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ENDOMORPHISMS OF TYPE II_1-FACTORS AND CUNTZ ALGEBRAS

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Abstract

Any unital *-endomorphism of a type II_1-factor is implemented by isometries of a Cuntz algebra outside the factor. If the Jones index of the range of the *-endomorphism is an integer and the algebras act on the standard space, the Jones index must agree with the number of the generators of the Cuntz algebra. We also study (outer) conjugacy of *-endomorphisms using Cuntz algebras.

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1. Introduction

In [1], W. Arveson showed the following fact: Let α be a nonzero normal *-endomorphism of the algebra $B(H)$ of all bounded linear operators on a (separable) Hilbert space H. Then there is a (finite or infinite) sequence of isometries $v_1, v_2, \dots$ in $B(H)$ having mutually orthogonal ranges such that

$$\alpha(a) = \sum_n v_n a v_n^*, \quad a \in B(H).$$

The linear space of operators

$$E_\alpha = \{t \in B(H) : ta = \alpha(a)t \quad \text{for any} \quad a \in B(H)\}$$

is a Hilbert space relative to the inner product defined by

$$t^*s = \langle s, t\rangle 1, \qquad s, t \in E,$$

and $\{v_1, v_2, \dots\}$ is an orthonormal basis for E and generates an extension of a Cuntz algebra [5]. In this paper we shall show an analogous fact for type II_1-factors on

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a (separable) Hilbert space. We can describe *-endomorphisms of a type II_1-factor M using the relative position between a Cuntz algebra $\mathcal{O}_n$ and the type II_1-factor M. More precisely, any unital *-endomorphism α of a type II_1-factor M is implemented by isometries of a Cuntz algebra $\mathcal{O}_n = C^*(\{s_i : i = 1, \dots, n\})$ outside the factor M such that $\alpha(x) = \sum_{i=1}^n s_i x s_i^*$, $x \in M$. In particular if the Jones index $[M : \alpha(M)]$ is an integer and M acts on a Hilbert space standardly, then the Jones index $[M : \alpha(M)]$ must agree with the number of generators of the Cuntz algebra $\mathcal{O}_n$. In general we can choose the number n of generators of the Cuntz algebra $\mathcal{O}_n$ such that $n \leq 4\{(\text{the integer part of } [M : \alpha(M)]) + 1\}$. We study the condition of conjugacy and outer conjugacy for *-endomorphisms of a type II_1-factor which is analogous with results of Laca [10, Proposition 2.2, 2.3 and 2.4]. Using Cuntz algebras, Doplicher and Roberts [6] presented a new duality theory for compact groups. Among other things they obtained the following result [6, Theorem 7.1] which is related to ours: for the *-endomorphism α of a factor $\mathscr{A}$ with a permutation symmetry of dimension $d > 1$ satisfying the special conjugate property, there exists a factor $\mathscr{B}$, a group $\mathscr{G} \subset \operatorname{Aut}\mathscr{B}$ and a Cuntz algebra $\mathcal{O}_d = C^*(\{s_i : i = 1, \dots, d\}) \subset \mathscr{B}$ such that $\mathscr{A} = \mathscr{B}^{\mathscr{G}}$ and $\mathscr{B}$ is generated by $\mathscr{A}$ and $\mathcal{O}_d$. Furthermore $\alpha(x) = \sum_{i=1}^d s_i x s_i^*$, $x \in \mathscr{A}$. But, in Remark 4 below, we note that there exists a *-endomorphism α of a hyperfinite type II_1-factor R such that $[R : \alpha(R)] = 2$ and α does not satisfy permutation symmetry, but α still has the form $\alpha(x) = \sum_i s_i x s_i^*$, $x \in R$. So our result generalize a part of [6]. Crossed products by *-endomorphisms are discussed in [2, 6, 15]. Related results on *-endomorphisms are also investigated in [3, 4, 7, 12, 13].

2. Implementation of *-endomorphisms

In the following, we shall show that any unital *-endomorphism of a type II_1-factor is implemented by isometries of a Cuntz algebra outside the factor.

LEMMA 1. *Let M be a type II_1-factor acting standardly on $L^2(M, \mathrm{tr})$. Let α be a unital *-endomorphism of M such that $[M : \alpha(M)]$ is an integer $n = 1, 2, 3, \dots$. Then there exists a representation ρ of a Cuntz algebra $\mathcal{O}_n = C^*(\{v_i : i = 1, \dots, n\})$ into $B(L^2(M, \mathrm{tr}))$ such that*

$$\alpha(x) = \sum_{i=1}^{n} \rho(v_i) x \rho(v_i)^* \quad \text{for any} \quad x \in M,$$

where we identify $\mathcal{O}_1$ with the algebra $C(T)$ of continuous functions on the torus T, which is generated by a single unitary v_1.

PROOF. Let η be a canonical embedding of M into $L^2(M, \mathrm{tr})$. In the case $n = 1$, α is an automorphism and then Lemma 1 is well known. In fact, if we put $u\eta(x) = \eta(\alpha(x))$,

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for $x \in M$, then we can take v_1 as $\rho(v_1) = u$. Next we consider the case $n \geq 2$. Put $\alpha(M) = N$. Let E_N be the unique trace preserving conditional expectation of M onto N. Let e_N be the orthogonal projection of $L^2(M, \mathrm{tr})$ onto $L^2(N, \mathrm{tr})$ (which is the closure in $L^2(M, \mathrm{tr})$ of N). Then we can choose a Pimsner-Popa basis $\{m_j\}_{j=1,\dots,n}$ for $N \subset M$ such that $m_1 = 1$ and $E_N(m_i m_j^*) = \delta_{i,j}$. Therefore every $m \in M$ has a unique decomposition $m = \sum_{j=1}^n y_j m_j$ with $y_j \in N$, and $m_j^* e_N$ are partial isometries $(1 \leq j \leq n)$,

$$\sum_{j=1}^n m_j^* e_N m_j = 1, \qquad \sum_{j=1}^n m_j^* m_j = [M : N].$$

Using the above relations, we have the following: for any $x = \sum_{j=1}^n x_j m_j \in M$ and $y = \sum_{j=1}^n y_j m_j \in M$, $x_j, y_j \in N$,

$$\begin{aligned}\langle \eta(x) \mid \eta(y) \rangle &= \left\langle \eta\left(\sum_{j=1}^n x_j m_j\right) \middle| \eta\left(\sum_{j=1}^n y_j m_j\right)\right\rangle = \mathrm{tr}\left(\left(\sum_{j=1}^n m_j^* y_j^*\right)\left(\sum_{j=1}^n x_j m_j\right)\right) \\ &= \mathrm{tr}\left(\sum_{j,k=1}^n y_j^* x_k E_N(m_k m_j^*)\right) = \mathrm{tr}\left(\sum_{j=1}^n y_j^* x_j E_N(m_j m_j^*)\right) = \mathrm{tr}\left(\sum_{j=1}^n y_j^* x_j\right).\end{aligned}$$

We shall define an operator s_1 on $L^2(M, \mathrm{tr})$ by

$$s_1 \eta(x) = \eta(\alpha(x)) \quad \text{for} \quad x \in M.$$

Since

$$\langle s_1 \eta(x) \mid s_1 \eta(y) \rangle = \langle \eta(\alpha(x)) \mid \eta(\alpha(y)) \rangle = \mathrm{tr}(\alpha(y^* x)) = \langle \eta(x) \mid \eta(y) \rangle,$$

s_1 is an isometry. Furthermore, we have

$$s_1^* \eta\left(\sum_{i=1}^n x_i m_i\right) = \eta\left(\alpha^{-1}(x_1)\right).$$

Since $\|\eta(x)\|^2 = \sum_{i=1}^n \|\eta(x_j)\|^2$ for $x = \sum_{i=1}^n x_i m_i \in M$, $x_i \in N$, we can define a self-adjoint unitary operator v_i $(i \geq 2)$ on $L^2(M, \mathrm{tr})$ by permutating the first component and the i-th component; that is,

$$v_i \eta\left(\sum_{j=1}^n x_j m_j\right) = \eta\left(x_i m_1 + x_1 m_i + \sum_{j \neq 1, i} x_j m_j\right).$$

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Consider the isometries $s_i = v_i s_1$ for $i = 1, \dots, n$, where $v_1 = 1$. Then for $x \in M$ and $y = \sum_{j=1}^n y_j m_j$,

$$s_i x s_i^* \eta(y) = s_i x s_i^* \eta\left(\sum_{j=1}^n y_j m_j\right) = v_i s_1 x s_1^* v_i^* \eta\left(\sum_{j=1}^n y_j m_j\right)$$

$$= v_i s_1 x \eta\left(\alpha^{-1}(y_i)\right) = v_i s_1 \eta\left(x\alpha^{-1}(y_i)\right)$$

$$= v_i \eta\left(\alpha(x) y_i\right) = v_i \eta\left(\alpha(x) y_i m_1\right) = \eta\left(\alpha(x) y_i m_i\right).$$

Therefore we have

$$\sum_{i=1}^n s_i x s_i^* \eta(y) = \sum_{i=1}^n \eta\left(\alpha(x) y_i m_i\right) = \eta\left(\alpha(x)\left(\sum_{i=1}^n y_i m_i\right)\right) = \eta(\alpha(x)y).$$

Thus we have

$$\alpha(x) = \sum_{i=1}^n s_i x s_i^* \quad \text{on} \quad L^2(M, \mathrm{tr}).$$

Furthermore we have $\sum_{i=1}^n s_i s_i^* \eta(y) = \eta(y)$. So $\sum_{i=1}^n s_i s_i^* = 1$. Thus this family $\{s_i : i = 1, \dots, n\}$ of isometries generates a Cuntz algebra $\mathcal{O}_n$.

Next we shall investigate the non integer case.

LEMMA 2. *Let M be a type II_1-factor and α be a unital $*$-endomorphism of M. Assume that $[M : \alpha(M)] < \infty$. Then there exist a II_1-factor L with $L \supseteq M$ and a $*$-endomorphism β of L such that*

(1) *$[L : \beta(L)]$ is an integer.*
(2) *$\beta|_M = \alpha$.*
(3) *following diagram is a commuting square.*

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ \alpha(M) & \subset & \beta(L) \end{array}$$

PROOF. We put

$$n = 4\,((\text{the integer part of} \quad [M : \alpha(M)]) + 1)$$

and

$$\mu = \frac{n}{[M : \alpha(M)]}.$$

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Then $\mu \geq 4$. By [9], we can take the sequence of Jones projections $\{e_i\}_{i\geq 1}$ which satisfy the relations

$$e_i e_{i\pm 1} e_i = \frac{1}{\mu} e_i.$$

Set $M_\mu = \{e_1, e_2, e_3, \dots\}''$ and $N_\mu = \{e_2, e_3, \dots\}''$. Then there exists a *-endomorphism γ from M_μ onto N_μ such that

$$\gamma(e_k) = e_{k+1} \quad \text{for} \quad k = 1, 2, 3, \dots$$

Put $L = M \otimes M_\mu$ and $\beta = \alpha \otimes \gamma$. Then β is a unital *-endomorphism of L. It is clear that β satisfies (2) and (3). We shall calculate $[L : \beta(L)]$.

$$\begin{aligned}[L : \beta(L)] &= [M \otimes M_\mu : \beta(M \otimes M_\mu)] = [M \otimes M_\mu : \alpha(M) \otimes N_\mu] \\ &= [M : \alpha(M)][M_\mu : N_\mu] = [M : \alpha(M)]\mu = [M : \alpha(M)]\frac{n}{[M : \alpha(M)]} = n.\end{aligned}$$

Using Lemma 2, any unital *-endomorphism on a type II_1-factor can be described by Cuntz algebras as follows:

THEOREM 3. *Let M be a type II_1-factor and α be a unital *-endomorphism of M. Assume that $[M : \alpha(M)] < \infty$. Then there exist a natural number n, a *-representation ρ of M on a Hilbert space H and a *-representation π of a Cuntz algebra $\mathcal{O}_n = C^*(\{v_i : i = 1, \dots, n\})$ into $B(H)$ such that*

$$n \leq 4\,((\text{the integer part of}\ \ [M : \alpha(M)]) + 1)$$

and

$$\rho(\alpha(m)) = \sum_{i=1}^{n} \pi(v_i)\rho(m)\pi(v_i)^*, \quad m \in M.$$

PROOF. If the Jones index $[M : \alpha(M)]$ is an integer, by Lemma 1 we can take $n = [M : \alpha(M)]$. So we shall consider the case in which $[M : \alpha(M)]$ is a non-integer. Take L and β in Lemma 2. We put $H = L^2(L)$. Since $n = [L : \beta(L)]$ is a natural number and $n \geq 2$, by Lemma 1, we have a *-representation π of a Cuntz algebra $\mathcal{O}_n$ into $B(H)$ such that

$$\beta(x) = \sum_{i=1}^{n} \pi(v_i)x\pi(v_i)^* \quad \text{for all} \quad x \in L.$$

As $L \supseteq M$, we can take ρ to be the restriction of the GNS representation of L. That is,

$$\rho(m)\eta(x) = \eta(mx) \quad \text{for} \quad x \in L \quad \text{and} \quad m \in M.$$

Thus

$$\rho(\alpha(m)) = \sum_{i=1}^{n} \pi(v_i)\rho(m)\pi(v_i)^*.$$

REMARK 4. Let $a(i)$, $i = 0, \pm 1, \pm 2, \ldots$ be a sequence such that $a(i) = 1$ if $i = \pm 2$ and $a(i) = 0$ if $i \neq \pm 2$. Let $\{u_i\}_{i \geq 0}$ be a family of self-adjoint unitaries such that

$$u_i u_j = (-1)^{a(i-j)} u_j u_i, \quad i, j \geq 0.$$

Then $\{u_i\}$ generates a hyperfinite type II_1-factor R and the map $\alpha : u_i \mapsto u_{i+1}$ induces a *-endomrphism α of R. For this α, we have $\alpha^2(R)' \cap R = \mathbb{C}I$ [8]. But *-endomorphisms ρ of a von Neumann algebra with permutation symmetry must satisfy the relation: $\rho^2(M)' \cap M \neq \mathbb{C}I$ [6]. Thus such a *-endomorphisms α does not satisfy permutation symmetry. But even for this *-endomorphism α, it has the form $\alpha(x) = \sum_i s_i x s_i^*$, $x \in R$.

3. Conjugacy and outer conjugacy of *-endomorphisms

In [10], Laca considered conjugacy between two *-endomorphisms of a type I-factor (and also for outer conjugacy) using the related Cuntz algebras. In the following, we shall consider a version of type II_1 case.

PROPOSITION 5. *Let M be a type II_1-factor on a Hilbert space H and $\mathcal{O}_n = C^*(\{s_i : i = 1, \ldots, n\})$ be the Cuntz algebra on H. Let α be a unital *-endomorphism of M such that $\alpha(x) = \sum_{i=1}^n s_i x s_i^*$, $x \in M$. Put*

$$E_\alpha = \{x \in B(H) : xa = \alpha(a)x \quad \text{for every} \quad a \in M\}$$

and

$$F(s_1, \ldots, s_n) = \left\{ \sum_{i=1}^{n} s_i n_i : n_i \in M' \right\}.$$

Then we have the following:

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(1) *E_α and $F(s_1, \ldots, s_n)$ are right M'-modules.*
(2) *$E_\alpha = F(s_1, \ldots, s_n)$ as a set.*
(3) *As a right M'-module, E_α is the free M'-module of rank n.*

PROOF. (1) is clear. We show (2). For any $\sum_{i=1}^n s_i n_i \in F(s_1, \ldots, s_n)$,

$$\alpha(a)\left(\sum_{j=1}^n s_j n_j\right) = \left(\sum_{i=1}^n s_i a s_i^*\right)\left(\sum_{j=1}^n s_j n_j\right) = \sum_{j=1}^n s_j a n_j = \left(\sum_{j=1}^n s_j n_j\right) a$$

Therefore $F(s_1, \ldots, s_n) \subset E_\alpha$. On the other hand, take $t \in E_\alpha$. Then

$$ta = \alpha(a)t \quad \text{for any} \quad a \in M.$$

Then

$$s_j^* t a = s_j^* \left(\sum_{i=1}^n s_i a s_i^*\right) t \quad \text{for any} \quad j = 1, \ldots, n$$

and

$$s_j^* t a = a s_j^* t \quad \text{for any} \quad j = 1, \ldots, n.$$

Hence $s_j^* t = n_j \in M'$. Then we have

$$t = \left(\sum_{j=1}^n s_j s_j^*\right) t = \sum_{j=1}^n s_j n_j \in F(s_1, \ldots, s_n).$$

Hence $E_\alpha \subset F(s_1, \ldots, s_n)$. We show (3). In order to do this, it is sufficient to show that

$$\text{if} \quad \sum_{i=1}^n s_i n_i = 0 \quad \text{then} \quad n_i = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

This follows from

$$s_k^* \left(\sum_{i=1}^n s_i n_i\right) = n_k \quad \text{for} \quad k = 1, \ldots, n.$$

Therefore we have

$$F(s_1, \ldots, s_n) \simeq \sum_{i=1}^n \oplus M'.$$

Hence $E_\alpha \simeq \sum_{i=1}^n \oplus M'$. Thus E_α is a free M' module of rank n.

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By Lemma 1 and Theorem 6 below, when M acts on $L^2(M, \mathrm{tr})$ standardly and the index $[M : \alpha(M)]$ is an integer, the index $[M : \alpha(M)]$ of *-endomorphism α of M necessarily agrees with the number of generators of the Cuntz algebra $\mathcal{O}_n$ related to α.

THEOREM 6. *Let M be a type II_1-factor acting on a Hibert space H, with finite commutant. Let $\mathcal{O}_m = C^*(\{s_i : i = 1, \dots, m\})$ and $\mathcal{O}_n = C^*(\{t_j : j = 1, \dots, n\})$ be the Cuntz algebras on H generated by isometries $\{s_i\}_i$ and $\{t_j\}_j$ respectively. Let α and β be unital *-endomorphisms of M such that $\alpha(x) = \sum_{i=1}^m s_i x s_i^*$ and $\beta(x) = \sum_{j=1}^n t_j x t_j^*$,$x \in M$. Then the following are equivalent:*

(1) $\alpha = \beta$.

(2) *$m = n$ and there exists a unitary matrix $(u_{ij}) \in M' \otimes M_n(\mathbb{C})$ such that $t_i = \sum_{j=1}^n s_j u_{ji}$.*

PROOF. (1) implies (2): If (1) holds, by proposition 5, we have

$$F(s_1, \dots, s_m) = E_\alpha = E_\beta = F(t_1, \dots, t_n).$$

Since M' is also type II_1-factor, the K-group $K_0(M')$ of M' is the set R of all real numbers. Considering the map $\mathrm{tr}^\sim$ from $K_0(M')$ to R which is induced from the normalized trace tr of M', we have

$$\mathrm{tr}^\sim\left(\left[\sum_{i=1}^n \oplus M'\right]\right) = n.$$

Since $F(s_1, \dots, s_m) = F(t_1, \dots, t_n)$, we have $m = n$. Since $\alpha = \beta$,

$$s_i^* \left(\sum s_\ell x s_\ell^*\right) t_j = s_i^* \left(\sum t_k x t_k^*\right) t_j.$$

Hence $x(s_i^* t_j) = (s_i^* t_j)x$. So $s_i^* t_j \in M'$. Put $u = (u_{ij}) = (s_i^* t_j) \in M' \otimes M_n(\mathbb{C})$. Since $(u^* u)_{ij} = \sum (s_k^* t_i)^* s_k^* t_j = \sum t_i^* s_k s_k^* t_j = t_i^* t_j = \delta_{ij}$, and similarly $uu^* = 1$, u is a unitary matrix. On the other hand we have $t_i = (\sum_j s_j s_j^*) t_i = \sum s_j u_{ji}$.

(2) implies (1): Conversely, assume that (2) holds. Then

$$\begin{aligned}
\beta(x) &= \sum_{k=1}^n t_k x t_k^* = \sum_k \left(\sum_j s_j u_{jk}\right) x \left(\sum_{j'} s_{j'} u_{j'k}\right)^* \\
&= \sum_{j,j'} s_j x \left(\sum_k u_{jk} (u_{j'k})^*\right) s_{j'}^* = \sum_{j,j'} s_j x \delta_{j,j'} s_{j'}^* \\
&= \sum_j s_j x s_j^* = \alpha(x).
\end{aligned}$$

REMARK 7. Let M be a type II_1-factor acting on a Hilbert space H such that the commutant M' of M is not finite. Let $\mathcal{O}_n = C^*(\{s_i : i = 1, \dots, n\})$ be a Cuntz algebra on H. Assume that the *-endomorphism α of M has a form $\alpha(x) = \sum_{i=1}^n s_i x s_i^*$, $x \in M$. Since M' is not finite, there exist isometries $t_1, t_2 \in M'$ such that $C^*(\{t_1, t_2\})$ is a Cuntz algebra. Considering $2n$ isometries $\{s_i t_j : i = 1, \dots, n, j = 1, 2\}$, we have $\alpha(x) = \sum_{i,j} s_i t_j x t_j^* s_i^*$, $x \in M$. Thus if M' is not finite, the number of isometries which implement α depends on the choice of isometries, in general.

On the conjugacy of *-endomorphisms, we have the following:

THEOREM 8. *Let M be a type II_1-factor acting standardly on a Hilbert space H. Let $\mathcal{O}_m = C^*(\{s_i : i = 1, \dots, m\})$ and $\mathcal{O}_n = C^*(\{t_j : j = 1, \dots, n\})$ be the Cuntz algebras generated by the isometries $\{s_i\}_i$ and $\{t_j\}_j$ respectively. Let α and β be unital *-endomorphisms of M such that $\alpha(x) = \sum_{i=1}^m s_i x s_i^*$ and $\beta(x) = \sum_{j=1}^n t_j x t_j^*$, $x \in M$. Then the following are equivalent:*

(1) *α and β are conjugate.*
(2) *$m = n$ and there exist a unitary operator $w \in B(L^2(M, \mathrm{tr}))$ and a unitary matrix $(u_{ij}) \in M' \otimes M_n(\mathbb{C})$ such that $t_i = \sum_{j=1}^n (w s_j w^*) u_{ji}$.*

PROOF. (1) implies (2): Suppose that α and β are conjugate. Then there exists $\gamma \in \mathrm{Aut}(M)$ such that $\beta = \gamma\alpha\gamma^{-1}$. This γ gives rise to a unitary operator w on $L^2(M, \mathrm{tr})$ such that $w\eta(x) = \eta(\gamma(x))$ for $x \in M$. Using this w, we have $\sum_{j=1}^n t_j x t_j^* = \sum_{i=1}^m (w s_i w^*) x (w s_i w^*)^*$. Applying Theorem 6 for $\beta = \gamma\alpha\gamma^{-1}$, we have $m = n$ and a unitary matrix $(u_{ij}) \in M' \otimes M_n(\mathbb{C})$ such that $t_i = \sum_{j=1}^n (w s_j w^*) u_{ji}$.

(2) implies (1): Assume the condition (2). Then

$$\begin{aligned}\beta(x) &= \sum_k t_k x t_k^* = \sum_k \left(\sum_j w s_j w^* u_{jk}\right) x \left(\sum_{j'} (w s_{j'} w^*) u_{j'k}\right)^* \\ &= \sum_{j,j'} w s_j w^* x \left(\sum_k u_{jk} (u_{j'k})^*\right) (w s_{j'} w^*)^* = \sum_{j,j'} w s_j w^* x \delta_{j,j'} (w s_{j'} w^*)^* \\ &= \sum_j w s_j (w^* x w) s_j w^* = \gamma\alpha\gamma^{-1}(x).\end{aligned}$$

On the outer conjugacy of *-endomorphisms we have the following:

THEOREM 9. *Let M be a type II_1-factor acting standardly on a Hilbert space H. Let $\mathcal{O}_m = C^*(\{s_i : i = 1, \dots, m\})$ and $\mathcal{O}_n = C^*(\{t_j : j = 1, \dots, n\})$ be Cuntz algebras generated by isometries $\{s_i\}_i$ and $\{t_j\}_j$ on H respectively. Let α and β be unital*-endomorphisms of M such that $\alpha(x) = \sum_{i=1}^m s_i x s_i^*$ and $\beta(x) = \sum_{j=1}^n t_j x t_j^*$, $x \in M$. Then the following are equivalent:*

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(1) *α and β are outer conjugate.*
(2) *$m = n$ and there exist a unitary operator $w \in B(L^2(M, \mathrm{tr}))$, a unitary operator $v \in M$ and a unitary matrix $(u_{ij}) \in M' \otimes M_n(\mathbb{C})$ such that $t_i = \sum_{j=1}^n v(ws_jw^*)u_{ji}$.*

PROOF. (1) implies (2): Assume that α and β are outer conjugate. Then there exist a unitary operator $v \in M$ and $\gamma \in \mathrm{Aut}(M)$ such that $\beta = \mathrm{Ad}(v)\gamma\alpha\gamma^{-1}$. This γ gives rise to a unitary operator w on $L^2(M, \mathrm{tr})$ such that $w\eta(x) = \eta(\gamma(x))$ for $x \in M$. Using this w, we have

$$\sum_{j=1}^{n} t_j x t_j^* = \sum_{i=1}^{m} (vws_iw^*)x(vws_iw^*)^*.$$

Applying Theorem 6 for $\beta = \mathrm{Ad}(v)\gamma\alpha\gamma^{-1}$, we have $m = n$ and a unitary matrix $(u_{ij}) \in M' \otimes M_n(\mathbb{C})$ such that $t_i = \sum_{j=1}^n v(ws_jw^*)u_{ij}$.

(2) implies (1): Assume the condition (2). Then

$$\begin{aligned}\beta(x) &= \sum_k t_k x t_k^* = \sum_k \left(\sum_j vws_jw^*u_{jk}\right) x \left(\sum_{j'} vws_{j'}w^*u_{j'k}\right)^* \\ &= \sum_{j,j'} vws_jw^*x\left(\sum_k u_{jk}u_{j'k}^*\right)(vws_{j'}w^*)^* = \sum_{j,j'} vws_jw^*x\delta_{j,j'}(vws_{j'}w^*)^* \\ &= \sum_j vws_j(w^*xw)s_j^*w^*v^* = \mathrm{Ad}(v)\gamma\alpha\gamma^{-1}(x).\end{aligned}$$

4. Ergodic endomorphisms and shifts

A unital*-endomorphism α of a von Neumann algebra M is called *ergodic* if $\{a \in M : \alpha(x) = x\} = \mathbb{C}I$ and a *shift* if $\cap_{i\geq 0}\alpha^i(M) = \mathbb{C}I$. If α is a shift, then α is an ergodic endomorphism. In the following we shall investigate ergodicity of endomorphisms of a type II_1-factor M as in [10]. We shall use the following notation. We assume that a family of isometries $\{v_i\}_{i=1,\dots,n}$ generates a Cuntz algebra. Put $W_k = \{(i_1, \dots, i_k) : i_j = 1, \dots, n,\ j = 1, \dots, k\}$. For $\beta \in W_k$, $v_\beta = v_{i_1}v_{i_2}\dots v_{i_k}$. We denote $\mathscr{F}_n$ by the C^*-algebra generated by $\cup_k\{v_\beta v_\gamma^* : \beta, \gamma \in W_k\}$.

PROPOSITION 10. *Let M be a type II_1-factor on a Hilbert space H and π be a *-representation of a Cuntz algebra $\mathcal{O}_n = C^*(\{v_j : j = 1, \dots, n\})$ on H. Put $s_j = \pi(v_j)$. Let α be a unital *-endomorphism of M such that*

$$\alpha(x) = \sum_{i=1}^{n} s_i x s_i^*, \quad x \in M.$$

Let M^α be the fixed point algebra of M under α. Then we have the following:

(1) $M^\alpha = \pi(\mathcal{O}_n)' \cap M$.

(2) $\cap_{k\geq 0}\alpha^k(M) = \pi(\mathcal{F}_n)' \cap \cap_{k\geq 0}\{y \in M : \textit{for any} \quad \gamma \in W_k, s_\gamma^* y s_\gamma \in M\}$.

In particular α is a shift if $\pi(\mathcal{F}_n)' \cap M = \mathbb{C}I$.

PROOF. (1): We put a *-endomorphism σ of $B(H)$ such that

$$\sigma(x) = \sum_{i=1}^{n} s_i x s_i^* \quad \text{for} \quad x \in B(H).$$

Then we have $\sigma|_M = \alpha$. By Laca [10], $B(H)^\sigma = \pi(\mathcal{O}_n)' \cap B(H)$. Then $M^\alpha = B(H)^\sigma \cap M = \pi(\mathcal{O}_n)' \cap M$.

(2): Take $y \in \alpha^k(M)$. Then there exists $x \in M$ such that

$$y = \sum_{\tau\in W_k} s_\tau x s_\tau^*.$$

Take $\beta, \gamma \in W_k$.

$$s_\beta s_\gamma^* y = s_\beta s_\gamma^* \left(\sum_{\tau\in W_k} s_\tau x s_\tau^*\right) = s_\beta x s_\gamma^*.$$

On the other hand

$$y s_\beta s_\gamma^* = \left(\sum_{\tau\in W_k} s_\tau x s_\tau^*\right) s_\beta s_\gamma^* = s_\beta x s_\gamma^*.$$

Furthermore, for $\gamma \in W_k$,

$$s_\gamma^* y s_\gamma = x$$

So, $s_\gamma^* y s_\gamma \in M$. Thus we have

$$\alpha^k(M) \subseteq C^*\{s_\beta s_\gamma^* : \beta, \gamma \in W_k\}' \cap \{y \in M : \text{for any } \gamma \in W_k, s_\gamma^* y s_\gamma \in M\}.$$

Conversely, take

$$y \in C^*\{s_\beta s_\gamma^* : \beta, \gamma \in W_k\}' \cap \{y \in M : \text{for any } \gamma \in W_k, s_\gamma^* y s_\gamma \in M\}.$$

Then for $\tau_0 \in W_k$,

$$\alpha^k(s_{\tau_0}^* y s_{\tau_0}) = \sum_{\tau\in W_k} s_\tau (s_{\tau_0}^* y s_{\tau_0}) s_\tau^* = \left(\sum_{\tau\in W_k} s_\tau s_\tau^*\right) y = y.$$

So $y \in \alpha^k(M)$, that is,

$$C^*(s_\beta s_\gamma^* : \beta, \gamma \in W_k)' \cap \{y \in M : \text{for any } \gamma \in W_k, s_\gamma^* y s_\gamma \in M\} \subseteq \alpha^k(M).$$

Therefore

$$\alpha^k(M) = C^*(s_\beta s_\gamma^* : \beta, \gamma \in W_k)' \cap \{y \in M : \text{for any } \gamma \in W_k, s_\gamma^* y s_\gamma \in M\}.$$

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Next remark describes a fact on a *-endomorphism of a type I-factor which was already obtained by Price [14, Definition 2.1].

REMARK 11. Let M be the algebra $B(H)$ of all bounded linear operators on a Hilbert space H. Let $C^*(\{s_i : i = 1, \ldots, n\})$ be a Cuntz algebra generated by isometries $\{s_i\}_i$. Let α be a unital *-endomorphism of M such that $\alpha(x) = \sum_{i=1}^n s_i x s_i^*$, $x \in M$. Then we have

$$[M : \alpha(M)] = n^2.$$

References

[1] W. Arveson, 'Continuous analogues of Fock space', *Memoirs Amer. Math. Soc.* 409 (1989).

[2] S. Boyd, N. Keswani and I. Raeburn, 'Faithful representations of crossed products by endomorphisms', *Proc. Amer. Math. Soc.* **118** (1993), 427–436.

[3] D. Bures and H. S. Yin, 'Outer conjugacy of shifts on the hyperfinite II_1 factor', *Pacific J. Math.* **142** (1990), 245–257.

[4] M. Choda, 'Shifts on the hyperfinite II_1-factor', *J. Operator Theory* **17** (1987), 223–235.

[5] J. Cuntz, 'Simple C^*-algebras generated by isometries', *Comm. Math. Phys.* **57** (1977), 173–185.

[6] S. Doplicher and J. E. Roberts, 'Endomorphisms of C^*-algebras, cross products and duality for compact groups', *Ann. of Math.* (2) **130** (1989), 75–119.

[7] M. Enomoto, M. Nagisa, Y. Watatani and H.Yoshida, 'Relative commutant algebras of Powers' binary shifts on the hyperfinite II_1 factor', *Math. Scand.* **68** (1991), 115–130.

[8] M. Enomoto and Y. Watatani, 'Powers' binary shifts on the hyperfinite factor of type II_1', *Proc. Amer. Math. Soc.* **105** (1989), 371–374.

[9] V. Jones, 'Index for subfactors', *Invent. Math.* **72** (1983), 1–25.

[10] M. Laca, 'Endomorphisms of $B(H)$ and Cuntz algebras', preprint, 1991.

[11] M. Pimsner and S. Popa, 'Entropy and index for subfactors', *Ann. Sc. École Norm. Sup.* **19** (1986), 57–106.

[12] R. T. Powers, 'An index theory for semigroups of *-endomorphisms of $B(H)$ and type II_1 factors', *Canad. J. Math.* **40** (1988), 86–114.

[13] G. Price, 'Shifts on type II_1 factors', *Canad. J. Math.* **39** (1987), 493–511.

[14] ———, 'Endomorphisms of certain operator algebras', *Publ. Res. Inst. Math. Sci.* **25** (1989), 45–57.

[15] P. J. Stacey, 'Crossed products of C^*-algebras by *-endomorphisms', *J. Austral. Math. Soc. (Series A)* **54** (1993), 204–212.

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