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DOMINATION OF THE SUPREMUM OF A BOUNDED HARMONIC FUNCTION BY ITS SUPREMUM OVER A COUNTABLE SUBSET

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1. Introduction

For what sequences $\{a_n\}$ of points of the open unit disc D does there exist a constant κ such that

$$\sup_{z\in D}|f(z)|\leqq\kappa\sup_{n\in\mathbb{N}}|f(a_n)| \tag{1}$$

for all bounded harmonic functions f on D?

This question is of interest because these are the sequences such that every integrable function f on the unit circle ∂D is of the form

$$f=\sum_{n=1}^{\infty}\lambda_n p_{a_n} \tag{2}$$

with $\sum_{n=1}^{\infty}|\lambda_n|<\infty$ (see [1]). Here

$$p_a(\zeta)=(1-|a|^2)|1-\bar{a}\zeta|^{-2}\quad(\zeta\in\partial D,\ a\in D),$$

that is $p_a(e^{i\theta})$ is the Poisson kernel $P_a(\theta)$.

Brown, Shields and Zeller [2] have proved the closely related result that

$$\sup_{z\in D}|f(z)|=\sup_{n\in\mathbb{N}}|f(a_n)| \tag{3}$$

for all $f\in H^\infty$ (the space of bounded analytic functions on D) if and only if $\{a_n\}$ is *non-tangentially dense* for ∂D, that is if and only if almost every point of ∂D is the non-tangential limit of some subsequence of $\{a_n\}$. Our main result, Theorem 2, is a list of equivalent conditions on the sequence $\{a_n\}$ which includes conditions (1) and (3).

In Theorem 3, we establish an elementary property of the harmonic measure $\chi_F(z)$ of a Lebesgue measurable subset F of $\mathbb{R}$; namely, $\chi_F(z)$ is arbitrarily small outside the union of certain triangular domains associated with the points of F. This shows that if the inequality (1) holds for all *positive* bounded harmonic functions, then $\{a_n\}$ is non-tangentially dense.

Theorem 2 describes the sequences $\{a_n\}$ for which the bounded linear mapping T of l^1 into L^1 given by $T\{\lambda_n\}=\sum_{n=1}^{\infty}\lambda_n p_{a_n}$ is surjective. It is an immediate consequence that T is never bijective. When is it injective? This question remains unanswered, but Theorem 6 shows that T has zero kernel and closed range if and only if $\{a_n\}$ is an interpolating sequence for H^∞.

I am indebted to W. K. Hayman for asking a question that provoked this work and also for an observation showing that there are no sequences $\{a_n\}$ for which the infimum in Theorem 2(ii) is always attained.

2. Results

In the following elementary lemma, G denotes a simply connected domain in the complex plane, $H^\infty(G)$ the space of bounded analytic functions on G, and $BH(G)$ the space of bounded complex valued harmonic functions on G.

Lemma 1. *Let A be a subset of G, and let there exist a constant κ such that*

$$\sup_{z\in G}|f(z)|\leqq \kappa \sup_{z\in A}|f(z)| \tag{4}$$

for all invertible elements f of $H^\infty(G)$. Then

$$\sup_{z\in G}|f(z)|=\sup_{z\in A}|f(z)| \tag{5}$$

for all $f\in BH(G)$.

Proof. Let u be a non-negative real valued element of $BH(G)$. Since G is simply connected, there exists a function g analytic on G with $\operatorname{Re} g=u$. Let

$$f(z)=\exp g(z) \quad (z\in G).$$

Since $|f(z)|=\exp u(z)$, we have $f\in H^\infty(G)$, and plainly $1/f$ is also in $H^\infty(G)$. Therefore inequality (4) holds, that is

$$\sup_{z\in G}\exp u(z)\leqq \kappa \sup_{z\in A}\exp u(z).$$

Therefore

$$\sup_{z\in G}u(z)\leqq \log\kappa+\sup_{z\in A}u(z).$$

This inequality also holds with u replaced by αu with positive α, and so

$$\sup_{z\in G}u(z)\leqq \frac{1}{\alpha}\log\kappa+\sup_{z\in A}u(z).$$

Therefore, u satisfies (5). Next, if h is any real valued bounded harmonic function on G, then $M \pm h$ is non-negative for suitable positive M, and so h satisfies (5). Finally, given any complex valued $f \in BH(G)$, and $\theta \in \mathbb{R}$, let $h_\theta(z) = \operatorname{Re}(e^{i\theta} f(z))$. Then h_θ satisfies (5). We choose z_0 in G with $|f(z_0)|$ close to $\sup_{z \in G} |f(z)|$, and then choose θ so that $h_\theta(z_0) = |f(z_0)|$ to complete the proof.

In the following theorem, we write L^p for $L^p(\partial D, d\theta/2\pi)$, and H^∞ for $H^\infty(D)$.

Theorem 2. *Given a sequence $\{a_n\}$ of points of D, the following conditions are equivalent to each other.*

(i) *Every $f \in L^1$ is of the form* (2) *with $\sum_{n=1}^\infty |\lambda_n| < \infty$.*

(ii) *Condition* (i) *holds and also*

$$\|f\|_1 = \inf \sum_{n=1}^{\infty} |\lambda_n|,$$

with the infimum taken over all sequences $\{\lambda_n\}$ satisfying (2).

(iii) *There exists a constant κ such that the inequality* (1) *holds for all $f \in BH(D)$.*

(iv) *The equality* (3) *holds for all $f \in BH(D)$.*

(v) *The equality* (3) *holds for all $f \in H^\infty$.*

(vi) *Almost every point of ∂D is the non-tangential limit of some subsequence of $\{a_n\}$.*

Proof. The order of proof is (i)→(iii)→(iv)→(v)→(vi)→(ii)→(i).

(i)→(iii). Suppose that (i) holds, and, given $\lambda = \{\lambda_n\} \in l^1$, let

$$T\lambda = \sum_{n=1}^{\infty} \lambda_n p_{a_n}.$$

Since $\|p_a\|_1 = 1$, T is a bounded linear mapping of l^1 onto L^1. It is therefore an open mapping, and there exists $\kappa > 0$ such that the image of the ball in l^1 with centre 0 and radius κ contains the unit ball in L^1. Thus (2) holds for all $f \in L^1$, and

$$\inf\{\|\lambda\|_1 : (2) \text{ holds}\} \leqq \kappa \|f\|_1. \tag{6}$$

Now let $g \in L^\infty$ with $g(z)$ its harmonic extension to D, and let $\varepsilon > 0$. Since $\|g\|_\infty$ is the norm of the linear functional on L^1 given by g, there exists $f \in L^1$ with $\|f\|_1 = 1$ and $|\langle f, g \rangle| > \|g\|_\infty - \varepsilon$. By (6), $f = \sum_{n=1}^\infty \lambda_n p_{a_n}$ with $\lambda = \{\lambda_n\} \in l^1$ and $\|\lambda\|_1 < \kappa + \varepsilon$. Therefore

$$\begin{aligned}\sup_{z \in D} |g(z)| - \varepsilon = \|g\|_\infty - \varepsilon &< \sum_{n=1}^{\infty} |\lambda_n| \, |\langle p_{a_n}, g \rangle| \\ &= \sum_{n=1}^{\infty} |\lambda_n| \, |g(a_n)| \leqq \|\lambda\|_1 \sup_n |g(a_n)| \leqq (\kappa + \varepsilon) \sup_n |g(a_n)|.\end{aligned}$$

(iii)→(iv)→(v). $H^\infty \subset BH(D)$ and Lemma 1.

(v)→(vi). Brown, Shields and Zeller [2].

(vi)→(ii). (See [1]). (ii)→(i). Clear.

Remarks. The equality $\|f\|_1 = \sum_{n=1}^\infty |\lambda_n|$ obviously holds if $f = \sum_{n=1}^\infty \lambda_n p_{a_n}$ with $\lambda_n \geqq 0$ for all n. However there is no sequence $\{a_n\}$ such that this equality holds for all $f \in L^1$. For let $f \in L^1$ with zero essential infimum on ∂D and $\|f\|_1 > 0$. By taking real parts, we may assume that $f = \sum_{n=1}^\infty \lambda_n p_{a_n}$ with all λ_n real. If $\lambda_n \geqq 0$ for all n, then $\lambda_n p_{a_n} \leqq f$ and so $\lambda_n = 0$, for all n. We may therefore assume that $\lambda_1 < 0$. Then, since $f \geqq 0$ almost everywhere,

$$\|f\|_1 \leqq \|f + |\lambda_1| p_{a_1}\|_1 = \|\sum_{n=2}^\infty \lambda_n p_{a_n}\|_1 \leqq \sum_{n=2}^\infty |\lambda_n|.$$

Theorem 2 also holds with the disc replaced by the upper half-plane. In fact, the non-trivial step (v)→(vi) is easier to prove in that context and then transfer to D by conformal mapping. See Corollary 5 below.

Notation. Let $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, let $P_z(t)$ denote the Poisson kernel for U, that is

$$P_z(t) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2} \qquad (t \in \mathbb{R},\ z = x + yi \in U),$$

and let $|E|$ denote the Lebesgue measure of a measurable set E in $\mathbb{R}$. With $0 < \delta < 1$, $0 < b \leqq \infty$, $t \in \mathbb{R}$, and $\kappa = \tan(\pi\delta/2)$, let $\Delta(t, b, \delta)$ denote the triangular domain

$$\Delta(t, b, \delta) = \{x + yi : \kappa|x - t| < y < b\}.$$

As usual, the harmonic measure $\chi_F(z)$ of a measurable subset F of $\mathbb{R}$ is the harmonic extension to U of the characteristic function χ_F, that is

$$\chi_F(z) = \int_{-\infty}^{\infty} \chi_F(t) P_z(t)\, dt \qquad (z \in U).$$

Theorem 3. *Let F be a Lebesgue measurable subset of $\mathbb{R}$, let $0 < \delta < 1$, and let $\pi\delta b \geqq |F|$. Then $\chi_F(z) \leqq \delta$ for all z in $U \setminus \bigcup_{t \in F} \Delta(t, b, \delta)$.*

Proof. As before, we take $\kappa = \tan(\pi\delta/2)$. If $J = (-\infty, \beta]$ with β real, we have for $x > \beta$,

$$\chi_J(z) = \int_{-\infty}^{\beta} P_z(t)\, dt = \frac{1}{\pi} \int_0^{y/(x-\beta)} \frac{du}{1+u^2} = \frac{1}{\pi} \arctan \frac{y}{x-\beta}.$$

Thus

$$0<y\leqq\kappa(x-\beta)\Rightarrow\chi_J(z)\leqq\frac{1}{\pi}\arctan\kappa=\frac{\delta}{2}.$$

Similarly, if $J=[\alpha,\infty)$ with α real, then

$$0<y\leqq\kappa(\alpha-x)\Rightarrow\chi_J(z)\leqq\frac{\delta}{2}.$$

Suppose first that F is a closed subset of $\mathbb{R}$, so that $\mathbb{R}\backslash F$ is a countable (perhaps finite or void) union of disjoint open intervals I_k. Let $z=x+yi\in U\backslash\bigcup_{t\in F}\Delta(t,b,\delta)$ with $0<y<b$. Then $x\in I_k$ for some k. If $I_k=(-\infty,d)$ with d real, we take $J=[d,\infty)$. Since $d\in F$, $z\notin\Delta(d,b,\delta)$; and, since $y<b$, it follows that $y\leqq\kappa(d-x)$. Since $F\subset J$, we therefore have

$$\chi_F(z)\leqq\chi_J(z)\leqq\frac{\delta}{2}.$$

The same inequality holds if $I_k=(c,\infty)$. If $I_k=(c,d)$ with $-\infty<c<d<\infty$, we take $J=(-\infty,c]$, $J'=[d,\infty)$. Since $c\in F$, we have $\chi_J(z)\leqq\delta/2$; and similarly for $\chi_{J'}$. Then since $F\subset J\cup J'$,

$$\chi_F(z)\leqq\chi_J(z)+\chi_{J'}(z)\leqq\delta.$$

Finally, if $y\geqq b$, then $P_z(t)\leqq 1/\pi b$ for all real t, and so

$$\chi_F(z)\leqq|F|/\pi b\leqq\delta,$$

and the theorem is proved for closed sets F.

Finally, given any Lebesgue measurable subset F of $\mathbb{R}$, there exists an increasing sequence $\{F_n\}$ of closed subsets of F with its union differing from F by a set of measure zero. We have $\chi_{F_n}(z)\leqq\delta$ for all z in $U\backslash\bigcup_{t\in F}\Delta(t,b,\delta)$, and the result follows.

Remark. The possibility of a result like Theorem 3 is suggested by the proof in Brown, Shields and Zeller [2] to which we have referred already.

Corollary 4. *Let $0<\delta<1$ and let the sequence $\{a_n\}$ of points of U fail to satisfy the following condition: for almost all $t\in\mathbb{R}$, $\Delta(t,b,\delta)\cap\{a_n:n\in\mathbb{N}\}$ is non-empty for every $b>0$.*

Then there exists a positive harmonic function g on U with $\sup_{z\in U}g(z)=1$ but $\sup_{n\in\mathbb{N}}g(a_n)<\delta$.

Proof. Let $E=\bigcup_{b>0}A(b)$, with

$$A(b)=\{t\in\mathbb{R}:\Delta(t,b,\delta)\cap\{a_n:n\in\mathbb{N}\}=\emptyset\}.$$

Since $A(b) \supset A(b')$ when $b < b'$, we have $E = \bigcup_{k \in \mathbb{N}} A(1/k)$; and, since each $A(b)$ is closed, it follows that E is measurable. By assumption, we now have $|E| > 0$, and so we can choose $b = 1/k$ with $|A(b)| > 0$. We take a closed interval I chosen so that, with $F = I \cap A(b)$, we have $0 < |F| \leqq \delta \pi b$. Theorem 3 now provides the required function $g = \chi_F$.

Corollary 5. *Let $\{a_n\}$ be a sequence of points of U such that there exists a constant κ with*

$$\sup_{z \in U} g(z) \leqq \kappa \sup_{n \in \mathbb{N}} g(a_n) \tag{7}$$

for all bounded positive harmonic functions g on U. Then almost every point of $\mathbb{R}$ is the non-tangential limit of some subsequence of $\{a_n\}$.

Proof. Immediate consequence of Corollary 4.

Corollary 5 can be transferred to the disc by conformal mapping. It is of interest, because it is not obvious that the inequality (7) for bounded positive harmonic functions g implies the same inequality for all $g \in BH(U)$, though this implication is obvious if $\kappa = 1$.

Let $\{a_n\}$ be a sequence of points of U, and let T be the bounded linear mapping of l^1 into $L^1 = L^1(\mathbb{R})$ defined by

$$T\lambda = \sum_{n=1}^{\infty} \lambda_n P_{a_n} \qquad (\lambda = \{\lambda_n\} \in l^1).$$

Theorem 2, for U in place of D, tells us that T is surjective if and only if $\{a_n\}$ is non-tangentially dense for $\mathbb{R}$. It is an immediate consequence that T is never bijective, for if $\{a_n\}$ is non-tangentially dense, then so is $\{a_{n+1}\}$ and we have

$$P_{a_1} = \sum_{n=2}^{\infty} \lambda_n P_{a_n}$$

with $\sum_{n=2}^{\infty} |\lambda_n| < \infty$. In these circumstances, it is natural to ask for what sequences $\{a_n\}$ the mapping T is injective. We do not know the answer to this question, but using an argument due to J. B. Garnett, it is easy to prove the following result.

Theorem 6. *T has zero kernel and closed range if and only if $\{a_n\}$ satisfies the geometric condition for H^∞ interpolation, that is there exists $\delta > 0$ such that*

$$\inf_k \prod_{j, j \neq k} |a_k - a_j| / |a_k - \bar{a}_j| \geqq \delta.$$

Proof. Since $T \in BL(l^1, L^1)$, the usual identification of dual spaces gives $T^* \in BL(L^\infty, l^\infty)$, and, with $g(z)$ denoting the harmonic extension of $g \in L^\infty$ to U, we have $T^* g = \{g(a_n)\} \in l^\infty$. If $\{a_n\}$ is an interpolation sequence for H^∞, then $T^* L^\infty = l^\infty$, and, by

Banach's closed range theorem [3, p. 488], T has closed range and zero kernel. On the other hand, if T has closed range and zero kernel, then there exists a constant M with

$$\|\lambda\|_1 \leqq M \left\| \sum_{n=1}^{\infty} \lambda_n P_{a_n} \right\|_1 \qquad (\lambda = \{\lambda_n\} \in l^1).$$

This is the inequality (4.5) in Garnett [4, p. 303] from which it is there deduced that $\{a_n\}$ satisfies the geometric condition for H^∞ interpolation.

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