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The mapping class group action on $\mathsf{SU}(3)$-character varieties

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Dedicated to the memory of Todd A. Drumm

Abstract. Let Σ be a compact orientable surface of genus $g = 1$ with $n = 1$ boundary component. The mapping class group Γ of Σ acts on the $\mathsf{SU}(3)$-character variety of Σ. We show that the action is ergodic with respect to the natural symplectic measure on the character variety.

Key words: character variety, ergodicity, simple closed curves
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1. *Introduction*

Let $\Sigma = \Sigma_{g,n}$ be the compact oriented surface of genus g with boundary $\partial\Sigma$ which has $n \geq 1$ components denoted $\partial_1\Sigma, \ldots, \partial_n\Sigma$. Fix a base point x_0 on Σ and let $\pi = \pi_1(\Sigma, x_0)$ denote its fundamental group. Let $\mathsf{Homeo}^+(\Sigma, \partial\Sigma)$ be the group of orientation-preserving homeomorphisms of Σ which fixes $\partial\Sigma$ pointwise. Define the *mapping class group* of Σ:

$$\Gamma := \pi_0(\mathsf{Homeo}^+(\Sigma, \partial\Sigma)).$$

Alternatively, choose base points $x_i \in \partial_i\Sigma$ and paths from x_0 to x_i. Define $\pi_1(\partial_i\Sigma)$ as the cyclic subgroup of π corresponding to $\pi_1(\partial_i\Sigma, x_i)$ and determined by the paths between x_0 and x_i. Let $\mathsf{Aut}^+(\pi, \partial)$ denote the subgroup of $\mathsf{Aut}(\pi)$ which preserves both the conjugacy classes of the subgroups $\pi_1(\partial_i(\Sigma))$ and the orientation of Σ. Then Γ is isomorphic to the image of $\mathsf{Aut}^+(\pi, \partial)$ under the quotient homomorphism

$$\mathsf{Aut}(\pi) \longrightarrow \mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi).$$

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Let G be an algebraic group over $\mathbb{R}$. Then the set of homomorphisms $\pi \to G$ enjoys the structure of an $\mathbb{R}$-algebraic set denoted $\mathsf{Hom}(\pi, G)$. Choose conjugacy classes $\mathcal{C}_i \subset G$ for each $i = 1, \ldots, n$ and let $\mathcal{C} := \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Denote by $\mathsf{Hom}_{\mathcal{C}}(\pi, G)$ the subset of $\mathsf{Hom}(\pi, G)$ comprising homomorphisms which send $\pi_1(\partial_i \Sigma)$ to $\mathcal{C}_i$.

The group $\mathsf{Inn}(G)$ of inner automorphisms of G acts on $\mathsf{Hom}_{\mathcal{C}}(\pi, G)$ by composition. Denote the resulting *relative character variety* by

$$\mathcal{M}_{\mathcal{C}}(G) := \mathsf{Hom}_{\mathcal{C}}(\pi, G)/\mathsf{Inn}(G).$$

The group $\mathsf{Aut}^+(\pi, \partial)$ acts on π and hence on $\mathsf{Hom}_{\mathcal{C}}(\pi, G)$ by composition. Furthermore, the action descends to a Γ-action on $\mathcal{M}_{\mathcal{C}}(G)$. The moduli space $\mathcal{M}_{\mathcal{C}}(G)$ has an invariant dense open subset $\mathcal{M}^o_{\mathcal{C}}(G)$, which is a smooth manifold. If, for example, G is $\mathbb{R}$-reductive (see [**Gol84**, **Gol97**]), this subset has an Γ-invariant symplectic structure Ω. In particular, $\mathcal{M}^o_{\mathcal{C}}(G)$ admits a natural smooth Γ-invariant measure μ.

This paper is part of the general program to understand the dynamics of the action of Γ and automorphism groups of free groups on character varieties when the Lie group is compact. Suppose that K is a compact Lie group. In [**Gol97**], Goldman conjectured that Γ acts ergodically on $\mathcal{M}_{\mathcal{C}}(K)$ and showed this to be the case when K is locally a product of $\mathsf{SU}(2)$'s and $\mathsf{U}(1)$'s. In [**PX02**] and [**PX03**], the conjecture was proved for $g \geq 2$ and also established for almost all boundary classes when $g = 1 = n$. In [**GX11**], Goldman and Xia offered an alternative and simpler proof for the case of $K = \mathsf{SU}(2)$.

In this paper, we consider the case $g = 1 = n$ with $K = \mathsf{SU}(3)$. Then $\Gamma \cong \mathsf{SL}(2, \mathbb{Z})$ (see [**FM12**]). In this special case, $\mathcal{M}_{\mathcal{C}}(K)$ is explicitly described by the *commutator map*

$$\begin{aligned} K \times K &\xrightarrow{\kappa} K \\ (a, b) &\longmapsto [a, b] := aba^{-1}b^{-1}. \end{aligned}$$

Indeed, $\mathcal{M}_{\mathcal{C}}(K) \cong \kappa^{-1}(\mathcal{C})/\mathsf{Inn}(K)$ for each conjugacy class $\mathcal{C} \subset K$.

THEOREM 1.1. *Let $K = \mathsf{SU}(3)$ and $\Sigma = \Sigma_{1,1}$. The Γ-action is ergodic on $\mathcal{M}_{\mathcal{C}}(K)$ with respect to the measure μ.*

We prove this theorem along the lines of the main results in [**GX11**]. If $\rho \in \mathsf{Hom}_{\mathcal{C}}(\pi, G)$, we denote its $\mathsf{Inn}(G)$-equivalence class as $[\rho]$. Similarly, if $S \subseteq \mathsf{Hom}_{\mathcal{C}}(\pi, G)$, then the corresponding set of $\mathsf{Inn}(G)$-equivalence classes is denoted by $[S]$. A simple closed curve α on Σ defines a function

$$\begin{aligned} \mathcal{M}_{\mathcal{C}}(K) &\xrightarrow{\mathsf{t}_\alpha} \mathbb{C} \\ [\rho] &\longmapsto \mathsf{Tr}(\rho(\alpha)). \end{aligned}$$

The symplectic structure Ω together with the real and imaginary parts of t_α give rise to Hamiltonian flows. The ring-theoretical results in [**Law07**] imply that the algebra of Hamiltonian vector fields is *infinitesimally* transitive. It follows that the group generated by these flows is *locally* transitive and hence ergodic.

Depending on the choice of α, these Hamiltonian flows preserve the sets $[\mathsf{H}(a, b)]$ or $[\mathsf{H}'(a, b)]$ defined in §5.1. On the other hand, Γ contains the Dehn twist τ_α along α. The τ_α-action also preserves $[\mathsf{H}(a, b)]$ and is ergodic in almost all $[\mathsf{H}(a, b)]$. It follows

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that a μ-measurable function is invariant under τ_α if and only if it is invariant under the Hamiltonian flows associated with t_α. By local transitivity, such a measurable function is almost everywhere constant.

1.1. *Notation and terminology.* Let G be a group. Denote the inner automorphism induced by $A \in G$ by

$$G \xrightarrow{\mathsf{Inn}(A)} G$$
$$B \longmapsto ABA^{-1}.$$

The set of conjugacy classes in G equals the quotient $G/\mathsf{Inn}(G)$, and we denote the image of a subset $S \subseteq G$ under the quotient map by $[S]$. Denote the *centralizer* of A in G by

$$G_A := \mathsf{Fix}(\mathsf{Inn}(A)) < G,$$

where $\mathsf{Fix}(S)$ is the set of fixed points of $S \subseteq G$.

Define the commutator map of G:

$$G \times G \xrightarrow{\kappa} G$$
$$(A, B) \longmapsto [A, B] := ABA^{-1}B^{-1}.$$

Suppose further that G is a Lie group with Lie algebra $\mathfrak{g}$, and denote the adjoint representation of G on $\mathfrak{g}$ by Ad. Identify $\mathfrak{g}$ with the Lie algebra of *right-invariant vector fields* on G; then, for any right-invariant vector field $X \in \mathfrak{g}$, the element $\mathsf{Ad}(a)(X)$ equals the image of X under left multiplication by a. Denote the *centralizer* of a in $\mathfrak{g}$ by

$$\mathfrak{g}_a := \mathsf{Fix}(\mathsf{Ad}(a)) \subseteq \mathfrak{g}.$$

Denote the trace of a matrix a by $\mathsf{Tr}(a)$ and the λ-eigenspace of a matrix a by $\mathsf{Eig}_\lambda(a)$ for a scalar $\lambda \in \mathbb{C}$.

The notation K and $\mathfrak{k}$ is reserved for compact Lie groups and their Lie algebras, respectively.

If (M, Ω) is a symplectic manifold and $M \xrightarrow{f} \mathbb{R}$ is a smooth function, denote its *Hamiltonian vector field* by $\mathsf{Ham}(f)$. We denote the tangent space to a smooth manifold M at a point $p \in M$ by T_pM.

When we say a set is *closed*, we mean it to be closed in the classical topology.

2. *Character varieties and the mapping class group*

We fix a base point on the boundary of $\Sigma := \Sigma_{1,1}$. The fundamental group $\pi := \pi_1(\Sigma)$ is isomorphic to the rank 2 free group F_2 generated by homotopy classes of oriented based loops α and β. We often do not distinguish elements in π from corresponding oriented based loops on Σ.

We write

$$\pi = \langle \alpha, \beta, \sigma | \kappa(\alpha, \beta) = \sigma \rangle,$$

where σ is the boundary element. In this way, we have

$$R := \mathsf{Hom}(\pi, K) \cong K \times K \quad \text{and} \quad \mathcal{M} := R/K,$$

where the K-action is by conjugation.

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In our case, we have only one boundary circle and we let $\mathcal{C} \subseteq K$ be a conjugacy class and $c \in \mathcal{C}$. Then the relative representation variety and character variety are

$$R_c := \mathsf{Hom}_{\mathcal{C}}(\pi, K) := \kappa^{-1}(c) \quad \text{and} \quad \mathcal{M}_c := R_c(K)/K_c.$$

Again, the K_c-action is by conjugation. In this way, a representation $\rho \in R_c$ corresponds to $(a, b) \in K \times K$ such that $\kappa(a, b) = c$. Notice that $\mathcal{M}_c$ is usually and equivalently defined as

$$\mathcal{M}_c = \kappa^{-1}(\mathcal{C})/K.$$

The space $\mathcal{M}_c$ has a natural symplectic structure Ω [**Gol84**, **Gol97**].

The diffeomorphism group of Σ (fixing the boundary and hence also the base point) acts on π and this action descends to a Γ-action on π, fixing the conjugacy class of σ. This further induces an action

$$\begin{aligned} \mathcal{M}_c \times \Gamma &\longrightarrow \mathcal{M}_c \\ ([\rho], \gamma) &\longmapsto [\rho \circ \gamma]. \end{aligned}$$

The Γ-action leaves Ω and μ invariant [**Gol97**]. For any oriented simple closed curve α on Σ, denote by τ_α the Dehn twist along α. The mapping class group Γ contains all Dehn twists; indeed, the Dehn twists *generate* Γ (although we do not need this fact). Denote by $\mathcal{S}$ the set of homotopy classes of oriented simple closed curves on Σ.

3. *Compact Lie groups*

This section reviews well-known facts that are used in the proofs. *Generic elements* are introduced; these are regular elements which are dense in their maximal tori and provide non-trivial dynamics.

3.1. *Regularity.*

Suppose that M is an irreducible algebraic set over $\mathbb{R}$ or $\mathbb{C}$ and $M^s \subset M$ its singular locus. Then $U = M \setminus M^s$ is a smooth manifold and Zariski dense in M. The smooth structure on U gives rise to the Lebesgue measure class on U and on M, by assigning M^s to be a null set. We shall always mean this class, which coincides with the measure class discussed in the introduction [**Hue95**].

Let G be a linear semi-simple algebraic group over $\mathbb{C}$ of rank r and $K < G$ a maximal compact subgroup. The corresponding Lie algebras are denoted by $\mathfrak{k}$ and $\mathfrak{g}$, respectively.

Recall that an element $a \in K$ is **regular** if K_a has dimension r. In general, $\dim(K_a) \geq r$ and K_a contains a maximal torus of K. An element $a \in K$ is regular if and only if K_a is a maximal torus (that is, a Cartan subgroup) in K.

Recall that an action on a topological space is *minimal* if every orbit is dense. If $a \in K$, denote the Haar measure on K_a by μ_{K_a} and the pushforward $(L_b)_*\mu_a$ under left multiplication $K_a \xrightarrow{L_b} bK_a$ by μ_{ba}.

3.2. *Genericity.*

In general, regularity is too weak a notion for dynamical complexity. We introduce a notion of *genericity*, which is more useful for constructing non-trivial dynamics.

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Let $(a, b) \in K \times K$. Then the cyclic group $\langle a \rangle$ acts on the left coset bK_a by

$$b\zeta \overset{a^n}{\longmapsto} b\zeta a^n,$$

where $n \in \mathbb{Z}$ and $\zeta \in K_a$.

PROPOSITION 3.1. *Let $a \in K$ be a regular element. For any $b \in K$, the following conditions are equivalent:*

- *the cyclic group $\langle a \rangle < K_a$ is Zariski dense in K_a;*
- *the cyclic group $\langle a \rangle < K_a$ is dense in K_a;*
- *the action of $\langle a \rangle$ on bK_a is minimal;*
- *the action of $\langle a \rangle$ on (bK_a, μ_{ba}) is ergodic.*

In this case, we say that a is *generic*. The proof of Proposition 3.1 uses standard facts about compact abelian Lie groups, such as the following lemma.

LEMMA 3.2. *A cyclic subgroup of K_a is dense in the classical topology if and only if it is dense in the Zariski topology.*

Proof. Any set that is classically dense is also Zariski dense since the Zariski topology is coarser than the classical topology. We now show the converse. Clearly the cyclic group $\langle a \rangle \subset K_a$, and its Zariski closure is also an abelian subgroup. Its closure in the classical topology

$$H := \overline{\langle a \rangle}$$

is a closed abelian subgroup of K_a. Now every compact linear group is Zariski closed (see Onishchik and Vinberg [**OV90**, §4.4, Theorem 5, pp. 133–134]). Hence, H is Zariski closed in K_a. Since $\langle a \rangle$ is Zariski dense in K_a and $H \supseteq \langle a \rangle$, $H = K_a$. □

Proof of Proposition 3.1. The proof now follows from Lemma 3.2 and the fact that dense subgroups of the torus act minimally (see Katok and Hasselblatt [**KH95**, §1.4, p. 28]) and ergodically (see Katok and Hasselblatt [**KH95**, Proposition 4.2.2, p. 147] or Walters [**Wal82**, Theorem 1.9, p. 30]). □

4. *Infinitesimal transitivity and Hamiltonian flows*

In this section, we let G be a semi-simple complex algebraic Lie group. Let M be a symplectic manifold and $M \xrightarrow{f} \mathbb{R}$ a smooth function. Denote by $\mathsf{Ham}(f)$ the associated Hamiltonian vector field.

Definition 4.1. Let M be a manifold and $\mathcal{F}$ be a set of real smooth $\mathbb{R}$-functions on M such that at $x \in M$, the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T_x^*(M)$. Then $\mathcal{F}$ is said to be infinitesimally transitive at x. $\mathcal{F}$ is infinitesimally transitive on M if $\mathcal{F}$ is infinitesimally transitive at all $x \in M$.

PROPOSITION 4.2. *Let M be a connected symplectic manifold and $\mathcal{F}$ be infinitesimally transitive on M. Then the group $\mathcal{H}$ generated by the Hamiltonian flows $\mathsf{Ham}(f)$ of the vector fields $\mathsf{Ham}(f)$, for $f \in \mathcal{F}$, acts transitively on M.*

Proof. See [**GX11**, Lemma 3.2]. □

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We now briefly review results of Goldman [**Gol86**], describing the flows generated by the Hamiltonian vector fields by *simple* closed curves on Σ. In this case, the local flow of this vector field on $\mathcal{M}_c$ lifts to a flow on the representation variety R_c. Furthermore, this flow admits a simple description [**Gol86**], as follows.

4.1. *Invariant functions and flows on groups.* Let G be a semi-simple complex Lie group with Lie algebra $\mathfrak{g}$. Then the adjoint representation Ad preserves a non-degenerate symmetric bilinear form $\langle \_, \_\rangle$ on $\mathfrak{g}$. In the case $G = \mathsf{SL}(n, \mathbb{C})$, this is

$$\langle X, Y\rangle := \mathsf{Tr}(XY).$$

Let $G \xrightarrow{\mathsf{t}} \mathbb{R}$ be a function invariant under the inner automorphisms $\mathsf{Inn}(G)$. Following [**Gol86**], we describe how t determines a way to associate to every element $x \in G$ a one-parameter subgroup

$$\zeta^t(x) = \exp(tF(x))$$

centralizing x. Given t, define its *variation function* $G \xrightarrow{F} \mathfrak{g}$ by

$$\langle F(x), \upsilon\rangle = \frac{d}{dt}\bigg|_{t=0} \mathsf{t}(x \exp(t\upsilon))$$

for all $\upsilon \in \mathfrak{g}$. Invariance of t under $\mathsf{Inn}(G)$ implies that F is G-equivariant:

$$F(gxg^{-1}) = \mathsf{Ad}(g)F(x).$$

Taking $g = x$ implies that the one-parameter subgroup

$$\zeta^t(x) := \exp(tF(x)) \tag{1}$$

lies in the centralizer of $x \in G$.

Intrinsically, $F(x) \in \mathfrak{g}$ is dual (by $\langle \_, \_\rangle$) to the element of $\mathfrak{g}^*$ corresponding to the left-invariant 1-form on G extending the covector $df(x) \in T_x^*(G)$.

4.2. *Invariant functions and centralizing one-parameter subgroups.* Recall that $\mathcal{S}$ denotes the set of homotopy classes of oriented simple closed curves on Σ. If $\alpha \in \mathcal{S}$ is an oriented homotopy class of based loops, then t_α, the *trace function* of α, is defined as

$$\begin{aligned} \mathsf{Hom}(\pi, G) &\xrightarrow{\mathsf{t}_\alpha} \mathbb{C} \\ \rho &\longmapsto \mathsf{Tr}(\rho(\alpha)). \end{aligned}$$

Since the function $G \xrightarrow{\mathsf{Tr}} \mathbb{C}$ is $\mathsf{Inn}(G)$-invariant, t_α defines a $\mathbb{C}$-valued function (also denoted by t_α) on $\mathcal{M}_c$. Let

$$\mathsf{t}_\alpha^R = \mathsf{Re}(\mathsf{t}_\alpha), \quad \mathsf{t}_\alpha^I = \mathsf{Im}(\mathsf{t}_\alpha).$$

Then t_α^R and t_α^I define $\mathbb{R}$-valued functions on $\mathcal{M}_c$.

Let $\alpha \in \mathcal{S}$ and $\Sigma|\alpha$ denote the compact surface obtained by *splitting* Σ along α. The two components $\alpha_\pm$ of $\partial\Sigma|\alpha$ corresponding to α are the preimages of $\alpha \subset \Sigma$ under the quotient mapping $\Sigma|\alpha \longrightarrow \Sigma$. The original surface Σ may be reconstructed as a quotient space under the identification of α_- with α_+.

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The fundamental group π can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension:

$$\pi \cong (\pi_1(\Sigma|\alpha) \amalg \langle\beta\rangle)/(\beta\alpha_-\beta^{-1} = \alpha_+). \tag{2}$$

A representation ρ of π is determined by:

- the restriction ρ' of ρ to the subgroup $\pi_1(\Sigma|\alpha) \subset \pi$; and
- the value $\beta' = \rho(\beta)$

which satisfies

$$\beta'\rho'(\alpha_-)\beta'^{-1} = \rho'(\alpha_+). \tag{3}$$

Furthermore, any pair (ρ', β'), where ρ' is a representation of $\pi_1(\Sigma|\alpha)$ and $\beta' \in G$ satisfies (3), determines a representation ρ of π.

Let $\mathsf{t} = \mathsf{t}_\alpha^R$ or $\mathsf{t} = \mathsf{t}_\alpha^I$. Define the *twist flow* ξ_α^t associated with t on $\mathsf{Hom}(\pi, \mathsf{SU}(3))$:

$$\xi_\alpha^t(\rho) : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha), \\ \rho(\beta)\zeta^t(\rho(\alpha_-)) & \text{if } \gamma = \beta, \end{cases} \tag{4}$$

where ζ^t is defined in (1). This flow covers the flow generated by $\mathsf{Ham}(\mathsf{t})$ on $\mathcal{M}_c$ (see [**Gol86**]).

4.3. *Infinitesimal transitivity.* Let $\mathfrak{X}$ be the geometric invariant theory (GIT) quotient of $\mathsf{Hom}(\mathsf{F}_2, \mathsf{SL}(3, \mathbb{C}))$ by $\mathsf{Inn}(\mathsf{SL}(3, \mathbb{C}))$. Choose $\alpha, \beta \in \mathcal{S}$ as in §4.2 to correspond to curves with geometric intersection number 1 (or equivalently a free basis of F_2). Let

$$\mathcal{S}_{\alpha,\beta} := \{\alpha, \beta, \alpha\beta, \alpha\beta^{-1}, \alpha\beta\alpha^{-1}\beta^{-1}\} \subset \mathcal{S}$$

and $\mathcal{S}_{\alpha,\beta}^{-1} := \{\gamma^{-1} : \gamma \in \mathcal{S}_{\alpha,\beta}\}$. Let

$$\mathcal{F}_{\alpha,\beta} := \{\mathsf{t}_\gamma : \gamma \in \mathcal{S}_{\alpha,\beta} \cup \mathcal{S}_{\alpha,\beta}^{-1}\}.$$

THEOREM 4.3. (Lawton [**Law06**, **Law07**]) *The coordinate ring $\mathbb{C}[\mathfrak{X}]$ is generated by $\mathcal{F}_{\alpha,\beta}$.*

Since $\mathsf{Tr}(A^{-1}) = \overline{\mathsf{Tr}(A)}$ for $A \in \mathsf{SU}(3)$, the set $\mathcal{F}_{\alpha,\beta}$ is invariant under complex conjugation.

The relative $\mathsf{SU}(3)$-character variety $\mathcal{M}_c$ of F_2 embeds in $\mathfrak{X}$ as a real semi-algebraic subset. The regular functions on this $\mathbb{R}$-semi-algebraic set are the real and imaginary parts of the restrictions to $\mathcal{M}_c$ of the regular functions on $\mathfrak{X}$.

COROLLARY 4.4. *The set*

$$\mathcal{F}_{\alpha,\beta}^{\mathbb{R}} := \{\mathsf{t}_\gamma^R, \mathsf{t}_\gamma^I : \gamma \in \mathcal{S}_{\alpha,\beta}\}$$

is infinitesimally transitive on $\mathcal{M}_c$.

Proof. Given the remarks preceding this corollary, the result follows from Theorem 4.3 and [**GX11**, Lemma 3.1]. □

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5. *The Dehn twists*

Let $\alpha \in \mathcal{S}$ and $\tau_\alpha \in \Gamma$ be the corresponding Dehn twist. The fundamental group π can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension as in (2). Then τ_α induces the automorphism $(\tau_\alpha)_* \in \mathsf{Aut}(\pi)$ defined by

$$(\tau_\alpha)_* : \gamma \longmapsto \begin{cases} \gamma & \text{if } \gamma \in \pi_1(\Sigma|\alpha), \\ \gamma\alpha & \text{if } \gamma = \beta. \end{cases}$$

This further induces the map $(\tau_\alpha)^*$ on $\mathsf{Hom}(\pi, G)$ mapping ρ to

$$(\rho)(\tau_\alpha)^* : \gamma \longmapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha), \\ \rho(\gamma)\rho(\alpha) & \text{if } \gamma = \beta. \end{cases} \tag{5}$$

(See [**Gol86**]).

5.1. *Dehn twists and Hamiltonian twist flows.*

Let $a := \rho(\alpha)$ and $b := \rho(\beta)$. Then

$$\begin{aligned} R_c &= \{\rho \in \mathsf{Hom}(\pi, K) : \kappa(\rho(\alpha), \rho(\beta)) = c\} \\ &= \{(a, b) \in K \times K : \kappa(a, b) = c\}. \end{aligned}$$

Let

$$\mathsf{H}(a, b) := \{a\} \times bK_a, \quad \mathsf{H}'(a, b) := aK_b \times \{b\}.$$

PROPOSITION 5.1. *If* $(a, b) \in R_c$, *then* $\mathsf{H}(a, b), \mathsf{H}'(a, b) \subseteq R_c$.

Proof. Suppose that $t \in K_b$. Then $at = ta$ and $t^{-1}a^{-1} = a^{-1}t^{-1}$. Then

$$k(a, bt) = a(bt)a^{-1}(bt)^{-1} = abta^{-1}t^{-1}b^{-1} = aba^{-1}b^{-1} = k(a, b) = c.$$

Hence, $(a, bt) \in R_c$. The proof of $aK_b \times \{b\} \subseteq R_c$ is similar. □

PROPOSITION 5.2. *If* $(a, b) \in R_c$, *then the Hamiltonian flows of the vector fields* $\mathsf{Ham}(\mathsf{t}^R_\alpha)$ *and* $\mathsf{Ham}(\mathsf{t}^I_\alpha)$ *preserve* $[\mathsf{H}(a, b)]$. *Similarly,* $\mathsf{Ham}(\mathsf{t}^R_\beta)$ *and* $\mathsf{Ham}(\mathsf{t}^I_\beta)$ *preserve* $[\mathsf{H}'(a, b)]$.

Proof. This follows from (4) and by exchanging α and β. □

COROLLARY 5.3. *If a is generic, then* $\langle\tau_\alpha\rangle$ *acts ergodically on* $\mathsf{H}(a, b)$. *If b is generic, then* $\langle\tau_\beta\rangle$ *acts ergodically on* $\mathsf{H}'(a, b)$.

Proof. By (5), $\tau_\alpha(a, b) \in \mathsf{H}(a, b)$ and $\tau_\beta(a, b) \in \mathsf{H}'(a, b)$. The corollary then follows from Proposition 3.1. □

6. *The case of* $K = \mathsf{SU}(3)$

For the rest of this paper, we denote $\omega := e^{2\pi i/3}$ and the identity transformation by $\mathbb{I}$. In this section, we fix $K = \mathsf{SU}(3)$.

The classification of conjugacy classes of K can be described in terms of the *trace* function

$$K \xrightarrow{\mathsf{Tr}} \mathbb{C}.$$

Let $\Delta = \mathsf{Tr}(K)$ (see Figure 1).

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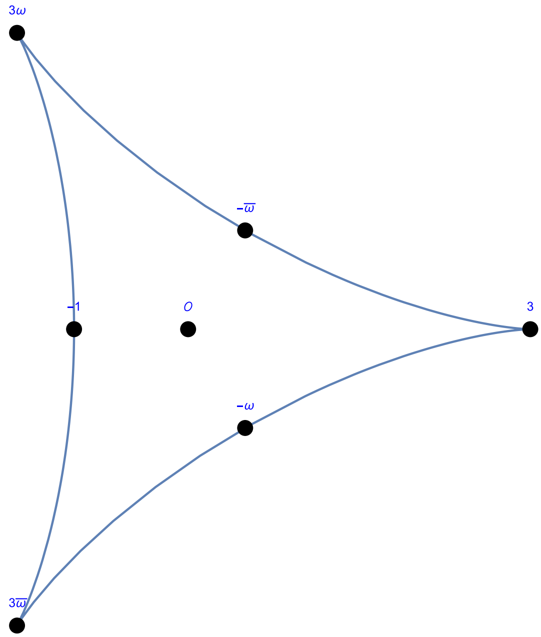

FIGURE 1. Δ, the traces in $\mathsf{SU}(3)$.

If $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ are the eigenvalues of $a \in K$, then they satisfy

$$|\zeta_1| = |\zeta_2| = |\zeta_3| = 1 \quad \text{and} \quad \zeta_1\zeta_2\zeta_3 = 1. \tag{6}$$

The coefficients of the character polynomial χ_a are

$$\begin{aligned} 1 &= 1, \\ \zeta_1 + \zeta_2 + \zeta_3 &= \mathsf{Tr}(a), \\ \zeta_2\zeta_3 + \zeta_3\zeta_1 + \zeta_1\zeta_2 &= \overline{\mathsf{Tr}(a)}, \\ \zeta_1\zeta_2\zeta_3 &= 1. \end{aligned}$$

Therefore, the characteristic polynomial is

$$\chi_A(\lambda) = \lambda^3 - z\,\lambda^2 + \bar{z}\,\lambda - 1,$$

where $z = \mathsf{Tr}(a) \in \mathbb{C}$. Furthermore, (6) is equivalent to the condition

$$|z|^4 - 8\mathsf{Re}(z^3) + 18|z|^2 - 27 \leq 0$$

and this real polynomial condition exactly describes the image $\Delta \subseteq \mathbb{C}$.

The traces of central elements are the vertices $3, 3\omega, 3\bar{\omega}$ of Δ. The trace of a regular element of order three is the center 0 of Δ. The traces of elements of order two are $-1, -\omega, -\bar{\omega}$, the mid-points of the edges of $\partial\Delta$.

PROPOSITION 6.1. *The map* Tr *is a local submersion at almost all points of* Δ.

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Proof. The map Tr is smooth. Hence, by Sard's theorem, almost all points of $\mathbb{C}$ are regular values of Tr [**GP10**, §1.7]. Hence, Tr is a local submersion at almost all points of Δ. □

Remark 6.2. It is not difficult to show that Tr has full rank in the interior of Δ, but we only need Proposition 6.1. For a general discussion of Δ and Weyl chambers, see [**DK00**].

PROPOSITION 6.3. *The image* $\mathsf{Tr}(N)$ *of the subset* $N \subseteq K$ *of generic elements is conull in* Δ.

Proof. Let

$$U = \left\{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha, \beta, \frac{\alpha}{\beta} \in \mathbb{R} \setminus \mathbb{Q}\right\}.$$

Let $\Delta' \subset \Delta$ be the image of U under the mapping

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{C} \\ (\alpha, \beta) &\longmapsto e^{2\pi i\alpha} + e^{2\pi i\beta} + e^{-2\pi i(\alpha+\beta)}. \end{aligned}$$

Then Δ' is conull in Δ and $\mathsf{Tr}^{-1}(\Delta') \subseteq N$. □

7. *Central fibers of* κ

Again in this section, we fix $K = \mathsf{SU}(3)$.

7.1. *The abelian representations.*

The fiber $R_{\mathbb{I}} = \kappa^{-1}(\mathbb{I})$ consists of commuting pairs (a, b). In this case, a and b lie in a maximal torus $\mathbb{T}^2$. Hence, $\mathcal{M}_{\mathbb{I}} \cong (\mathbb{T}^2 \times \mathbb{T}^2)/W$, where W is the Weyl group of K acting diagonally (see [**FL14**] for more discussion of these abelian character varieties).

7.2. *The non-abelian cases.*

PROPOSITION 7.1. *If* $k = \omega\mathbb{I}$, *where* $\omega \neq 1$, *then* $\mathcal{M}_k$ *consists of a single point. Specifically, if* $(a, b) \in R_k$, *then there exists* $g \in K$ *such that*

$$a = ga_0g^{-1}, \quad b = gb_0g^{-1},$$

where

$$a_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad b_0 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{7}$$

Proof. Suppose that $(a, b) \in R_k$, that is,

$$aba^{-1}b^{-1} = \omega\mathbb{I}. \tag{8}$$

We first prove the following lemma.

LEMMA 7.2. $a^3 = b^3 = \mathbb{I}$.

Proof of Lemma 7.2. By (8) and taking traces,

$$\mathsf{Tr}(a) = \mathsf{Tr}(\omega bab^{-1}) = \omega\mathsf{Tr}(a) \quad \text{and} \quad \mathsf{Tr}(a^{-1}) = \mathsf{Tr}(\omega b^{-1}a^{-1}b) = \omega\mathsf{Tr}(a^{-1}).$$

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Hence, $\omega \neq 1$ implies that $\mathsf{Tr}(a) = \mathsf{Tr}(a^{-1}) = 0$. Now apply the Cayley–Hamilton theorem:

$$a^3 - \mathbb{I} = a^3 - \mathsf{Tr}(a)a^2 + \mathsf{Tr}(a^{-1})a - \mathsf{Det}(a)\mathbb{I} = 0.$$

The same argument applied to $b = \omega a^{-1}ba$ implies that $b^3 = \mathbb{I}$, as claimed. □

Returning to the proof of Proposition 7.1, Lemma 7.2 implies that $a = \mathsf{Inn}(g)a_0$ for some $g \in K$, since $a \neq \mathbb{I}$.

We claim that $b = \mathsf{Inn}(g)b_0$. For convenience, assume that $a = a_0$; then we show that (8) implies that b is the permutation matrix b_0 defined in (7).

Recall that $\mathsf{Eig}_\lambda(a)$ is the λ-eigenspace of a. We claim that (8) implies that

$$b(\mathsf{Eig}_\lambda(a)) = \mathsf{Eig}_{\omega\lambda}(a). \tag{9}$$

To see this, rewrite (8) as

$$ab = \omega ba. \tag{10}$$

Suppose that $v \in \mathsf{Eig}_\lambda(a)$, that is,

$$av = \lambda v,$$

so, applying (10),

$$a(bv) = \omega bav = \omega\lambda(bv),$$

whence $bv \in \mathsf{Eig}_{\omega\lambda}(a)$, as claimed.

Since a is the diagonal matrix a_0 defined by (7), the lines $\mathsf{Eig}_1(a)$, $\mathsf{Eig}_\omega(a)$ and $\mathsf{Eig}_{\bar{\omega}}(a)$ are the three coordinate lines in $\mathbb{C}^3$. Thus, (9) implies that $b = b_0$, concluding the proof of Proposition 7.1. □

In particular, both a and b have order 3. Since $\kappa(a, b)$ has order 3 and is central, the subgroup $\langle a, b\rangle \subset K$ is a non-abelian group of order 27, a non-trivial central extension of $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ by $\mathbb{Z}/3$.

8. *Ergodicity*

Let K be any compact Lie group. Each tangent space $T_a K$ identifies with the Lie algebra $\mathfrak{k}$ of *right-invariant vector fields:* namely, a tangent vector $v \in T_a K$ identifies with the right-invariant vector $X \in \mathfrak{k}$ such that $X(a) = v$. In this way, the differential of the commutator map κ at $(a, b) \in K \times K$ identifies with the linear map (see [**Gol84**, **Gol20**, **PX02**]):

$$\begin{aligned} \mathsf{D}\kappa_{(a,b)} : \mathfrak{k} \oplus \mathfrak{k} &\longrightarrow \mathfrak{k} \\ (X, Y) &\longmapsto \mathsf{Ad}(ba)(\mathsf{Ad}(b^{-1}) - \mathbb{I})X \\ &\qquad + (\mathbb{I} - \mathsf{Ad}(a^{-1}))Y. \end{aligned}$$

From this formula, the following proposition holds.

PROPOSITION 8.1. *κ is submersive at (a, b) if and only if $\mathfrak{k}_a \cap \mathfrak{k}_b = 0$.*

For the rest of this section, let $K = \mathsf{SU}(3)$, which comes with the standard representation Π on $\mathbb{C}^3$. An element in $(a, b) \in R_c$ corresponds to a representation ρ of π (see §2). Hence, $\rho \circ \Pi$ is a representation of π. Denote by $\mathcal{M}_c^i \subseteq \mathcal{M}_c$ the subspace of irreducible representation classes.

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8.1. *Relation to the moduli space of K-bundles.* If we fix a complex structure on Σ, then we can construct the coarse moduli space $\mathcal{N}^{ss}$ of semi-stable parabolic K-bundles [**BR89**] on Σ, endowed with a complex structure. $\mathcal{N}^{ss}$ contains the subspace $\mathcal{N}^{s}$ of stable parabolic K-bundles.

PROPOSITION 8.2. *The set of smooth points of $\mathcal{M}_c$ is a connected manifold.*

Proof. There is a homeomorphism $\mathcal{M}_c \cong \mathcal{N}^{ss}$, restricting to a diffeomorphism $\mathcal{M}^i_c \cong \mathcal{N}^s$ [**BR89**, Theorem 1]. The moduli space $\mathcal{N}^{ss}$ is irreducible and contains $\mathcal{N}^s$ as an open subvariety [**BR89**, Theorem II]. Hence, $\mathcal{N}^s$ is open and connected. Hence, $\mathcal{M}^i_c$ is open and connected. Proposition 8.1 implies that a point $[\rho] = [(a, b)] \in \mathcal{M}_c$ is a smooth point if and only if ρ is irreducible, i.e. $\mathcal{M}^i_c$ is also the set of smooth points of $\mathcal{M}_c$. The proposition follows. □

For almost every conjugacy class c, the action $\mathcal{M}_c \times \Gamma \to \mathcal{M}_c$ is ergodic [**PX02**]. This section proves that this is true for *all* c.

Let $c \in K$. Up to conjugation,

$$c = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}.$$

For $(a, b) \in R_c$, we have the natural map

$$\iota : K_a \longrightarrow \mathsf{H}(a, b), \quad \iota(t) = (a, bt).$$

Let $P_2 : K \times K \longrightarrow K$ be the projection to the second factor. An element $\rho \in R_c$ corresponds to a pair $(a, b) \in K \times K$. Let

$$\mathsf{T} = \mathsf{Tr} \circ P_2 : R_c \longrightarrow \Delta \quad \text{and} \quad T := \iota \circ \mathsf{T} : K_a \longrightarrow \Delta.$$

PROPOSITION 8.3. *Let $c \notin Z(K)$. Then $(a, b) \in R_c$ exists such that:*
(1) *(a, b) is a smooth point;*
(2) *T is a submersion at (a, b).*

Proof. Let $(a, b) \in R_c$ be such that

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}. \tag{11}$$

Then

$$c = \kappa(a, b) = \begin{bmatrix} \frac{b_2}{b_3} & 0 & 0 \\ 0 & \frac{b_3}{b_1} & 0 \\ 0 & 0 & \frac{b_1}{b_2} \end{bmatrix}. \tag{12}$$

This formula for c implies that κ is onto K.

Remark 8.4. A result of Gotô [**Got49**] states that for any compact semi-simple Lie group K, $K \times K \xrightarrow{\kappa} K$ is surjective. Hence, $R_c \neq \emptyset$ for all $c \in K$.

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Note that a is regular. For (1), there are three cases for $b \in Z(K)$, $b_1 = b_2 \neq b_3$ (and its permutation variation) and b being regular.

If $b \in Z(K)$, then $\kappa(a, b) = \mathbb{I} = c$ and this violates our hypothesis of $c \notin Z(K)$.

If b is regular, then $t \in \mathfrak{k}_b$ implies that

$$t = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}, \quad t_1 + t_2 + t_3 = 0. \tag{13}$$

Then

$$(\mathsf{Ad}(a) - \mathbb{I})t = \begin{bmatrix} t_2 - t_1 & 0 & 0 \\ 0 & t_3 - t_2 & 0 \\ 0 & 0 & t_1 - t_3 \end{bmatrix}.$$

Hence, if $t \in \mathfrak{k}_a$, then $(\mathsf{Ad}(a) - \mathbb{I})t = 0$. This implies that $\mathfrak{k}_a \cap \mathfrak{k}_b = 0$. Hence, by Proposition 8.1, we conclude that κ is regular at (a, b).

If $b_1 = b_2 \neq b_3$ and $t \in \mathfrak{k}_b$, then

$$t = \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{bmatrix}, \quad t_1 + t_2 + t_3 = 0.$$

Then

$$(\mathsf{Ad}(a) - \mathbb{I})t = \begin{bmatrix} t_{22} - t_{11} & -t_{12} & t_{21} \\ -t_{21} & t_{33} - t_{22} & 0 \\ t_{12} & 0 & t_{11} - t_{33} \end{bmatrix}.$$

Hence, if $t \in \mathfrak{k}_a$, then $(\mathsf{Ad}(a) - \mathbb{I})t = 0$. This implies that $t = 0$. Hence, $\mathfrak{k}_a \cap \mathfrak{k}_b = 0$. By Proposition 8.1, κ is regular at (a, b). We conclude that in all cases $(a, b) \in R_c$ is a smooth point.

Notice that $\mathsf{H}(a, b) \subseteq R_c$. We consider T restricted to $\mathsf{H}(a, b)$. Let

$$p = \begin{bmatrix} 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}.$$

Then $a = pqp^{-1}$. The element $t \in K_a$ has the form $t = pt_b p^{-1}$, where $t_b \in K_b$ is diagonal. Then

$$T(t) = \mathsf{Tr}(bt) = \tfrac{1}{3}\mathsf{Tr}(b)\mathsf{Tr}(t).$$

By Proposition 6.1, T is a local submersion for almost all $t \in \Delta$ unless $\mathsf{Tr}(b) = 0$. However, $\mathsf{Tr}(b) = 0$ implies that $c \in Z(K)$, which is a contradiction. Hence, $\mathsf{Tr}(b) \neq 0$ and T is a local submersion for almost all t. □

COROLLARY 8.5. *T is a local submersion for almost all points in $\mathsf{H}(a, b)$.*

Proof. Since $\mathsf{D}T = \mathsf{DT} \circ \mathsf{D}\iota$, $\mathsf{D}T_t$ being surjective implies that $\mathsf{DT}_{(a,bt)}$ is surjective. □

COROLLARY 8.6. *There is a conull set $V \subset R_c$ such that b is generic for almost all $(a, b) \in V$.*

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Proof. The subset containing points at which a map is locally submersive is Zariski open. Hence, by Proposition 8.3, there exists a smooth and Zariski open $V \subseteq R_c$ such that $T|_V$ is a submersion. Let $Q \subseteq R_c$ be the smooth part. By Proposition 8.2, Q is connected and hence irreducible. Since a Zariski open subset of a smooth irreducible variety is conull in the Lebesgue class, V is conull. The map $T|_V$ is a fibration over an open domain of Δ. The corollary follows from Proposition 6.3. □

COROLLARY 8.7. *Suppose that $c \notin Z(K)$. Suppose that $\beta \in \mathcal{S}$ and $\phi : \mathcal{M}_c \longrightarrow \mathbb{R}$ is a μ-measurable function. If ϕ is τ_β-invariant, then ϕ is $\mathsf{Ham}(\mathsf{t}^R_\beta)$-invariant and $\mathsf{Ham}(\mathsf{t}^I_\beta)$-invariant.*

Proof. By Proposition 8.3, R_c contains a smooth point (a, b) with b generic. By Proposition 8.2 and Corollary 8.6, $b \in K$ is generic for almost all $(a, b) \in R_c$. Hence, $b \in K$ is generic for almost all $[(a, b)] \in \mathcal{M}_c$. By Proposition 3.1, 5.1 and Corollary 5.3, τ_β-orbit is dense in $\mathsf{H}'(a, b)$ and ϕ is $\mathsf{Ham}(\mathsf{t}^R_\beta)$-invariant and $\mathsf{Ham}(\mathsf{t}^I_\beta)$-invariant. □

With the notation we have adopted, we restate Theorem 1.1 as follows.

THEOREM 8.8. *The action $\mathcal{M}_c \times \Gamma \to \mathcal{M}_c$ is ergodic.*

Proof. Suppose that $c = \mathbb{I}$. Identifying $\mathbb{R}^2$ with its dual $(\mathbb{R}^2)^*$, the group $\mathsf{SL}(2, \mathbb{Z})$ has the standard dual linear action on $\mathbb{R}^2$ which induces the diagonal action on $\mathbb{T}^2 \times \mathbb{T}^2$. This $\mathsf{SL}(2, \mathbb{Z})$-action is known to be ergodic because $\mathsf{SL}(2, \mathbb{Z})$ contains hyperbolic elements, meaning that the eigenvalues of these elements do not have absolute value 1 [**BS15**, §4].

There is an isomorphism [**FM12**] $\iota : \Gamma \xrightarrow{\cong} \mathsf{SL}(2, \mathbb{Z})$. By §7.1, $\mathcal{M}_c \cong (\mathbb{T}^2 \times \mathbb{T}^2)/W$. The Γ-action on $\mathcal{M}_c$ lifts to an action on $\mathbb{T}^2 \times \mathbb{T}^2$. Moreover, this Γ-action is equivariant with respect to ι. Hence, the Γ-action on $\mathcal{M}_c$ is ergodic.

Suppose that $c \in Z(K)$ and $c \neq \mathbb{I}$; then $\mathcal{M}_c$ is a single point by Proposition 7.1 and the statement is trivially true.

Suppose that $c \notin Z(K)$. Recall that $\mathcal{H}$ is the group generated by all Hamiltonian flows $\mathsf{Ham}(\mathsf{t}^R_\beta)$ and $\mathsf{Ham}(\mathsf{t}^I_\beta)$, where $\beta \in \mathcal{S}$. Let $\phi : \mathcal{M}_c \longrightarrow \mathbb{R}$ be a Γ-invariant μ-measurable function. By Corollary 8.7, ϕ is $\mathcal{H}$-invariant. By Corollary 4.4 and Proposition 4.2, ϕ is constant almost everywhere. Our theorem follows. □

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REFERENCES

[BR89] U. Bhosle and A. Ramanathan. Moduli of parabolic G-bundles on curves. *Math. Z.* **202**(2) (1989), 161–180.

[BS15] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, Cambridge, 2015.

[DK00] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups (Universitext)*. Springer, Berlin, 2000.

[FL14] C. Florentino and S. Lawton. Topology of character varieties of Abelian groups. *Topology Appl.* **173** (2014), 32–58.

[FM12] B. Farb and D. Margalit. *A Primer on Mapping Class Groups (Princeton Mathematical Series, 49)*. Princeton University Press, Princeton, NJ, 2012.

[Gol20] W. Goldman. Parallelism on Lie groups and Fox's free differential calculus. *Characters in Low-Dimensional Topology (Centre de Recherches Mathématiques Proc.) (Contemporary Mathematics, 760)*. Eds. O. Collin, S. Friedl, C. Gordon, S. Tillmann and L. Watson. American Mathematical Society, Providence, RI, 2020, to appear.

[Gol84] W. M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. Math.* **54**(2) (1984), 200–225.

[Gol86] W. M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Invent. Math.* **85**(2) (1986), 263–302.

[Gol97] W. M. Goldman. Ergodic theory on moduli spaces. *Ann. of Math. (2)* **146**(3) (1997), 475–507.

[Got49] M. Gotô. A theorem on compact semi-simple groups. *J. Math. Soc. Japan* **1** (1949), 270–272.

[GP10] V. Guillemin and A. Pollack. *Differential Topology*. AMS Chelsea, Providence, RI, 2010.

[GX11] W. M. Goldman and E. Z. Xia. Ergodicity of mapping class group actions on SU(2)-character varieties. *Geometry, Rigidity, and Group Actions (Chicago Lectures in Mathematics)*. University of Chicago Press, Chicago, IL, 2011, pp. 591–608.

[Hue95] J. Huebschmann. Symplectic and Poisson structures of certain moduli spaces. I. *Duke Math. J.* **80**(3) (1995), 737–756.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[Law06] S. Lawton. SL(3, C)-character varieties and RP2-structures on a trinion. *PhD Thesis*, University of Maryland, College Park, ProQuest LLC, Ann Arbor, MI, 2006.

[Law07] S. Lawton. Generators, relations and symmetries in pairs of 3×3 unimodular matrices. *J. Algebra* **313**(2) (2007), 782–801.

[OV90] A. L. Onishchik and È. B. Vinberg. *Lie Groups and Algebraic Groups (Springer Series in Soviet Mathematics)*. Springer, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[PX02] D. Pickrell and E. Z. Xia. Ergodicity of mapping class group actions on representation varieties. I. Closed surfaces. *Comment. Math. Helv.* **77**(2) (2002), 339–362.

[PX03] D. Pickrell and E. Z. Xia. Ergodicity of mapping class group actions on representation varieties. II. Surfaces with boundary. *Transform. Groups* **8**(4) (2003), 397–402.

[Wal82] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, 1982.

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