BULL. AUSTRAL. MATH. SOC. 20D45, 20D10
VOL. 9 (1973), 363-366.

On groups admitting a noncyclic abelian automorphism group

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It is shown that a condition of Kurzweil concerning fixed-points of certain operators on a finite group G is sufficient to ensure that G is soluble. The result generalizes those of Martineau on elementary abelian fixed-point-free operator groups.

Suppose that V is an abelian group of operators on the finite group G. Kurzweil [3] studied soluble groups which satisfy the condition:

(*) for each prime divisor q of $|C_G(V)|$ and for each $v \in V^{\#}$ the q-elements of $C_G(V)$ centralize the q'-elements of $C_G(v)$.

The purpose of this note is to prove the following theorem.

THEOREM. *Suppose that V is a noncyclic elementary abelian r-group of operators on the finite r'-group G for some prime r. Assume that the condition (*) is satisfied. Then G is soluble. If $|V| = r^n$ then $G = F_n(G)$.*

Here $F_n(G)$ denotes the n-th term of the upper Fitting series of G. The other notation is standard and, in any case, agrees with that of [2].

It is hardly necessary to remark that the condition (*) is satisfied whenever $C_G(V) = 1$. Thus the theorem directly extends the theorems of Martineau ([4] and [5]) on groups with elementary abelian fixed-point-free

Received 21 June 1973.

operator groups.

Another special case of interest is that in which $C_G(v)$ is nilpotent for each $v \in V^{\#}$. In this case we conclude that G is soluble. In fact more can be said and the relevant results appear in [7] and [8] for the cases $m(V) = 2$ and $m(V) \geq 3$ respectively.

The theorem also has an application in the study of rank 3 signalizer functors on finite groups.

The proof of the theorem which is given below reduces the proof of solubility to the special case which was considered by Martineau. I am grateful to Mr P. Rowley for suggesting this reduction.

Proof. The first step is to show that a group which satisfies the hypothesis of the theorem is soluble. We use the notation of the statement of the theorem and proceed by induction on $|G|$. Clearly we may assume that G is characteristically simple.

By the corollary to the main theorem of [5] we may suppose that $C_G(V) \neq 1$. Choose in $C_G(V)$ an element x which has prime order - say x has order q. Let p denote any prime divisor of $|G|$ different from q and let P denote a V-invariant Sylow p-subgroup of G. (Such a subgroup P exists by Theorem 6.2.2 of [2]). Since V is noncyclic it follows that

$$P = \left\langle C_P(v) \mid v \in V^{\#} \right\rangle .$$

But by our hypothesis (*) we know that $C_P(v) \leq C_G(x)$. Hence P is contained in $C_G(x)$ so that the number of conjugates of x is prime to p. From our arbitrary choice of p, we deduce that the number of conjugates of x in G is a power of q.

It now follows from Burnside's Lemma ([2], Lemma 4.3.2) that G is not a nonabelian simple group. A slight extension of this lemma yields that G must be elementary abelian and hence soluble. This completes the proof of the solubility of G.

We now assume that $|V| = r^n$ and prove that $G = F_n(G)$. Again we

https://doi.org/10.1017/S0004972700043379 Published online by Cambridge University Press

proceed by induction on $|G|$. By Lemma 2, p. 482 of [6], $F(G)$ is the unique minimal V-invariant normal subgroup of G . Hence $F(G)$ is an elementary abelian p-group for some prime p and $F_2(G)/F(G)$ has order prime to p .

Now the above proof shows that if $x \in C_G(V)$ and x has prime order q , which is distinct from p , then $x \in C_G(F(G))$. By a well known property of soluble groups x is contained in $F(G)$. But this contradicts the choice of x . Hence $C_G(V)$ is a p-group.

The same argument now shows that $C_G(V)$ centralizes $F_2(G)/F(G)$. Hence $C_G(V) \leq F_2(G)$. Since $F(G)$ is the Sylow p-subgroup of $F_2(G)$ we may conclude that $C_G(V) \leq Z(F_2(G))$.

On the other hand, our characterization of $F(G)$ as the unique minimal normal V-invariant subgroup of G forces us to conclude that $Z(F_2(G)) = 1$. Hence $C_G(V) = 1$. We may now apply the main theorem of [1], to obtain the required conclusion.

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