriction of ω to $\mathfrak{o}_F^\times$ is of order e. Since $e \mid (q-1)$ and $(q-1)/2$ is odd, we have

$$\frac{q-1}{2} \equiv \frac{e}{2} \mod e.$$

Hence, we have

$$\omega(-1) = \omega(\zeta'^{(q-1)/2}) = \omega(\zeta')^{e/2} = -1.$$

We finally show that

$$\omega_+\big(\sigma^{f/2}(\zeta) - \zeta\big) = \begin{cases} -1 & \text{if } q \equiv 1 \bmod 4, \\ -1 & \text{if } q \equiv 3 \bmod 4 \text{ and } f \equiv 0 \bmod 4, \\ 1 & \text{if } q \equiv 3 \bmod 4 \text{ and } f \equiv 2 \bmod 4. \end{cases}$$

Put $\beta = \sigma^{f/2}(\zeta) - \zeta \in \mathfrak{o}_{F''}^\times$. Then we have $\sigma^{f/2}(\beta) = -\beta$, and hence $(\beta^2)^{q^{f/2}-1} \equiv 1 \bmod \mathfrak{p}_{F''}$. Let r be the order of β^2 in $(\mathfrak{o}_{F''}/\mathfrak{p}_{F''})^\times$. Then we have $r \mid (q^{f/2}-1)$. Since $2 \nmid q$, we have $\sigma^{f/2}(\beta) \not\equiv \beta \bmod \mathfrak{p}_{F''}$. Hence, $2r$ is the order of β in $(\mathfrak{o}_{F''}/\mathfrak{p}_{F''})^\times$ and $2r \nmid (q^{f/2}-1)$. Let b be an integer such that $(b, 2r) = 1$ and $\beta \equiv \zeta^{b(q^f-1)/2r} \bmod \mathfrak{p}_{F''}$. Then we have

$$\omega_+(\beta) = \omega_+(\zeta^{b(q^f-1)/2r}) = (-1)^{b(q^f-1)/2r} = (-1)^{(q^f-1)/2r} = (-1)^{(q^{f/2}+1)/2}.$$

Note that b, $(q^{f/2}-1)/r$ are odd and

$$\frac{q^f-1}{2r} = \frac{q^{f/2}-1}{r} \cdot \frac{q^{f/2}+1}{2}.$$

Hence, $\omega_+(\beta) = 1$ if and only if $4 \mid (q^{f/2}+1)$. This yields the assertion. □

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

LEMMA 10.6. *Let F be a nonarchimedean local field of characteristic zero. Let f be a positive integer. We fix a primitive (q^f-1)th root of unity ζ in $\bar{F}$. Let $F''=F(\zeta)$ be the unramified extension of F of degree f. Let e be a positive integer such that $e \mid (q-1)$.*

Let ϖ_E be a root of

$$X^e - \varpi_F \zeta^a,$$

where ϖ_F is a uniformizer of F and a is an integer such that

$$\left(a, \frac{q^f-1}{q-1}, e\right) = 1.$$

Then $E=F''(\varpi_E)$ is a tamely ramified cyclic extension of F with ramification index e and residual degree f. Conversely, any such E is of this form.

Proof. If E is a tamely ramified cyclic extension of F with ramification index e and residual degree f, then there exist a uniformizer ϖ_F of F and an integer a such that $E=F''(\varpi_E)$, where ϖ_E is a root of $X^e-\varpi_F\zeta^a$.

Conversely, let $E=F''(\varpi_E)$, where ϖ_E is a root of $X^e-\varpi_F\zeta^a$, ϖ_F is a uniformizer of F, and a is an integer. For each $0 \le i \le e-1$, $\varpi_E\zeta^{a(q-1)/e+i(q^f-1)/e}$ is a root of $X^e-\varpi_F\zeta^{aq}$. Hence, there exists an automorphism σ of E over F given by

$$\sigma(\zeta) = \zeta^q,$$

$$\sigma(\varpi_E) = \varpi_E \zeta^{a(q-1)/e+i(q^f-1)/e}.$$

Note that any extension to E of the Frobenius automorphism of F'' over F is of this form.

Let n be the order of σ. Since σ is an extension of the Frobenius automorphism of F'' over F, we have $f \mid n$. We have

$$\begin{aligned}\sigma^f(\varpi_E) &= \varpi_E\big(\zeta^{a(q-1)/e+i(q^f-1)/e}\big)^{1+q+\cdots+q^{f-1}} \\ &= \varpi_E\big(\zeta^{a(q-1)/e+i(q^f-1)/e}\big)^{(q^f-1)/(q-1)} \\ &= \varpi_E\big(\zeta^{a+i(q^f-1)/(q-1)}\big)^{(q^f-1)/e}.\end{aligned}$$

We have $\big(\zeta^{a(q-1)/e+i(q^f-1)/e}\big)^{(q^f-1)/(q-1)} \in F^\times$ since its order divides $q-1$. Hence, we have

$$\sigma^{jf}(\varpi_E) = \varpi_E\big(\zeta^{a+i(q^f-1)/(q-1)}\big)^{j(q^f-1)/e}$$

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

for all j, so that $n = ef$ if and only if

$$\left(a + i\frac{q^f - 1}{q - 1}, e\right) = 1.$$

Hence, there exists $0 \leq i \leq e - 1$ such that σ is of order ef if and only if

$$\left(a, \frac{q^f - 1}{q - 1}, e\right) = 1.$$

This completes the proof. □

This completes the proof of Theorem 1.4.

References

[1] J. Arthur and L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, Ann. of Math. Stud. **120**, Princeton University Press, Princeton, NJ, 1989.

[2] E. Artin and J. Tate, *Class Field Theory*, AMS Chelsea Publishing, Providence, 2009.

[3] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Ann. of Math. Stud. **151**, Princeton University Press, Princeton, NJ, 2001.

[4] G. Henniart, *La conjecture de Langlands locale pour* GL(3), Mém. Soc. Math. Fr. (N.S.) **11–12**, Math. Soc. Fr., Paris, 1984.

[5] ———, *On the local Langlands conjecture for* GL(n)*: The cyclic case*, Ann. of Math. (2) **123** (1986), 145–203.

[6] ———, *Induction automorphe pour* GL($n, \mathbb{C}$), J. Funct. Anal. **258** (2010), 3082–3096.

[7] G. Henniart and R. Herb, *Automorphic induction for* GL(n) *(over local non-Archimedean fields)*, Duke Math. J. **78** (1995), 131–192.

[8] G. Henniart and B. Lemaire, *Intégrales orbitales tordues sur* GL(n, F) *et corps locaux proches: Applications*, Canad. J. Math. **58** (2006), 1229–1267.

[9] ———, *Formules de caractères pour l'induction automorphe*, J. Reine Angew. Math. **645** (2010), 41–84.

[10] ———, *Changement de base et induction automorphe pour* GL_n *en caractéristique non nulle*, Mém. Soc. Math. Fr. (N.S.) **124**, Math. Soc. Fr., Paris, 2011.

[11] K. Hiraga and H. Saito, *On L-packets for inner forms of* SL_n, Mem. Amer. Math. Soc. **215** (2012), no. 1013.

[12] R. E. Howe, *Tamely ramified supercuspidal representations of* Gl_n, Pacific J. Math. **73** (1977), 437–460.

[13] R. E. Kottwitz and D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque **255**, Soc. Math. France, Paris, 1999.

[14] J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* SL(2), Canad. J. Math. **31** (1979), 726–785.

[15] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219–271.

[16] F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–355.

https://doi.org/10.1017/S0027763000010606 Published online by Cambridge University Press

[17] ———, *Fourier transforms of intertwining operators and Plancherel measures for* GL(n), Amer. J. Math. **106** (1984), 67–111.
[18] J. Tate, "Number theoretic background" in *Automorphic forms, representations and L-functions (Corvallis, Orea, 1977), Pt. 2*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, 1979, 3–26.
[19] J.-L. Waldspurger, *Sur les intégrales orbitales tordues pour les groupes linéaires: un lemme fondamental*, Canad. J. Math. **43** (1991), 852–896.
[20] ———, *Le lemme fondamental implique le transfert*, Compos. Math. **105** (1997), 153–236.
[21] ———, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spéciaux orthogonaux*, preprint, arXiv:0911.4568 [math.RT].

Kaoru Hiraga
Department of Mathematics
Kyoto University
Kyoto 606-8502
Japan
hiraga@math.kyoto-u.ac.jp

Atsushi Ichino
Department of Mathematics
Kyoto University
Kyoto 606-8502
Japan
ichino@math.kyoto-u.ac.jp