



A Note on the Exactness of Operator Spaces

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Abstract. In this paper, we give two characterizations of the exactness of operator spaces.

1 Introduction

Operator space theory, a natural non-commutative quantization of Banach space theory, is an emerging area in modern analysis. Recently, there have been important developments in the local theory of operator spaces [6, 7, 11]. We will be concerned mainly with the “geometry” of finite dimensional operator spaces. In the Banach space category it is well known that every separable space embeds isometrically into l_∞ . Moreover, if E is a finite dimensional normed space, then for each $\epsilon > 0$, there is an integer n and a subspace $F \subseteq l_\infty^n$ that is $(1 + \epsilon)$ -isomorphic to E , *i.e.*, there is an isomorphism $u: E \rightarrow F$ such that $\|u\| \cdot \|u^{-1}\| \leq 1 + \epsilon$. Quite interestingly it turns out that this fact is not valid in the category of operator spaces: although every operator space embeds completely isometrically into $\mathcal{B}(\mathcal{H})$ (the non-commutative analogue of l_∞), it is not true that a finite dimensional operator space must be close to a subspace of M_n (the non-commutative analogue of l_∞^n) for some n . The main object of this phenomenon is called the exactness of operator spaces. The exactness of C^* -algebras was first introduced by Kirchberg [9] and this concept was extended to the “purely” operator space setting by Pisier [11].

To state our main results, we first recall some basic notations and terminologies in operator spaces; the details can be found in [5, 12]. Given a Hilbert space \mathcal{H} , we let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators on \mathcal{H} . For each natural number $n \in \mathbf{N}$, there is a canonical norm $\|\cdot\|_n$ on the $n \times n$ matrix space $M_n(\mathcal{B}(\mathcal{H}))$ given by identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$. We call this family of norms $\{\|\cdot\|_n\}$ an operator space matrix norm on $\mathcal{B}(\mathcal{H})$. An operator space V is a norm closed subspace of some $\mathcal{B}(\mathcal{H})$ equipped with the distinguished operator space matrix norm inherited from $\mathcal{B}(\mathcal{H})$. An abstract matrix norm characterization of operator spaces was given in [13]. The morphisms in the category of operator spaces are the completely bounded linear maps. Given operator spaces V and W , a linear map $\varphi: V \rightarrow W$ is completely bounded if the corresponding linear mappings $\varphi_n: M_n(V) \rightarrow M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, *i.e.*,

$$\|\varphi\|_{\text{cb}} = \sup\{\|\varphi_n\| : n \in \mathbf{N}\} < \infty.$$

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A map φ is completely contractive (respectively, completely isometric, completely quotient) if $\|\varphi\|_{cb} \leq 1$ (respectively, for each $n \in \mathbf{N}$, φ_n is an isometry, a quotient map). We denote by $CB(V, W)$ the space of all completely bounded maps from V into W . It is known that $CB(V, W)$ is an operator space with the operator space matrix norm given by identifying $M_n(CB(V, W)) = CB(V, M_n(W))$. In particular, if V is an operator space, then its dual space V^* is an operator space with operator space matrix norm given by the identification $M_n(V^*) = CB(V, M_n)$. Given operator spaces V and W and a completely bounded mapping $\varphi: V \rightarrow W$, the corresponding adjoint mapping $\varphi^*: W^* \rightarrow V^*$ is completely bounded with $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$. Furthermore, $\varphi: V \rightarrow W$ is a completely isometric injection if and only if φ^* is a completely quotient mapping. On the other hand, if $\varphi: V \rightarrow W$ is a surjection, then φ is a completely quotient mapping if and only if φ^* is a completely isometric injection. We use the notations $V \overset{\circ}{\otimes} W$ and $V \overset{\hat{\circ}}{\otimes} W$ for the injective, projective operator space tensor products (see [1, 2]). The operator space tensor products share many of the properties of the Banach space analogues. For example, we have the natural complete isometries

$$(V \overset{\hat{\circ}}{\otimes} W)^* = CB(V, W^*), \quad (V \overset{\circ}{\otimes} W)^* = CB(W, V^*),$$

and the completely isometric injection $V^* \overset{\circ}{\otimes} W \hookrightarrow CB(V, W)$. The tensor product $\overset{\circ}{\otimes}$ is injective in the sense that if $\varphi: W \rightarrow Y$ is a completely isometric injection, then so is $\text{id}_V \overset{\circ}{\otimes} \varphi: V \overset{\circ}{\otimes} W \rightarrow V \overset{\circ}{\otimes} Y$. On the other hand, the tensor product $\overset{\hat{\circ}}{\otimes}$ is projective in the sense that if $\varphi: W \rightarrow Y$ is a completely quotient mapping, then so is

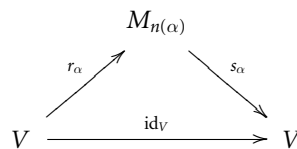
$$\text{id}_V \overset{\hat{\circ}}{\otimes} \varphi: V \overset{\hat{\circ}}{\otimes} W \rightarrow V \overset{\hat{\circ}}{\otimes} Y.$$

In the following, we give some definitions of local properties for an operator space V .

Exactness. If for any finite dimensional subspace L of V and every $\epsilon > 0$, there exist an integer n and a subspace $S \subseteq M_n$ such that $d_{cb}(L, S) < 1 + \epsilon$.

Local reflexivity. If for any finite dimensional operator space L , every complete contraction $\varphi: L \rightarrow V^{**}$ is the point-weak* limit of a net of complete contractions $\varphi_\alpha: L \rightarrow V$.

Nuclearity. If there exists a diagram of complete contractions



that approximately commute in the point-norm topology, we say V is nuclear. We say that a diagram of operator spaces and complete contractions

$$0 \rightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \rightarrow 0$$

is 1-exact if φ is a complete isometry, ψ is a completely quotient mapping, and $\ker \psi = \text{Im } \varphi$.

Effros and Ruan [5, Theorem 14.4.1] gave a characterization of exactness: an operator space V is exact if for any C^* -algebra \mathcal{A} and closed ideal $\mathcal{J} \subseteq \mathcal{A}$,

$$0 \rightarrow \mathcal{J} \check{\otimes} V \rightarrow \mathcal{A} \check{\otimes} V \rightarrow \mathcal{A}/\mathcal{J} \check{\otimes} V \rightarrow 0$$

is 1-exact. In this paper, we will use complete M -ideals to give a similar characterization of exactness. The notion of complete M -ideals was introduced in [4]. For an operator space V , if a linear map $P: V \rightarrow V$ satisfies $P^2 = P$ and

$$\|v\| = \max\{\|P_n(v)\|, \|(I - P)_n(v)\|\}$$

for all $v \in M_n(V)$, then P is called a complete M -projection. We say that a closed subspace $W \subseteq V$ is a complete M -summand if $W = PV$ for some complete M -projection, and that it is a complete M -ideal in V if the weak* closure W^- is a complete M -summand in V^{**} . It is clear that complete M -summands are M -summands, and similarly that complete M -ideals are M -ideals.

The second main result of this paper is that given a finite dimensional operator space L , then L is exact if and only if for any operator space W and any complete contraction $\varphi: L^* \rightarrow W^{**}$ is the point-weak* limit of a net of linear mappings $\varphi_\alpha: L^* \rightarrow W$ with $\|\varphi_\alpha\| \leq 1$. By virtue of this result, we can prove that an operator space V is exact if and only if $J(V, W^*) = (V \check{\otimes} W)^*$ for any separable operator space W .

2 Characterization of Exactness

Lemma 2.1 *Suppose that X, Y, Z are operator spaces, $X \subseteq Y$ and $\pi: Y \rightarrow Y/X$ is the canonical completely quotient mapping. If for any finite dimensional operator subspace F of Z , the mapping $\pi \otimes \text{id}_F: Y \check{\otimes} F \rightarrow Y/X \check{\otimes} F$ is a completely quotient mapping, then*

$$\ker(\pi \otimes \text{id}_Z: Y \check{\otimes} Z \rightarrow Y/X \check{\otimes} Z) = X \check{\otimes} Z.$$

Proof Suppose that $u \in Y \check{\otimes} Z$ satisfies $(\pi \check{\otimes} \text{id}_Z)(u) = 0$. Then given $\epsilon > 0$, we may choose an element

$$u_0 = \sum_{i=1}^n h_i \otimes v_i \in Y \otimes_\vee Z$$

such that $\|u - u_0\| < \epsilon$. It follows that $u_0 \in Y \check{\otimes} F$, where F is the finite dimensional subspace of Z spanned by v_1, \dots, v_n . Since the obvious mapping $Y/X \check{\otimes} F \rightarrow Y/X \check{\otimes} Z$ is isometric, we have

$$\begin{aligned} \|(\pi \otimes \text{id}_F)(u_0)\| &= \|(\pi \otimes \text{id}_Z)(u_0)\| \\ &\leq \|(\pi \otimes \text{id}_Z)(u_0) - (\pi \otimes \text{id}_Z)(u)\| + \|(\pi \otimes \text{id}_Z)(u)\| \\ &= \|(\pi \otimes \text{id}_Z)(u_0 - u)\| + 0 \\ &\leq \|u_0 - u\| < \epsilon. \end{aligned}$$

From the hypothesis, $\pi \otimes \text{id}_F: Y \check{\otimes} F \rightarrow Y/X \check{\otimes} F$ is a quotient mapping, and thus there is an element $u_1 \in Y \check{\otimes} F$ with $\|u_1\| < \epsilon$ and $(\pi \otimes \text{id}_F)(u_1) = (\pi \otimes \text{id}_F)(u_0)$. We have

$$\|u - (u_0 - u_1)\| \leq \|u - u_0\| + \|u_1\| < 2\epsilon,$$

where $u_0 - u_1 \in \ker \pi \otimes \text{id}_F = X \check{\otimes} F \subseteq X \check{\otimes} Z$ and thus $\text{dist}(u, X \check{\otimes} Z) < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $u \in X \check{\otimes} Z$. The converse inclusion is obvious. ■

Theorem 2.2 *Suppose that V is an operator space; then the following are equivalent.*

- (i) V is exact;
- (ii) for each finite dimensional subspace $L \subseteq V$ and for every operator space W with complete M -ideal $J \subseteq W$, the natural mapping $W \check{\otimes} L \rightarrow (W/J) \check{\otimes} L$ is a completely quotient mapping;
- (iii) for any operator space W with complete M -ideal $J \subseteq W$,

$$0 \rightarrow J \check{\otimes} V \xrightarrow{\iota \otimes \text{id}_V} W \check{\otimes} V \xrightarrow{\pi \otimes \text{id}_V} W/J \check{\otimes} V \rightarrow 0$$

is 1-exact.

Proof (i) \Rightarrow (ii). Since V is exact, so every finite dimensional operator subspace L of V is also exact. It follows from the condition C' for L (see [5, Theorem 14.4.1]) that

$$(W \check{\otimes} L)^{**} = W^{**} \check{\otimes} L \quad \text{and} \quad (W/J \check{\otimes} L)^{**} = (W/J)^{**} \check{\otimes} L.$$

Thus we have the following commutative diagram

$$\begin{array}{ccc} W \check{\otimes} L & \xrightarrow{\pi \otimes \text{id}_L} & W/J \check{\otimes} L \\ \downarrow & & \downarrow \\ W^{**} \check{\otimes} L & \xrightarrow{\pi^{**} \otimes \text{id}_L} & (W/J)^{**} \check{\otimes} L \\ \parallel & & \parallel \\ (W \check{\otimes} L)^{**} & \xrightarrow{(\pi \otimes \text{id}_L)^{**}} & (W/J \check{\otimes} L)^{**} \end{array}$$

Since J is a complete M -ideal in W , J^\perp is a complete M -summand in W^{**} , and we may assume that $J^\perp = PW^{**}$, where P is the complete M -projection determined by J^\perp . This gives the following complete isometries:

$$(W/J)^{**} = (J^\perp)^\perp = W^{**}/J^{\perp\perp} = W^{**}/J^\perp = (I - P)W^{**}.$$

So the completely quotient mapping $\pi^{**}: W^{**} \rightarrow (W/J)^{**}$ has a completely contractive lifting given by the canonical inclusion $(I - P)W^{**} \hookrightarrow W^{**}$. It follows from [5, Proposition 8.1.5] that $\pi^{**} \otimes \text{id}_L: W^{**} \check{\otimes} L \rightarrow (W/J)^{**} \check{\otimes} L$ is a completely quotient mapping. From the above commutative diagram, the bottom mapping

$(\pi \otimes \text{id}_L)^{**}$ is a completely quotient mapping. Since it is the second adjoint of the first row, the first row $\pi \otimes \text{id}_L$ is also a complete quotient mapping.

(ii) \Rightarrow (iii). From [5, Proposition 8.1.5], $\iota \otimes \text{id}_V$ is a complete isometry. It follows from Lemma 2.1 and the hypothesis (ii) that

$$\ker(\pi \otimes \text{id}_V : W \check{\otimes} V \rightarrow W/J \check{\otimes} V) = J \check{\otimes} V.$$

In the following, we first show that $\pi \otimes \text{id}_V$ is a quotient mapping. It is enough to prove that $\pi \otimes \text{id}_V$ maps $(W \otimes_V V)_{\|\cdot\| < 1}$ onto a dense subset of $(W/J \check{\otimes} V)_{\|\cdot\| < 1}$. Given an element \tilde{u} in the latter set, there exists a finite dimensional subspace $L \subseteq V$ with $\tilde{u} \in W/J \otimes_V L = W/J \check{\otimes} L$. From (ii) there exists an element $u \in W \check{\otimes} L$ with $\|u\| < 1$ and $(\pi \otimes \text{id}_L)(u) = \tilde{u}$, and we may regard u as an element of $W \check{\otimes} V$. We have the desired result.

If J is a complete M -ideal of W , certainly $M_n(J)$ is a complete M -ideal of $M_n(W)$. We have the commutative diagram

$$\begin{array}{ccc} M_n(W \check{\otimes} V) & \longrightarrow & M_n(W/J \check{\otimes} V) \\ \parallel & & \parallel \\ M_n(W) \check{\otimes} V & \longrightarrow & M_n(W)/M_n(J) \check{\otimes} V. \end{array}$$

Since the bottom row is a quotient mapping by the above proof, so is the top row. This shows that $\pi \otimes \text{id}_V$ is a completely quotient mapping.

(iii) \Rightarrow (i). It is well known that the M -ideals in a C^* -algebra are just the norm-closed two-sided algebraic ideals. Thus the M -ideals in C^* -algebras are automatically complete M -ideals. Therefore (iii) implies [5, Theorem 14.4.1(3)] and the exactness of V . ■

Corollary 2.3 *Suppose that V is an operator space. Then the following are equivalent.*

- (i) V is exact;
- (ii) for each finite dimensional subspace $L \subseteq V$ and for every unital operator algebra \mathcal{B} (self-adjoint or non-self-adjoint) with closed two-sided ideals \mathcal{J} having a contractive approximate identity, the natural mapping $\mathcal{B} \check{\otimes} L \rightarrow \mathcal{B}/\mathcal{J} \check{\otimes} L$ is a completely quotient mapping;
- (iii) for any unital operator algebra \mathcal{B} with closed two-sided ideals \mathcal{J} having a contractive approximate identity $0 \rightarrow \mathcal{J} \check{\otimes} V \rightarrow \mathcal{B} \check{\otimes} V \rightarrow \mathcal{B}/\mathcal{J} \check{\otimes} V \rightarrow 0$ is 1-exact.

Proof M -ideals in unital operator algebras coincide with the closed two-sided ideals having a contractive approximate identity (see [3]). Thus M -ideals in unital operator algebras are automatically complete M -ideals. So from Theorem 2.2, we have (i) \Rightarrow (ii). The arguments of (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are similar to those in Theorem 2.2. ■

The following characterization of nuclearity was discussed in Pisier [11], who attributed the result to Kirchberg and Valliant: an operator space V is nuclear if and

only if it has the following property. For any operator space W and operator subspace $X \subseteq W$, $0 \rightarrow X \check{\otimes} V \rightarrow W \check{\otimes} V \rightarrow (W/X) \check{\otimes} V \rightarrow 0$ is 1-exact.

Comparing Theorem 2.2 and the above result, we can see some difference and relation between exactness and nuclearity. The following result gives another characterization of exactness which is similar to the definition of local reflexivity.

Theorem 2.4 *Suppose that L is a finite dimensional operator space. Then L is exact if and only if, for any operator space W , every complete contraction $\varphi: L^* \rightarrow W^{**}$ is the point-weak* limit of a net of linear mapping $\varphi_\alpha: L^* \rightarrow W$ with $\|\varphi_\alpha\|_{cb} \leq 1$.*

Proof It follows from [5, Theorem 14.4.1, Corollary 14.2.3] that L is exact if and only if for any operator space W , we have completely isometric isomorphisms

$$(L \check{\otimes} W)^* = \mathcal{J}(L, W^*) = \mathcal{N}(L, W^*) = L^* \hat{\otimes} W^*.$$

Since $T_n(L^* \hat{\otimes} W^*) = L^* \hat{\otimes} M_n(W)^*$ and $T_n((L \check{\otimes} W)^*) = (L \check{\otimes} M_n(W))^*$,

$$\begin{aligned} L \text{ is exact} &\Leftrightarrow \text{for any operator space } W, (L \check{\otimes} W)^* = L^* \hat{\otimes} W^* \quad (\text{isometric}) \\ &\Leftrightarrow \text{for any operator space } W, (L \check{\otimes} W)^{**} = L \check{\otimes} W^{**} \quad (\text{isometric}), \end{aligned}$$

where the second equivalence follows from $(L^* \hat{\otimes} W^*)^* = L \check{\otimes} W^{**}$. This correspondence is explicitly given by the norm-increasing linear isomorphism

$$\tau: L \check{\otimes} W^{**} \rightarrow (L \check{\otimes} W)^{**}.$$

Thus the relation is isometric if and only if

$$\varphi \in (L \check{\otimes} W^{**})_{\|\cdot\| \leq 1} = \text{CB}(L^*, W^{**})_{\|\cdot\|_{cb} \leq 1}$$

implies that

$$\varphi \in (L \check{\otimes} W)^{**}_{\|\cdot\| \leq 1}.$$

From the bipolar theorem, the latter is the case if and only if φ is a weak* limit of elements in

$$(L \check{\otimes} W)_{\|\cdot\| \leq 1} = \text{CB}(L^*, W)_{\|\cdot\|_{cb} \leq 1}.$$

It follows from [10, Lemma 7.2] that $\tau: \text{CB}(L^*, W^{**}) \rightarrow (L \check{\otimes} W)^{**}$ is a homeomorphism in the point-weak* and weak* topologies. ■

From the above result and the definition of local reflexivity and exactness, it follows that all operator spaces are exact if and only if all operator spaces are locally reflexive. In other words, there exists a non-locally reflexive operator space if and only if there exists a non-exact operator space.

Lemma 2.5 *A finite dimensional operator space L is exact if and only if for any operator space W and any finite dimensional operator subspace $F \subseteq W^*$ and $\epsilon > 0$, we have that for every complete contraction $\varphi: L^* \rightarrow W^{**}$ there exists a mapping $\psi: L^* \rightarrow W$ such that $\|\psi\|_{cb} < 1 + \epsilon$ and $\langle \psi(x), f \rangle = \langle \varphi(x), f \rangle$ for all $x \in L^*$ and $f \in F$.*

Proof Since L is exact, we have $(L\check{\otimes}W)^* = L^*\hat{\otimes}W^*$ and $(L\check{\otimes}W)^{**} = L\check{\otimes}W^{**}$. Thus φ is a contractive element of $\text{CB}(L^*, W^{**}) = L\check{\otimes}W^{**} = (L\check{\otimes}W)^{**}$, and $E = L^* \otimes F$ as a finite dimensional subspace of $L^*\hat{\otimes}W^* = (L\check{\otimes}W)^*$. From Helly's Lemma, we can choose an element $\psi \in L\check{\otimes}W = \text{CB}(L^*, W)$ such that $\|\psi\|_{\text{cb}} < 1 + \epsilon$ and

$$\langle \psi(x), f \rangle = \langle \psi, x \otimes f \rangle = \langle \varphi, x \otimes f \rangle = \langle \varphi(x), f \rangle$$

for all $x \in L^*$ and $f \in F$.

To prove the converse, it is enough to consider a net of complete contraction of the form $\psi_{(F,\epsilon)} = \frac{\psi}{1+\epsilon}$, with ψ chosen as above. Then Theorem 2.4 implies that L is exact. ■

Theorem 2.6 *Given a finite dimensional operator space L , the following are equivalent.*

- (i) L is exact;
- (ii) for any separable operator space W , $(L\check{\otimes}W)^* = L^*\hat{\otimes}W^*$;
- (iii) for any separable operator space W , $(L\check{\otimes}W)^{**} = L\check{\otimes}W^{**}$.

Proof Clearly, (i) \Rightarrow (ii) and (ii) \Leftrightarrow (iii).

(ii) \Rightarrow (i). Owing to Lemma 2.5, it suffices to show that for any operator space W , if $\varphi: L^* \rightarrow W^{**}$ is a complete contraction and $F \subseteq W^*$ is finite dimensional, then for each $\epsilon > 0$ there exists a mapping $\psi_\epsilon: L^* \rightarrow W$ such that $\|\psi_\epsilon\|_{\text{cb}} < 1 + \epsilon$ and $\langle \psi_\epsilon(x), f \rangle = \langle \varphi(x), f \rangle$ for all $x \in L^*$ and $f \in F$.

From [5, Lemma 14.3.4] (a result of Ge and Hadwin), we may find a mapping $\psi^{(n)}: L^* \rightarrow W$ such that $\|(\psi^{(n)})_n\| < 1 + 1/n$ and $\langle \psi^{(n)}(x), f \rangle = \langle \varphi(x), f \rangle$ for all $x \in L^*$ and $f \in F$. The norm closed linear span W_0 of the union of the subspaces $\psi^{(n)}(L^*)$ with $n \in \mathbf{N}$ is separable in the norm topology, and we can regard $\psi^{(n)}$ as a sequence in $B(L^*, W_0^{**})$. Since the closed ball of radius 2 is compact in the point-weak* topology on $B(L^*, W_0^{**})$, we may choose a limit point $\psi: L^* \rightarrow W_0^{**}$ of the sequence $\psi^{(n)}$. If $r \leq n$, then $\|(\psi^{(n)})_r\| \leq \|(\psi^{(n)})_n\| \leq 1 + 1/n$, and thus $\|\psi_r\| \leq 1$. It follows that $\|\psi\|_{\text{cb}} \leq 1$. Furthermore, $\langle \psi(x), f \rangle = \langle \varphi(x), f \rangle$ for all $x \in L^*$ and $f \in F$.

By assumption and a similar argument to that of Lemma 2.5, for given $\epsilon > 0$ we may find a mapping $\psi_\epsilon: L^* \rightarrow W_0(\subseteq W)$ such that $\|\psi_\epsilon\|_{\text{cb}} < 1 + \epsilon$ such that

$$\langle \psi_\epsilon(x), f \rangle = \langle \psi(x), f \rangle = \langle \varphi(x), f \rangle$$

for any $x \in L^*$ and $f \in F$. Thus Lemma 2.5 implies that L is exact. ■

It is well known that for any finite dimensional operator space W ,

$$\mathcal{J}(V, W^*) = (V\check{\otimes}W)^*.$$

In the following result, we will consider the case when W is any separable operator space. The analogue on local reflexivity of the following result is that given an operator space V , V is locally reflexive if and only if every separable operator subspace of V is locally reflexive.

Corollary 2.7 Given an operator space V , the following are equivalent.

- (i) V is exact;
- (ii) for any separable operator space W , $\mathcal{J}(V, W^*) = (V \check{\otimes} W)^*$;
- (iii) for any separable operator space W , $V \check{\otimes} : W^{**} = V \check{\otimes} W^{**}$.

Proof Clearly (i) \Rightarrow (ii), (ii) \Rightarrow (iii): It follows from [5, Theorem 14.2.2]. (iii) \Rightarrow (i): For any separable operator space W and any finite dimensional subspace $L \subseteq V$, we have the following commutative diagram

$$\begin{array}{ccc} L \otimes_V W^{**} & \longrightarrow & (L \check{\otimes} W)^{**} \\ \downarrow & & \downarrow \\ V \otimes_V W^{**} & \longrightarrow & (V \check{\otimes} W)^{**}. \end{array}$$

Since the columns are automatically completely isometric and the bottom row is completely isometric by assumption, the top row is also completely isometric. Theorem 2.6 shows that L is exact and so is V . ■

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