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Dynamics of exp (z)

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Abstract. We describe the dynamical behaviour of the entire transcendental function exp (z). We use symbolic dynamics to describe the complicated orbit structure of this map whose Julia Set is the entire complex plane. Bifurcations occurring in the family c exp (z) are discussed in the final section.

0. *Introduction*

The study of the dynamics of complex analytic maps has a long history. The iteration of these maps was studied extensively in the early part of the twentieth century, notably by Fatou [**3**] and Julia [**4**]. See [**1**] for a good summary of their work. Recently, there has been renewed interest in this subject. Work of Mañé, Sad, and Sullivan [**6**], [**8**], [**9**], Mandelbrot [**5**], and Douady and Hubbard [**2**] has shed new light on the richness of the dynamics of complex analytic maps.

Most of the recent work has centred on rational maps of the complex plane and the structure of the Julia set, $J(f)$, of such a map. Recall that the Julia set of a map is the set of points at which the family of iterates of a given map fails to be a normal family. Equivalently, the Julia set is the closure of the set of expanding periodic orbits of f. The Julia set is often a rather complicated set and the dynamics of the map restricted to $J(f)$ is extremely rich.

Our goal in this paper is to examine the dynamics of a specific complex analytic map that is not rational, namely, the complex exponential map $z \to \exp(z)$. The Julia set for this map has been studied by Misiurewicz [**7**], who showed that $J(f) = \mathbb{C}$. It follows that periodic points of exp (z) are dense in $\mathbb{C}$ and that the map is topologically transitive. Our aim is to describe further the dynamics of this map.

Our main result is a topological classification of the orbits of exp (z). Using symbolic dynamics, we assign an infinite sequence of integers to each orbit. Roughly speaking, this sequence gives the itinerary of the orbit – which of the various fundamental domains successive points on the orbit fall into. Not all sequences are possible; sequences which grow too fast must be excluded. In §§ 1 and 2 we give necessary and sufficient conditions for the existence of an orbit with prescribed itinerary. This then gives an interesting semi-conjugacy between exp (z) and the well known shift automorphism.

Our methods show that there is a unique periodic point corresponding to each repeating sequence of non-zero integers. Moreover, each such periodic point (with two exceptions) comes equipped with 'strings' attached. These are continuous curves in the plane which connect the periodic point to ∞. Each string is preserved by the map and consists of all points which share the same itinerary as the periodic point.

The two exceptions are the fixed points which lie in the primary fundamental domains containing the unit circle. Here the dynamics are quite different from other fundamental domains. In § 5 we show that the strings associated to these fixed points contain a Cantor set of curves.

We conjecture that all other orbits come equipped with similar strings (or Cantor sets of strings). In § 3 we show that the set of points which share the same itinerary and which proceed monotonically to the right to ∞ lie on a continuous curve or 'tail'. We can in fact show that this curve is Lipschitz.

In the final section, we discuss the dynamics of $\lambda \exp(z)$ for λ real. Much of what we say goes over verbatim to this case. However, a spectacular bifurcation occurs as λ decreases through $1/e$. The Julia set changes at this parameter value into a Cantor set of curves. All periodic points discussed above remain, but there is also an attractive fixed point or sink whose basin of attraction is open and dense in the plane.

1. *Dynamics of* $\exp(z)$.

Let $f(z) = \exp(z)$. Our goal in this paper is to describe the behaviour of points under iteration of f. The orbit of z is the set of points $\{f^n(z) | n \geq 0\}$ where $f^n = f \circ \cdots \circ f$ is the n'th iterate of f. The point z is *periodic* if there is $n > 0$ such that $f^n(z) = z$. The least such n is the period of z. If z is periodic with period n, then $f^{in}(z) = z$ for all $i \in \mathbb{Z}^+$. Periodic points play a central role in any dynamical system. One of our aims is to provide a complete description of the periodic point structure for $\exp(z)$.

A periodic point p of period n is a *sink* if $|(f^n)'(p)| < 1$. It is well known that if p is a sink of period n, then there exists a neighbourhood U of p such that $f^n(U) \subset U$. Moreover, if $z \in U$, then $f^{in}(z) \to p$ as $i \to \infty$. A periodic point p is a *source* if $|(f^n)'(p)| > 1$; p is *indifferent* if $|(f^n)'(p)| = 1$.

For a complex analytic map, most of the interesting dynamics occur on the *Julia set*, $J(f)$. $J(f)$ consists of all points at which the family of iterates of f fails to be a normal family of functions. More precisely, $z \in J(f)$ iff there exists no neighbourhood U of z such that the maps $f^n | U$ form a normal family of functions. The following characterization of the Julia set was proved by Fatou [**3**].

THEOREM. *Let g be an entire transcendental function.*

(1) *$J(g)$ is a closed, non-empty perfect set.*

(2) *Periodic points are dense in $J(g)$.*

(3) *$J(g)$ is both forward and backward invariant under g.*

It is known that periodic sinks are never in the Julia set, while periodic sources always are. Indifferent points may or may not be in the Julia set. We refer to [**1**] for a complete discussion of these ideas for a rational map.

The following theorem, originally conjectured by Fatou, was proved by Misiurewicz [7] in 1981.

THEOREM. $J(\exp(z)) = \mathbb{C}$.

Thus the dynamics of $\exp(z)$ are quite complicated. Periodic points are dense in $\mathbb{C}$ and one can also show that there is a dense orbit (see also [7]). Our aim is to shed more light on this complicated orbit structure.

We first recall some elementary facts about $f(z) = \exp(z)$. $f(z)$ is $2\pi i$-periodic, and if $z \in \mathbb{R}$, then so is $f^n(z)$ for all $n > 0$. Hence the real line is invariant under f. Moreover, if $\operatorname{Im} z = 2k\pi$ for $k \in \mathbb{Z}$ then $f(z) \in \mathbb{R}^+$. If $\operatorname{Im}(z) = (2k+1)\pi$, then $f(z) \in \mathbb{R}^-$. We therefore divide the plane into infinitely many fundamental domains bounded by the lines $\operatorname{Im} z = 2k\pi$. If $k \geq 1$, we define the strip $R(k)$ by

$$R(k) = \{z \in \mathbb{C} \mid 2(k-1)\pi < \operatorname{Im} z < 2k\pi\}.$$

If $k \leq -1$, let

$$R(k) = \{z \in \mathbb{C} \mid 2k\pi < \operatorname{Im} z < 2(k+1)\pi\}.$$

Note that f maps each $R(k)$ diffeomorphically onto $\mathbb{C} - (\mathbb{R}^+ \cup 0)$. Finally, let $R(0)$ be the real axis. We also let $R(0^+) = \mathbb{R}^+$ and $R(0^-) = \mathbb{R}^-$.

Our aim for the remainder of this section is to describe the set of possible orbits for $\exp(z)$. Toward that end we introduce symbolic dynamics into the problem as follows. Let A denote the set of non-zero integers augmented by the symbols 0^+ and 0^-. To each $z \in \mathbb{C}$ we assign an infinite sequence $\underline{s}$ of elements of A, $\underline{s} = s_0 s_1 s_2 \ldots$, as follows. Suppose first that $\operatorname{Im} f^j(z) \neq 2k\pi$ for any $j, k \in \mathbb{Z}$. If $f^j(z) \in R(k)$ we then set $s_j = k$. If $f^j(z) \in \mathbb{R}^\pm$, we set $s_j = 0^\pm$. Thus the sequence associated to z is determined by listing the successive fundamental domains in which successive iterates land. The only possible confusion arises when $\operatorname{Im} f^j(z) = 2k\pi$, $k \neq 0$. In this case we set $s_j = k$.

We denote the sequence associated to z by $S(z)$ and its j'th entry by $s_j(z)$. $S(z)$ is called the *itinerary* of z. The proof of the following proposition is straightforward.

PROPOSITION 1.1. *Let* $S(z) = s_0 s_1 s_2 \ldots$.

(1) *If* $s_j = 0^-$, *then* $s_{j+1} = 0^+$.

(2) *If* $s_j = 0^+$, *then* $s_{j+k} = 0^+$ *for all* $k \in \mathbb{Z}^+$.

(3) $S(f(z)) = s_1 s_2 s_3 \ldots$.

(3) of this proposition relates the dynamics of $\exp(z)$ to the well known shift automorphism of symbolic dynamics. This map is defined by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

Thus (3) above gives $S(f(z)) = \sigma(S(z))$.

Not all sequences of elements of A actually correspond to orbits of f. For example, by the above proposition, 0^+ must always follow 0^+. There is another less obvious necessary condition.

Definition. A sequence $s_0 s_1 s_2 \ldots$ is of *exponential order* if there exists a real number x such that $2\pi |s_j| \leq f^j(x)$ for all j.

Remark. Repeating sequences and bounded sequences are clearly of exponential order. However, not all sequences are. For example, it is easy to check that the sequence

$$3, 3^3, 3^{3^3} \ldots$$

is not of exponential order.

Let Σ denote the set of all sequences $s_0 s_1 s_2 \ldots$ with $s_j \in A$ and which satisfy:

(1) if $s_j = 0^-$, then $s_{j+1} = 0^+$;

(2) if $s_j = 0^+$, then $s_{j+1} = 0^+$;

(3) $s_0 s_1 s_2 \ldots$ *is of exponential order.*

A sequence of Σ is called *allowable.* The following proposition shows that these three conditions are necessary for a sequence to correspond to an actual orbit of f.

PROPOSITION 1.2. *Let $z \in \mathbb{C}$ have itinerary $\underline{s}$. Then $\underline{s} \in \Sigma$.*

Proof. Proposition 1.1. gives the necessity of (1) and (2). For (3), first let $y = |\mathrm{Re}\,(z)|$. Then $|f(z)| \leq e^y$, and by induction it follows that $|f^j(z)| \leq f^j(y)$.

Now let $x = y + 2\pi$. Since $y > 0$,

$$f(x) = e^y e^{2\pi} \geq e^y + 2\pi.$$

By induction, we have $f^j(x) \geq f^j(y) + 2\pi$ as well. Therefore, we have

$$\begin{aligned} 2\pi |s_j| &\leq |f^j(z)| + 2\pi \\ &\leq f^j(y) + 2\pi \\ &\leq f^j(x). \end{aligned}$$

Hence $\underline{s}$ is of exponential order. □

S gives a map from $\mathbb{C}$ to Σ. There is a topology on Σ which makes this map continuous. Hence we have

THEOREM. *$f(z)$ is topologically semi-conjugate to the shift automorphism on Σ.*

Remark. Let $\lambda > 0$. Let $f_\lambda(z) = \lambda \exp(z)$. One may also describe the dynamics of f_λ using symbolic dynamics. A sequence of elements of A is exponential of order λ if there exists $x \in \mathbb{R}$ such that

$$2\pi |s_j| \leq f_\lambda^j(x) \qquad \text{for all } j.$$

Let Σ_λ denote the set of all sequences which are exponential of order λ and which also satisfy (1) and (2) of proposition 1.2. One may then easily show that f_λ is topologically semiconjugate to the shift map on Σ_λ.

2. *Itineraries of* exp (z)

The goal of this section is to prove the converse of proposition 1.2, i.e. to show that for any $\underline{s} \in \Sigma$, there exists $z \in \mathbb{C}$ with $S(z) = \underline{s}$. The case where $\underline{s}$ ends in a tail of 1's (or of -1's) will necessitate special arguments both here and elsewhere, so we begin with this case.

PROPOSITION 2.1. *There exists a unique fixed point in $R(1)$ (resp. in $R(-1)$.) This fixed point is a source.*

Proof. There exists a fixed point in $R(1)$ iff $e^x \cos y = x$ and $e^x \sin y = y$ with $0 < y < 2\pi$. One checks easily that these two curves meet transversely at a unique point outside the unit circle, at approximately $0.3 + 1.3i$. □

Remark. One can use similar methods to find a unique fixed source in each $R(j)$. We will use different methods, however, which generalize to the case of periodic points. Let $s_0, \ldots, s_n \in \mathbb{Z} - \{0\}$. Let

$$V(s_0, \ldots, s_n) = \{z \in R(s_0) | f^j(z) \in R(s_j),\ 1 \le j \le n\}$$

$V(s_0, \ldots, s_n)$ consists of all points whose itinerary begins with $s_0 s_1 \cdots s_n$.

PROPOSITION 2.2 *Suppose $s_i \neq 0$ for all i. Then $V(s_0, \ldots, s_n)$ is non-empty and is mapped onto $R(s_n)$ by f^n.*

Proof. f maps $R(s_0)$ onto $\mathbb{C} - (\mathbb{R}^+ \cup 0)$. Let f_0^{-1} be the inverse of $f|R(s_0)$. By induction,

$$V(s_0, \ldots, s_n) = R(s_0) \cap f_0^{-1}(V(s_1, \ldots, s_n))$$

is non-empty. Also, $f(V(s_0, \ldots, s_n)) = V(s_1, \ldots, s_n)$, and so the second part follows. □

It follows that $f^{n+1}(V(s_0, \ldots, s_n)) = \mathbb{C} - (\mathbb{R}^+ \cup 0)$. Hence we have

PROPOSITION 2.3. *Let $\underline{s} = s_0 s_1 \cdots s_n 1\ 1\ 1\ 1 \ldots$ (resp. $s_0 \ldots s_n -1\ -1\ -1 \ldots$) with $s_i \neq 0$. Then there exists z with $S(z) = \underline{s}$.*

Proof. Some point in $V(s_0, \ldots, s_n)$ is mapped onto the fixed point in $R(1)$ given by proposition 2.1. □

As another special case, consider sequences in Σ of the form $s_0 \cdots s_n 0^-\ 0^+\ 0^+\ 0^+ \ldots$ or $s_0 \cdots s_n 0^+\ 0^+\ 0^+ \ldots$. Using arguments as above, it follows that there are two curves in $V(s_0, \ldots, s_n)$ whose itineraries are of this form. We remark that this situation is typical: the set of points which share the same itinerary usually forms a curve. We will discuss this phenomenon in more detail later.

The remainder of this section is devoted to proving the existence result in the general case. Let $\underline{s} \in \Sigma$ and suppose $\underline{s}$ does not end in a tail of 0's, 1's, or -1's. Since $\underline{s}$ is of exponential order, there exists $\hat{x} \in \mathbb{R}$ such that

$$f^j(\hat{x}) \ge 2\pi |s_j| \qquad \text{for all } j.$$

In the following construction, we assume that $\hat{x} > 2\pi$. We first construct a sequence of squares B_i defined as follows:

(1) $B_i \subset \operatorname{clos} R(s_i)$;

(2) each side of B_i is parallel to the x- or y-axis and has length 2π;

(3) the leftmost vertical side of B_i lies in $\operatorname{Re} z = f^i(\hat{x})$.

A similar set of boxes can be constructed for each $x > \hat{x}$.

PROPOSITION 2.4. *Each B_j lies in the sector $|\operatorname{Im} z| \le \operatorname{Re} z$. Moreover, $|f'(z)| > 1$ for any $z \in B_j$.*

Proof. The upper (resp. lower) left corner of B_j is given by $f^j(\hat{x}) + i2\pi s_j$ when $s_j \ge 1$ (resp. $s_j \le -1$). Since $f^j(x) \ge 2\pi|s_j|$, the result follows. □

PROPOSITION 2.5. $f(B_j) \supset B_{j+1}$.

Proof. The f-image of B_j is contained in the annulus of inner radius $f^{j+1}(\hat{x})$ and outer radius $f(f^j(\hat{x}) + 2\pi)$. Note that the outer circle meets the lines $y = \pm x$ at points with real part $(\sqrt{2}/2)\, e^{2\pi} f^{j+1}(\hat{x})$. One checks easily that this number is larger than $f^{j+1}(\hat{x}) + 2\pi$ provided $f^{j+1}(\hat{x}) > 1$. This completes the proof. □

Let us make the restrictive assumption that $|s_i| \neq 1$ for all i. Then we define

$$V_n = \{z \in B_0 | f^i(z) \in B_i \text{ for } i \leq n\}.$$

Clearly, V_n is closed. Since $|s_n| \neq 1$, V_{n+1} is contained in the interior of V_n. Hence the intersection $\bigcap_{n\geq 0} V_n$ is non-empty. It is trivial to check that any point in this intersection has the desired itinerary. Moreover, this point is unique, since $|(f^n)'(z)| > (f^n)'(\hat{x})$ for $z \in f^{-n}(B_n)$.

The above argument breaks down if some of the s_i are ± 1. There are two problems. First, if $s_n = \pm 1$ then V_n need not be connected. Indeed, V_n consists of two components, the first of which is a curve mapped into $\mathbb{R}$, and the second a 'rectangular' components mapped by f^n onto B_n. The above construction must be modified by choosing W_n to be this latter component of V_n.

The second problem is that W_{n+1} is not contained in the interior of W_n if $s_{n+1} = 1$. If arbitrarily large s_i satisfy $|s_i| \neq 1$, then the above argument goes through. If, on the other hand, all s_i satisfy $|s_i| = 1$, then the intersection may yield a point with itinerary eventually all 0's. This indeed happens if $\underline{s}$ ends in a tail of +1's or of −1's. However, this case was handled earlier. If the sequence continually changes from +1 to −1 and vice versa, then the above argument works. We leave the details to the reader. This completes the proof of the following theorem.

THEOREM. *Let* $\underline{s} \in \Sigma$. *There exists* $z \in \mathbb{C}$ *with* $S(z) = \underline{s}$.

Remark 1. The point z with itinerary $\underline{s}$ constructed above has orbit which tends to infinity with monotonically increasing real part. Even if $\underline{s}$ is bounded or repeating, the orbit of z is unbounded. This suggests two questions. For any periodic itinerary, is there a corresponding periodic point for f? More generally, what is the nature of the set of points in $\mathbb{C}$ which share the same itinerary? We will answer these questions in subsequent sections.

Remark 2. A similar result is valid for $f_\lambda(z) = \lambda \exp(z)$ with $\lambda > 0$. We must of course assume that $\underline{s}$ is exponential of order λ.

3. *Tails of itineraries*

In this section we discuss in more detail the set of points which share the same itinerary. We will deal exclusively with $\underline{s} \in \Sigma$ which does not end in the constant sequence $1, 1, 1 \ldots$ or $-1, -1, -1 \ldots$.

Fix $\underline{s} \in \Sigma$ of the above type and let $\eta \in \mathbb{R}$. Define

$$\gamma_\eta(\underline{s}) = \{z \in R(s_0) | S(z) = \underline{s} \quad \text{and} \quad \operatorname{Re} f^j(z) \geq f^j(\eta)\}.$$

$\gamma_\eta(\underline{s})$ consists of the set of points in $S^{-1}(\underline{s})$ which move sufficiently quickly to the right. Our goal in this section is to show that $\gamma_\eta(\underline{s})$ is Lipschitz if η is sufficiently large.

We first recall the construction of the previous section. For each $x \geq \hat{x}$ there is defined a set of squares $B_j(x) \subset R(s_j)$ with the property that $\bigcap_{j\geq 0} f^{-j}(B_j(x))$ is a unique point (where the appropriate inverse is chosen at each stage). Let $\zeta(x)$ be this unique point. One checks easily that $\zeta(x)$ is a parametrization of $\gamma_{\hat{x}}(\underline{s})$. Moreover,

$\zeta(x)$ is continuous. This can be seen as follows. Let $z_0=\zeta(x_0)$ and take any neighbourhood U of z_0. There exists N such that $f^{-N}(B_N(x_0))\subset U$. If x is close enough to x_0 then $f^{-N}(B_N(x))\subset U$ as well.

To prove that γ_η is Lipschitz we need several lemmas.

LEMMA 3.1. *If* $\lambda>2$ *then* $f^j(\lambda\hat{x})>\lambda^{j+1}f^j(\hat{x})$.

Proof. Recall that $\hat{x}>2\pi$. By induction, for $j\geq 1$ we have

$$\begin{aligned} f^{j+1}(\lambda\hat{x}) &= \exp(f^j(\lambda\hat{x}) \\ &> \exp(\lambda^{j+1}f^j(\hat{x})) \\ &> \exp(\lambda^{j+1}+f^j(\hat{x})), \end{aligned}$$

since the product of two numbers larger than 2 is larger than the sum.
Therefore

$$\begin{aligned} f^{j+1}(\lambda\hat{x}) &> \exp(\lambda^{j+1})\cdot f^{j+1}(\hat{x}) \\ &> \lambda^{j+2}f^{j+1}(\hat{x}). \end{aligned}$$ □

Let $\theta_j(z)=\arg(f^j(z))$.

LEMMA 3.2. *Let* $\varepsilon>0$. *There exists* $\eta>0$ *such that, if* $f^j(z)\in R(s_j)$ *and* $\operatorname{Re} f^j(z)>f^j(\eta)$, *then*

$$\sum_{j=0}^{n}|\tan\theta_j(z)|<\varepsilon.$$

Proof. Let z be such that $\operatorname{Re} z=\lambda\hat{x}$ with λ as chosen below. We have, using lemma 3.1

$$\begin{aligned} |\tan\theta_j(z)| &\leq \frac{2\pi|s_j|}{\operatorname{Re} f^j(z)} \\ &\leq \frac{f^j(\hat{x})}{\lambda^{j+1}f^j(\hat{x})}. \end{aligned}$$

Hence we may choose λ so that $\sum_{j=0}^{n}|\tan\theta_j(z)|$ is arbitrarily small. □

In particular, we have shown

PROPOSITION 3.3 *Given* $\varepsilon>0$, *there exists* $\eta>0$ *such that if* $z\in\gamma_\eta(\underline{s})$, *then* $\sum_{i=0}^{\infty}|\theta_i(z)|<\varepsilon$.

Now define the semi-infinite strip

$$C_i(\eta)=\{z\in R(s_i)|\operatorname{Re} z>f^i(\eta)\}.$$

If $\eta>\hat{x}+2\pi$, it is easily seen that $f(C_i)$ covers C_{i+1}. Let

$$D_i(\eta)=f^{-i}(C_i)$$

where the inverses are chosen so that $z\in D_i\Rightarrow f^j(z)\in C_j$ for $0\leq j\leq i$. The D_i's form a nested sequence of strips in $R(s_0)$. One boundary component of each D_j lies in the line $\operatorname{Re} z=\eta$. The others are smooth curves which tend to infinity.

Let $\varepsilon>0$. Choose $\eta=\lambda\hat{x}$ as in lemma 3.2 so that $\sum_{i=0}^{\infty}|\theta_j(z)|<\varepsilon$ for any $z\in\gamma_\eta(\underline{s})$. We now show that γ_η is Lipschitz.

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Let $\Gamma_\eta^\pm$ denote the upper and lower boundaries of D_η. By lemma 3.2 these curves have the form $t+i\mu_\pm(t)$ for $t \geq \eta$ where $\mu_\pm$ is smooth and satisfies

$$|\mu'_\pm(t)| < \varepsilon.$$

The vertical widths of the D_i tend to zero as was shown in § 2. Hence $\Gamma_\eta^\pm \to \gamma_\eta(\underline{s})$ as $n \to \infty$. Since the slopes of the $\Gamma_\eta^\pm$ are bounded by ε, it follows that $\gamma_\eta(\underline{s})$ is Lipschitz with Lipschitz constant ε.

Remark. This proves that the set of points corresponding to a given itinerary (not ending with $1, 1, 1, \ldots$ or $-1, -1, -1 \ldots$) which tend monotonically to infinity (relative to the real axis) forms a nice curve in $\mathbb{C}$. We conjecture that this tail is in fact smooth.

4. *Periodic points of* exp (z)

Our goal in this section is to prove that there exists a unique periodic point corresponding to each repeating sequence in Σ. Moreover, each such point is expanding and comes equipped with a 'string', i.e. a continuous curve of points which limits on the periodic point and which consists of all points which share the same itinerary, provided $\underline{s} \neq 1, 1, 1, \ldots$ or $-1, -1, -1 \ldots$.

Definition. A region H in $\mathbb{C}$ is called *horseshoe-shaped* if:

(1) $H \subset \text{int}(R(i))$ for some i;

(2) H is bounded by two smooth curves $\zeta_i(t)$, $i = 1, 2$ defined for $-\infty < t < \infty$;

(3) $\lim_{t\to\pm\infty} \text{Re}\, \zeta_i(t) = \infty$;

(4) $\lim_{t\to\pm\infty}$ slope $\zeta'_i(t) = 0$.

See figure 1.

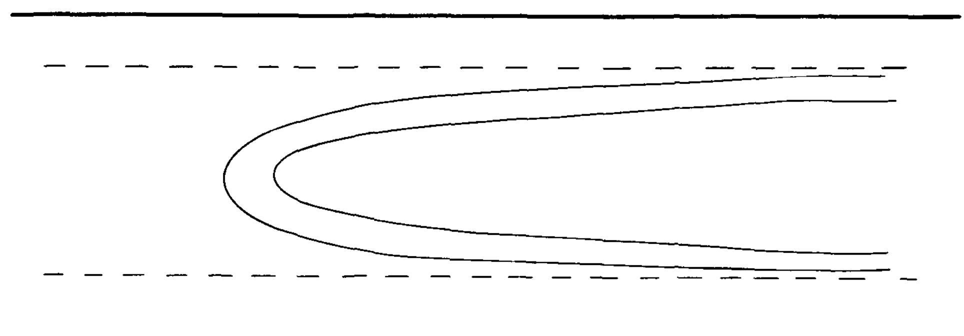

FIGURE 1

PROPOSITION 4.1. *If H is horseshoe-shaped, then $f_j^{-1}(H)$ is also horseshoe-shaped, where $f_j = f|R(j)$.*

Proof. Since f_j: int $R(j) \to \mathbb{C} - (\mathbb{R}^+ - 0)$ is a diffeomorphism, $f_j^{-1}(H)$ is bounded by two smooth curves in $R(j)$. Let $\Gamma_\alpha = \{z \in R(j) | \text{Re}\, z = \alpha\}$. Γ_α is mapped onto the circle of radius α by f_j. Therefore $f_j(\Gamma_\alpha)$ meets H in two arcs for large enough α.

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Moreover, the angles between $f_j(\Gamma_\alpha)$ and the boundary of H tend to $\pi/2$ as $\alpha \to \infty$. Since f_j is conformal, the same is true for the angle between Γ_α and $f_j^{-1}(H)$. This completes the proof. □

Recall that $V(s_0, \ldots, s_n) = \{z \in R(s_0) | f^i(z) \in R(s_i),\ 1 \le i \le n\}$.

PROPOSITION 4.2. *Suppose $s_1 \neq \pm 1$. Then $V(s_0, s_1)$ is a horseshoe-shaped region.*

Proof. Since f maps $R(s_0)$ diffeomorphically onto $\mathbb{C} - \mathbb{R}^+$, it follows that $f^{-1}(\partial(\overline{R(s_1)}))$ consists of two smooth curves in $R(s_0)$. As in proposition 4.1, the image of Γ_α cuts $R(s_1)$ in two arcs of circles for large enough α. Therefore, if $f^{-1}(\partial\overline{R(s_1)})$ is parametrized by the real part along $\partial\overline{R(s_1)}$, then it follows that $\text{Re}\, f^{-1}(\partial\overline{R(s_1)}) \to \infty$ as $t \to \pm\infty$. Moreover, the angle between $f_0(\Gamma_\alpha)$ and $\partial\overline{R(s_1)}$ tends to $\pi/2$ as $\alpha \to \infty$. Hence the slope of $f_0^{-1}(\partial\overline{R(s_1)})$ tends to zero as $\alpha \to \infty$. By conformality, the slopes of $f^{-1}(\partial(\overline{R(s_1)}))$ tend to zero as required. □

PROPOSITION 4.3. *Suppose $s_n \neq \pm 1$. Then $V(s_0, \ldots, s_n)$ is horseshoe-shaped.*

Proof. By induction, suppose $V(s_1, \ldots, s_n)$ is horseshoe-shaped. Then $V(s_0, \ldots, s_n) = f_0^{-1} V(s_1, \ldots, s_n)$ is horseshoe-shaped by proposition 4.1. □

The following result is due to Misiurewicz; (see [**6**, p. 103]).

PROPOSITION 4.4. $|(f^n)'(z)| \ge |\text{Im}\, f^n(z)|$.

We now prove the existence of strings in the special case where at least one of the $s_i \neq \pm 1$. Without loss of generality we may assume $s_0 \neq \pm 1$. Let $\underset{\sim}{s} = \overline{s_0 \cdots s_{n-1}}$ be a repeating sequence. Since $|s_0| \neq 1$, $V(s_0, \ldots, s_{n-1}, s_0)$ is a horseshoe-shaped region in $R(s_0)$, and f^n maps $V(s_0, \ldots, s_{n-1}, s_0)$ diffeomorphically onto $R(s_0)$. Let

$$V_\alpha = \{z \in V(s_0, \ldots, s_{n-1}, s_0) | \text{Re}\, z \le \alpha\}.$$

Let

$$W_\alpha = \{z \in V_\alpha | \text{Re}\, z = \alpha\}.$$

Since $V(s_0, \ldots, s_n, s_0)$ is horseshoe-shaped, it follows that if α is large enough, then W_α consists of two disjoint intervals. Moreover the results of § 3 imply that these two intervals are mapped into the region $\{z \in R(s_0) | |\text{Re}\, z| > \alpha\}$, one to the left of $\text{Re}\, z = -\alpha$ and one to the right of $\text{Re}\, z = \alpha$. See figure 2.

It follows that $f^n(V_\alpha) \supset V_\alpha$. Moreover, if $z \in V_\alpha$, then $|(f^n)'(z)| \ge 2\pi$ by proposition 4.4. Hence $f^n | V_\alpha$ has an inverse which is a contraction. Therefore there is a unique fixed point for f^n in V_α. Since the argument is independent of α for α large, it follows that there is a unique periodic point in $R(s_0)$ with itinerary $\underset{\sim}{s}$.

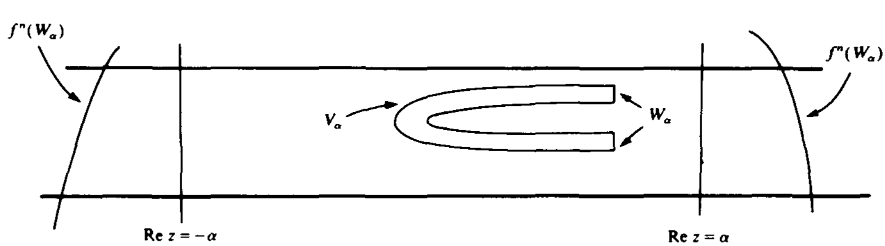

FIGURE 2

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Remark. This argument fails in the case where all of the s_i are ± 1's for two reasons. First, the $V(s_0, \ldots, s_{n-1}, s_0)$ will not be horseshoe-shaped in this case. And secondly, f^n need not be an expansion in this case. Nevertheless, we can adapt the previous argument to most itineraries of ± 1's as follows.

Let $\underline{s} = \overline{s_0, \ldots, s_{n-1}} \ldots$ be a repeating sequence with $|s_i| = 1$ for all i. Let us assume that not all $s_i = 1$ (or $= -1$). So $\underline{s}$ is not the constant sequence identically equal to $+1$ or to -1. We may assume that $s_0 = 1$ and $s_1 = -1$ by applying f several times.

If $S(z) = \underline{s}$, then we must have that $\pi < \operatorname{Im} z < 2\pi$, i.e. z lies in the upper half strip of $R(1)$. Let us denote this strip by H. Let

$$V = \{z \in H \mid f^i(z) \in R(s_i) \text{ for } i < n \text{ and } f^n(z) \in H\}.$$

One checks easily that V is a horseshoe shaped region. The previous argument then applies with V in place of $V(s_0, \ldots, s_{n-1}, s_0)$.

Hence the only unsettled cases correspond to the constant sequences $1, 1, 1 \ldots$ and $-1, -1, -1 \ldots$. We are indebted to E. Ghys and L. Goldberg who showed us the following argument to prove that f^{-1} has a global sink in each of $R(1)$ and $R(-1)$.

Since $R(1)$ is conformally equivalent to the disk and f^{-1} has a sink in $R(1)$, it follows from the Schwarz Lemma that f^{-1} is a contraction, i.e. $\|f^{-1}(z)\| < \|z\|$. It follows that there is a unique periodic point whose orbit lies in $R(1)$. This point is the fixed source described in § 2. □

We now prove the following:

THEOREM. *Let $\underline{s}$ be a repeating sequence in Σ and suppose $\underline{s} \neq 1, 1, 1 \ldots$ or $-1, -1, -1 \ldots$. Let $\gamma = \{z \in \mathbb{C} \mid S(z) = \underline{s}\}$. Then γ is a continuous curve in the plane.*

Proof. Simply take $\gamma_n(\underline{s})$ defined in § 3 and iterate it backwards (using the appropriate inverses at each stage). All points in $\gamma_n(\underline{s})$ are backward asymptotic to the unique periodic point with itinerary $\underline{s}$. □

Remark. It follows that $\gamma(\underline{s})$ is a curve or 'string' which spirals into the periodic point with itinerary $\underline{s}$. See figure 3.

Remark. It is not true that the set of points which share the exceptional itineraries $1, 1, 1, \ldots$ and $-1, -1, -1 \ldots$ form strings. Indeed we show below that these points contain a Cantor set of curves in $\mathbb{C}$.

5. *Dynamics in* $R(1)$

In this section we describe the dynamics of $f(z)$ in $R(1)$. More precisely, we attempt to describe the set of points whose forward orbits remain in $R(1)$. Our results are valid for $R(-1)$ as well, since $f(\bar{z}) = \overline{f(z)}$. It turns out that the set of points which remain for all iterates in $R(1)$ or in $R(-1)$ is remarkably different from those which have other repeating itineraries. In particular, there is a Cantor set of curves consisting of points whose orbits remain in $R(1)$, as opposed to a single curve corresponding to other repeating itineraries.

We remark that our results are valid for $f_\lambda(z) = \lambda e^z$ as long as $\lambda > 1/e$.

Recall that there exists a unique fixed point in $R(1)$, and that this point is a source (proposition 2.1). We denote this fixed point by p_0.

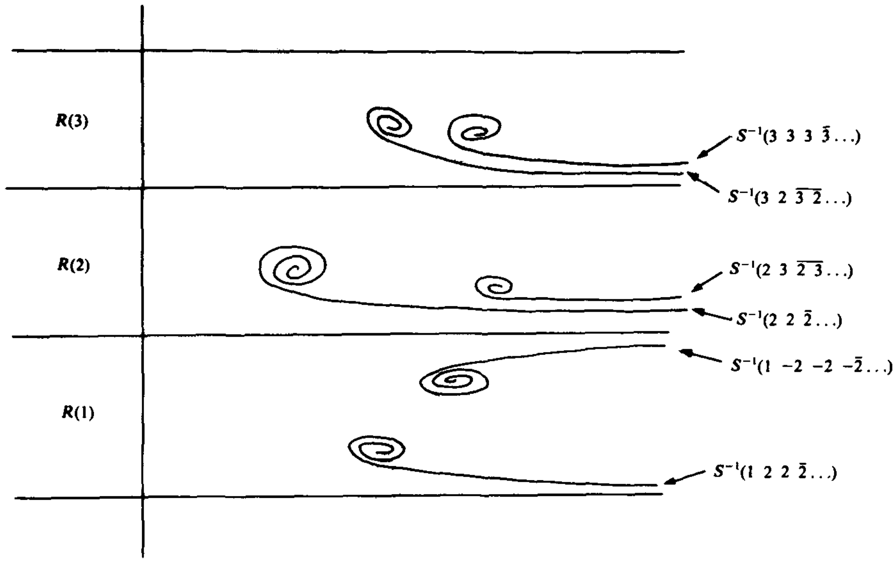

FIGURE 3

PROPOSITION 5.1. *The unstable manifold of* p_0, $W^u(p_0)$, *contains* $R(1)$. *Moreover, if* $\{z_n\}=\{f^n(z)\}$ *is an orbit of* f *completely contained in* $R(1)$, *then* z_n *approaches the boundary of* $R(1)$ *as* $n\to\infty$.

Proof. Let h be a conformal mapping of $R(1)$ onto the unit disk with $h(p_0)=0$. Recall that f is 1–1 on $R(1)$ and that $f(R(1))\supset R(1)$. Let g be the restriction of the branch of f^{-1} satisfying $g(p_0)=p_0$ to $R(1)$. Obviously, $g(R(1))\subset R(1)$. Let $\tilde{g}=h\circ g\circ h^{-1}$. $\tilde{g}$ maps the unit disk into itself, $|\tilde{g}'(0)|<1$, and $\tilde{g}(0)=0$. Therefore, by the Schwarz Lemma, $|\tilde{g}(z)|<|z|$ for all $z\neq 0$ in the unit disk. If $\{z_n\}$ is an orbit of f contained in $R(1)$, then, by the above, $|h(z_{n+1})|>|h(z_n)|$, so that $|h(z_n)|\to 1$ as $n\to\infty$. □

We now describe the behaviour of orbits living in $R(1)$ in more detail.

PROPOSITION 5.2. (1) *There exists* $k>0$ *such that if* $0\le y=\operatorname{Im}(z)<\frac{1}{2}\arccos(1/\lambda e)$, *then*

$$\lambda\, e^x\cos y=\operatorname{Re} f(z)\ge \operatorname{Re} z+k=x+k.$$

(2) *For each* $m>0$ *there is a constant* $K=K(m)>0$ *such that if* $\operatorname{Re} z=x>K$ *and* $0\le y=\operatorname{Im} z<\pi/3$, *then*

$$\lambda\, e^x\cos y=\operatorname{Re} f(z)\ge \operatorname{Re} z+m=x+m.$$

Proof. (1) By hypothesis, $\cos y=\sqrt{\frac{1}{2}(1+\cos 2y)}>a=\sqrt{\frac{1}{2}(1+(1/\lambda e))}$. It is easy to check that $a>1/e$. By elementary calculus we have

$$k=\inf(a\, e^x-x)>0$$

Thus

$$\lambda\, e^x\cos y-x\ge a\, e^x-x\ge k.$$

This proves (1). For (2), since $0<y<\pi/3$, we have $\frac{1}{2}<\cos y<1$. Hence

$$\lambda\, e^x \cos y>(\lambda/2)\, e^x.$$

Clearly, for sufficiently large x, $(\lambda/2)\, e^x>x+m$. □

PROPOSITION 5.3. *For each $M>0$ there exists $c(M)$ such that if* $\operatorname{Re} z=x\geq c(M)$ *and* $0<y<\pi/2$, *then*

$$\operatorname{Im} f(z)=\lambda\, e^x \sin y\geq My$$

Proof. We have $\lambda\, e^x \sin y>(2/\pi)\lambda\, e^x y\geq My$ for $x\geq\ln(M\pi)-\ln(2\lambda)$. □

PROPOSITION 5.4. *If $0<y<\pi$ and $\lambda\, e^x \sin y>\pi$, then at least one of the points $f(z)$ or $f^2(z)$ is not in $R(1)$.*

Proof. Assume $f(z)\in R(1)$. Then $\pi<\lambda\, e^x \sin y<2\pi$. Hence $\operatorname{Im} f^2(z)<0$ so $f^2(z)\notin R(1)$. □

We can now describe the behaviour of a point whose entire orbit lies in $R(1)$. Suppose that p is such a point and that p lies inside the unit circle. Typically, the first few iterations of f move p to the right but close to the x-axis. Eventually, an iteration of f maps p close to $y=\pi$. Then p is mapped into the left half plane, and finally the next iterate of p lands inside the unit circle again. It turns out that this last iterate brings p much closer to the x-axis than p was originally. Then the whole process begins again, with p taking a longer time to jump to $y=\pi$.

PROPOSITION 5.5. *Suppose $z_n\in R(1)$ for all n. Then*

(1) *there exists a subsequence (z_{n_k}) converging to 0, and satisfying*

(2) *for k sufficiently large, the points*

$$z_{n_k+1}, z_{n_k+2}, \ldots z_{n_{k+1}-3}$$

lie in the right half plane below the curve $\lambda\, e^x \sin y=\pi$. The point $z_{n_{k+1}-2}$ lies in the right half plane above the curve $\lambda\, e^x \sin y=\pi$. The point $z_{n_{k+1}-1}$ lies in the left half plane.

(3) $\lim_{k\to\infty} \operatorname{Re} z_{n_{k+1}-2}=+\infty$; $\lim_{k\to\infty} \operatorname{Re} z_{n_{k+1}-1}=-\infty$.

Proof. Recall that the plane $\operatorname{Re} z<-a<0$ is mapped inside the circle of radius $\lambda\, e^{-a}<\lambda$ centred at the origin. Also, since $z_n\in R(1)$ for all n, we have that $0<\operatorname{Im}(z_n)<\pi$. Let $a>10$ satisfy $a>K(k)$ as in proposition 5.2 and $a>c(4)$ as in proposition 5.3 and finally $\lambda\, e^{-a}<1$, $a>\ln\pi-\ln\lambda$. Choose ε such that $0<\varepsilon<\frac{1}{2}\arccos(1/\lambda e)$. Let $\delta>0$ be such that if $|x|<1$ and $|y|<\delta$, then $|f^n(z)-f^n(x)|<\varepsilon$ for $n=0, 1, \ldots n(a)$, where $n(a)k>a$. Let C be a rectangle with sides lying on the lines $y=\delta$, $y=\pi$, $x=\pm a$.

By proposition 5.1 only finitely many of the z_n's can be found in C. Thus for sufficiently large n there are three possibilities:

(a) z_n is to the left of $x=-a$;

(b) z_n is below $y=\delta$;

(c) z_n is to the right of $x=a$.

In case (a), $|z_{n+1}|<1$, so that for z_{n+1} we have case (b). In this case, the point moves to the right for at least $n(a)$ steps and, moreover, goes beyond $x=a$. Hence we have case (c) for $z_{n+n(a)}$. In this case, the imaginary part is multiplied by at least 4 at each step and the point moves to the right by k at least as long as z_n is below

$\lambda\, e^x \sin y = \pi$. Thus at some point the orbit is above $\lambda\, e^x \sin y = \pi$ but below $y = \pi$, i.e. there exists $i > 0$ such that

$$\pi - \arcsin\,(\pi/\lambda)\, e^{-x} < \operatorname{Im} z_{n_i} < \pi.$$

Hence Re $z_{n_i+1} < 0$. We have

$$x_{n_i+1} = \lambda \exp(x_{n_i}) \cos y_{n_i}.$$

It is easy to see that $-1 < \cos y_n < -0.9$, so

$$-\lambda < \lambda \cos y_{n_i} < -0.9\lambda < -0.9e^{-1} < -0.3$$

If $x \ge 10$ then

$$e^x > x^3/6 \ge (100/6)x > 16x.$$

Therefore $x_{n_i+1} < -0.3(16x_{n_i}) < -4x_{n_i} \le -4a$, so that x_{n_i+1} is to the left of $x = -a$. This completes the proof. □

The next proposition shows that points near the boundary of $R(1)$ keep returning to the unit circle at successively lower and lower points.

PROPOSITION 5.6. *For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, if* $|z_0| \le 1$, $|x_0| \le 1$, *and* $0 < y_0 < \delta$, *then*

(1) $z_0, z_1, \ldots, z_n$ *lie below* $y = \pi/3$; z_{n+1} *lies above* $y = 2\pi/3$, *but below* $y = \pi$; *and*
(2) $\operatorname{Im} z_{n+3} < \varepsilon \operatorname{Im} z_0$;
(3) $\operatorname{Im} z_{n+4} < \varepsilon \operatorname{Im} z_0$.

Proof. (1) follows immediately from proposition 5.5, and (3) follows from (2) with, perhaps, a different ε. For (2), there exists j such that if $m \ge j$, then $x_m \ge 10 > \ln(\pi/\lambda)$. We may assume that $0 < y_m < \arcsin((\pi/\lambda)\, e^{-x})$. If $x \ge 10$, then

$$(\pi/\lambda)\, e^{-x} < \pi e\, e^{-10} < 10\, e^{-10} < 0.001.$$

If $0 < t < 0.001$, then $t < \arcsin t < 1.1t$, so that

$$\arcsin((\pi/\lambda)\, e^{-x}) < 1.1\,\pi\lambda\, e^{-x}$$

for $x \ge 10$. Hence we have that, if $n \ge m \ge j$, then

$$\text{(A)} \qquad 0 < y_m < 1.1(\pi/\lambda)\, e^{-x_m} \quad \text{and} \quad 0 < \pi - y_{n+1} < 1.1(\pi/\lambda)\, e^{-x_{n+1}}.$$

It is easy to see that $1.1(\pi/\lambda)\, e^{-x} < 0.01$. If $0 < t < 0.01$ then $1 > \cos t > 1 - (t^2/2) > 0.9$. This proves that $-1 < \cos y_{n+1} = -\cos(\pi - y_{n+1}) < -0.9$. Hence

$$\text{(B)} \qquad \begin{aligned} y_{n+3} &= \lambda \exp(x_{n+2}) \sin y_{n+2} \\ &< \lambda \exp(x_{n+2}) y_{n+2} \\ &= \lambda^2 \exp(\lambda \exp(x_{n+1}) \cdot \cos y_{n+1}) \exp(x_{n+1}) \sin y_{n+1} \\ &< \lambda^2 y_{n+1} \exp(x_{n+1} - 0.9\lambda \exp(x_{n+1})). \end{aligned}$$

Analogously, $y_{m+1} < \lambda y_m \exp(x_m)$. Therefore

$$\text{(C)} \qquad \begin{aligned} y_j &> \lambda^{-1} \exp(-x_j) y_{j+1} \\ &> \lambda^{-2} \exp(-x_j - x_{j+1}) y_{j+2} \\ &> \cdots > \lambda^{-(n+1-j)} y_{n+1} \exp(-x_j - x_{j+1} - \cdots - x_n). \end{aligned}$$

We also have

(D)
$$\begin{aligned} y_0 &> y_1\lambda^{-1}\exp(-x_0)\\ &> y_2\lambda^{-2}\exp(-x_0-x_1)\\ &> \cdots > y_j\lambda^{-j}\exp(-x_0-x_1-\cdots-x_{j-1})\\ &> y_j(\lambda^{-1}\exp(-x_0))^j. \end{aligned}$$

In view of (A), (B), (C), in order to prove that $y_{n+3}<\varepsilon y_0$, it suffices to show that

$$\begin{aligned} y_{n+1} &> (\lambda^{-1}\exp(-x_0))^j\cdot\lambda^{-(n+1-j)}\exp(-x_0-\cdots-x_n)\\ &> y_{n+1}\lambda^2\exp(x_{n+1}-0.9\lambda\exp(x_{n+1})). \end{aligned}$$

Equivalently

$$0.9\lambda\exp(x_{n+1})>x_j+x_{j-1}+\cdots+x_n+x_{n+1}+(n+3)\ln\lambda+jx_0-\ln\varepsilon$$

Since $x_j>10$, $x_{m+1}\geq 2x_m$ for $n\geq m\geq j$, since $x_{m+1}=\exp(x_m)\cos y_m\geq 0.9\exp(x_m)\geq 2x_m$. Hence $x_j+\cdots+x_{n+1}<2x_{n+1}$. Note that $x_{n+1}>2^{n+1-j}x_j$, so that for n sufficiently large (and δ small), we have

$$(n+3)\ln\lambda+jx_0-\ln\varepsilon<x_{n+1}.$$

This proves that

$$x_j+\cdots+x_{n-1}+(n+3)\ln\lambda+jx_0-\ln\varepsilon<3x_{n+1}$$

Now we need $0.9\lambda\, e^x>3x$, which is clear since $\lim_{x\to\infty}x\,e^{-x}=0$. This completes the proof. □

It follows from proposition 5.6 that, if δ is small, then $(f^n)'(z_0)$ turns vectors by only a small angle. Indeed, $(f^n)'(x_0)=f'(x_n)\cdot\ldots\cdot f'(x_0)$ and $f'(x_i)=f(x_i)$. This shows that there exists an 'almost' horizontal curve C_n, $C^1-\varepsilon$ close to the interval $(-1, 1)$, which is mapped into $\lambda\, e^x\sin y=\pi$, $y<\pi/2$ by f^n, so into $y=\pi$ by f^{n+1}. There is also a curve B_n mapped into $\lambda\, e^x\sin y=\pi$ by f^{n+1}, and these two curves are $C^1-\varepsilon$ close. As f is applied to them several times, the vertical distance between them grows quickly at a rate described in proposition 5.6. Since this distance is small to begin with, it follows that the vertical distance d_n from B_n to C_n divided by the vertical distance D_n from C_n to the x-axis goes to 0 as $n\to\infty$, i.e. $\lim_{n\to\infty}(d_n/D_n)=0$.

Consider the rectangular region S_n bounded by C_n, B_n, and two arcs of the unit circle. By proposition 5.6, if $\varepsilon<1/2$, $f^{n+3}(S_n)\cap S_n=\emptyset$. One checks easily that the image of S_n is as depicted in figure 4.

Note that the horizontal boundaries of S_n are mapped to the x-axis and the vertical boundaries are mapped to near-semi-circles with small radius. Precise estimates are given in proposition 5.6.

We remark that one may also see this by looking at the curvature of a parametrized curve $r(t)$, $|r'(t)|\neq 0$. Recall that the curvature may be written

$$K(t)=|r'|^{-3}\operatorname{Im}(r''\cdot\bar r').$$

Hence the curve $f\circ r$ has curvature

$$K_f(t)=|e^{-r}|\cdot(|r'|^{-1}\operatorname{Im} r'+K(t)).$$

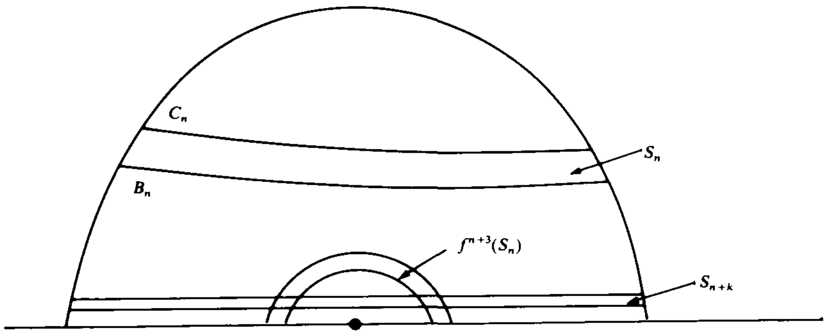

FIGURE 4

Now $f^{n+3}(S_n)$ intersects S_m for all sufficiently large $m > n$. ($f^{n+3}(S_n)$ does not meet S_n). Hence $f^{n+3}(S_n) \cap S_m$ has two components whose pre-images in S_n are thin, almost horizontal strips. Continuing this process yields a Cantor set of horizontal lines contained in the S_n. This process is similar in spirit to the construction of the Smale horseshoe map, but of course all of the horizontal curves are mapped downward by f^{n+3}. One can back these curves up by applying f^{-1} repeatedly. We then get a Cantor set of curves winding down to p_0, the fixed source in $R(1)$. We have proved:

THEOREM. *There is a Cantor set of curves in* $R(1)$ *consisting of points whose orbits are entirely contained in* $R(1)$.

Remark. By Misiurewicz's result [7], the set of all points whose orbits lie in $R(1)$ has empty interior. It is unclear what the Lebesgue measure of this set is. It is also unclear what the topological structure of the set of all such points is.

Remark. A similar result obviously holds for itineraries of the form $s_0 \cdots s_n 1\ 1\ 1\ 1 \ldots$ and $s_0 \cdots s_n -1\ -1\ -1 \ldots$.

6. *Bifurcations and* $\lambda\, e^z$

Let $\lambda > 0$ and consider the map $\lambda\, e^z$. When $\lambda \geq 1$, the Julia set of $\lambda\, e^z$ is $\mathbb{C}$, as was proved by Misiurewicz. We conjecture that, in fact $J(\lambda\, e^z) = \mathbb{C}$ for $\lambda > 1/e$. The arguments below indicate why this should be true.

We wish to show the rather spectacular bifurcation which occurs at $\lambda = 1/e$.

Consider first the dynamics of $\lambda\, e^x$ on the real line. Figure 5 shows the three possibilities. When $\lambda > 1/e$, all points tend to infinity under iteration of $\lambda\, e^x$. When $\lambda = 1/e$ there is a unique fixed point for $\lambda\, e^x$ at $x = 1$. For $0 < \lambda < 1/e$ this point separates into two fixed points: a sink p and a source q. Note that $0 < p < q$.

Thus we have a saddle-node bifurcation as λ decreases through $1/e$. We claim that the Julia set of $\lambda\, e^z$ changes dramatically as λ decreases through $1/e$. Indeed, let V be the vertical line through the source at q. Clearly, the image of V under f_λ is a circle centred at 0 which also passes through q. Hence all points to the left of V are mapped into the disk of radius q about 0. In fact, all such points tend asymptotically to the sink p. Therefore there are no points in the Julia set to the left of V; indeed the Julia set has changed dramatically at this point.

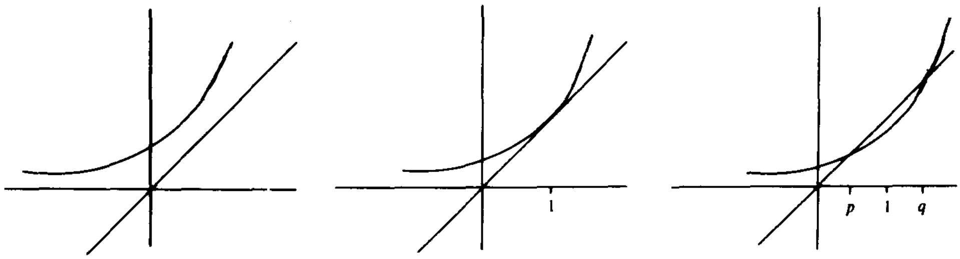

FIGURE 5

One can show that $J(\lambda e^z)$ is a Cantor set of lines when $\lambda \leq {}^1/e$. We sketch the argument. Consider the inverse images of V. These are an infinite collection of parabolic curves opening right with vertices over the source q. See figure 6.

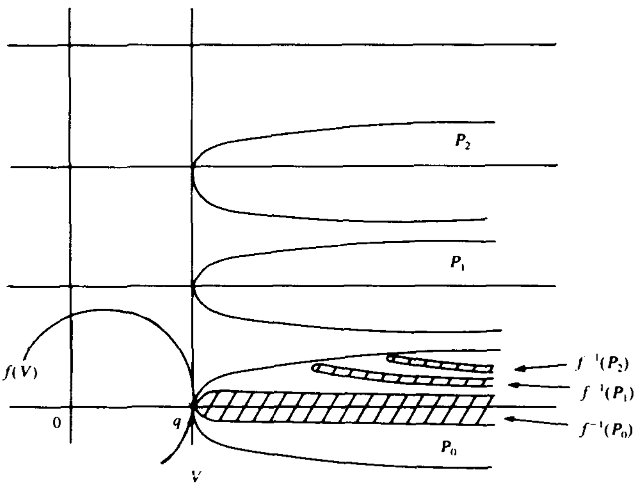

FIGURE 6

One may do symbolic dynamics in these regions exactly as before. Let $P(k)$ be the parabolic region which contains the line $y = 2k\pi i$ for $k \in \mathbb{Z}$ and $x \geq q$. We no longer need the special symbols 0^+ and 0^-. One may check that the set of points which visit the $P(k)$ according to an allowable itinerary forms a curve lying in the Julia set and that J is a Cantor set of such curves. We leave the details to the reader. We have the following:

THEOREM. *If $0 < \lambda < 1/e$, then $J(f_\lambda)$ is a Cantor set of curves in $\mathbb{C}$. There is a unique sink for f_λ and the basin of attraction of this point is open, dense and connected in $\mathbb{C}$.*

Remark. When $\lambda < 0$, the following situation occurs. If $0 > \lambda > -e$, there is a fixed sink as above. When $\lambda = -e$ we have a period doubling bifurcation and, for $\lambda < -e$, there is a periodic sink of period 2. The basin of this sink is open and dense in $\mathbb{C}$.

Remark. Let ζ be a point on the unit circle. The equations

$$f_\lambda(z) = \lambda\, e^z = z$$
$$f'_\lambda(z) = \lambda\, e^z = \zeta$$

define the set of parameters λ for which $\lambda\, e^z$ has an indifferent fixed point. It is clear that $\lambda = \zeta\, e^{-\zeta}$ defines a simple closed curve in the λ-plane for which a fixed point of $\lambda\, e^z$ is indifferent. We therefore have 'tongues' emanating from each point on this curve for which ζ is a rational angle wherein f_λ has a periodic sink. This suggests a bifurcation diagram for f_λ which looks locally like figure 7.

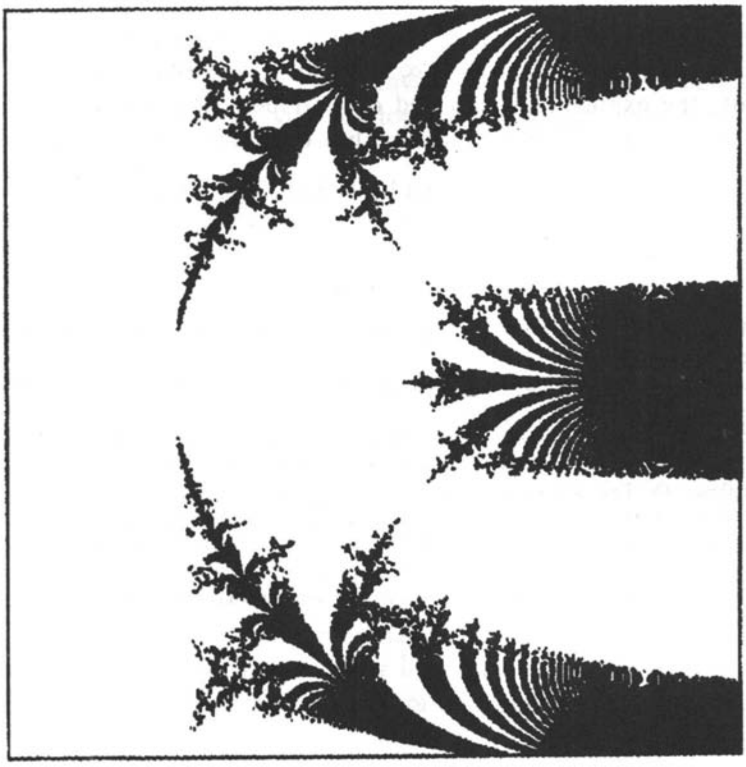

FIGURE 7. The bifurcation diagram in the λ-plane.

7. *Conclusion*

The dynamics of the family of maps λ exp (z) is clearly very complicated but very interesting. We conclude with several problems that we cannot solve at the present time.

Question 1. Is exp (z) ergodic?

Question 2. Given a non-repeating sequence, what can be said about the set of points which share this itinerary? Is it a curve as in the repeating sequence case? We have shown that the 'tail' is indeed a Lipschitz curve, but what happens nearer the imaginary axis?

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Question 3. What is the topological nature of the set of points which share the special itinerary 1, 1, 1, ... We have shown that this set contains a Cantor set of curves. Can one say more? Does this set have positive measure?

Question 4. Are the 'strings' associated to a given itinerary smooth? Are they analytic?

Finally, we propose that the study of entire functions seems to be a rich area for further study. There are significant differences between entire functions and rational maps. For example, entire functions may possess wandering domains and infinitely many sinks, both of which cannot happen for rational maps. The family

$$f(z) = z + \lambda \sin(z)$$

is easily seen to have examples of both phenomena, for particular values of λ. Moreover, the essential singularity at ∞ adds further complication to the dynamics. In summary, the exponential map and other important transcendental functions seem to be very interesting dynamically. They certainly warrant further study.

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