Canad. Math. Bull. Vol. **19** (1), 1976

DERIVED FUNCTORS OF TORSION

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ABSTRACT. In this note we compute the derived functors of "torsion submodule" using a certain duality result for semi-exact functors on an abelian category of finite global dimension.

Given a left-exact functor on a category of R-modules the program of homological algebra tells us to compute the right derived functors. These functors measure the obstruction to right exactness in a well-known fashion. One candidate for computation is the operation of taking the torsion submodule of a module over an integral domain. The object of this note is to compute these right derived functors for an arbitrary commutative integral domain. We use a certain duality result between left and right derived functors on an abelian category of finite global dimension.

We establish some terminology and notation. R will always denote a commutative integral domain with identity. If M is an R-module, and p is in R, we define a functor:

$$p\text{-tor}_R : R\text{-Mod} \to R\text{-Mod}$$

by:

$$p\text{-tor}_R(M) = \{x \in M : \exists k \geq 0\ p^k x = 0\},$$

called *p-torsion*. Similarly we have:

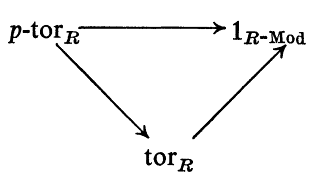
$$\text{tor}_R : R\text{-Mod} \to R\text{-Mod}$$

defined by:

$$\text{tor}_R(M) = \{x \in M : \exists r \in R rx = 0\},$$

called *torsion*.

It is easily checked that both are left-exact functors and there exists a commutative diagram of natural transformations:



using the obvious inclusion maps.

We now prove the basic lemma:

LEMMA. *Let* $\mathfrak{A}$ *be an abelian category with enough projectives and injectives of global dimension* n, $1 \leq n < \infty$, *and* $\mathfrak{B}$ *an arbitrary abelian category. Let* $F : \mathfrak{A} \to \mathfrak{B}$

https://doi.org/10.4153/CMB-1976-008-5 Published online by Cambridge University Press

be a right-exact functor. Then $L_kF:\mathfrak{A}\to\mathfrak{B}$ *vanishes on injectives*, $0\le k<n$, *if and only if there exist natural equivalences*:

$$R^k(L_nF)\xrightarrow{\approx}L_{n-k}F \qquad 0<k\le n.$$

Proof. One direction follows trivially from the fact that positive right-derived functors vanish on injectives. Conversely, let

$$0\to A'\to A\to A''\to 0$$

be a short exact sequence in $\mathfrak{A}$. We have the following diagram in $\mathfrak{B}$:

$$\begin{array}{ccccccccccccccc}
0\to & L_nFA' & \to & L_nFA & \to & L_nFA'' & \to & L_{n-1}FA' & \to & L_{n-1}FA & \to & L_{n-1}FA'' & \to\cdots \\
 & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \vdots & & \vdots & & \vdots & \\
0\to & L_nFA' & \to & L_nFA & \to & L_nFA'' & \to & R^1L_nFA' & \to & R^1L_nFA & \to & R^1L_nFA'' & \to\cdots
\end{array}$$

$$\begin{array}{ccccccc}
\to & FA' & \to & FA & \to & FA'' & \to 0 \\
 & \vdots & & \searrow & & \searrow & \\
\to & R^nL_nFA' & \to & R^nL_nFA & \to & R^nL_nFA'' & \to 0
\end{array}$$

The top row is the long exact homology sequence of left derived functors of F and the bottom row is the long exact cohomology sequence of right derived functors of L_nF. The assumption on the global dimension of $\mathfrak{A}$ ensures:

$$L_{n+1}F=0=R^{n+1}L_nF.$$

Combining the assumption of the L_kF's and the standard technique of dimension-shifting (see [1], p. 164), we obtain the desired isomorphisms and their naturality.

REMARKS 1. This Lemma seems to indicate that $L_{\mathrm{gl.dim}(\mathfrak{A})}F$ can, in some sense, be construed as the "dual" of F, analogous to the notion of duality in topological manifolds.

2. There exists a dual proposition for left-exact functors and the proof is immediate.

3. We can replace the assumption on the global dimension of $\mathfrak{A}$ by the much milder hypothesis $L_{n+1}F=0$ (which implies $R^{n+1}L_nF=0$ by another simple dimension-shifting argument). It is this form of the Lemma which we refer to below.

We proceed to our computations.

COROLLARY 1. *Let R be an arbitrary commutative integral domain and M an R-module. Then the right-derived functors of p-tor$_R$ are;*

$$R^q(p\text{-tor}_R)(M)=\begin{cases}\varinjlim(M\otimes_R R/p^nR) & \text{if } q=1\\ 0 & \textit{otherwise}\end{cases}$$

Proof. Let $n=1$ and $F(M)=M\otimes_R(\varinjlim R/p^nR)$ in the Lemma. Since F

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is right-exact $L_0F=F$ and F vanishes on injectives since the tensor product of a divisible module and a torsion module is zero. As in Remark 3, we must check that:

$$\operatorname{Tor}_2^R(-, \varinjlim R/p^nR) = 0.$$

Since Tor_2^R commutes with direct limits it suffices to show:

$$\operatorname{Tor}_2^R(-, R/p^nR) = 0.$$

The familiar short exact sequence:

$$0 \to R \to R \to R/p^nR \to 0$$

induces:

$$\cdots \to \operatorname{Tor}_2^R(-, R) \to \operatorname{Tor}_2^R(-, R/p^nR) \to \operatorname{Tor}_1^R(-, R) \to \cdots$$

Since R is flat the extreme terms are zero and the above claim follows.

Now we observe the well-known fact:

$$p\text{-tor}_R(M) = \operatorname{Tor}_1^R(M, \varinjlim R/p^nR)$$

from which the result follows.

COROLLARY 2. *Let $f: M \to N$ be an epimorphism of R-modules. Then;*

$$p\text{-tor}_R(f): p\text{-tor}_R(M) \to p\text{-tor}_R(N)$$

is epic if:

$$\varinjlim(\ker(f) \underset{R}{\otimes} R/p^nR) = 0.$$

Proof. Look at the short exact sequence:

$$0 \to \ker(f) \to M \to N \to 0$$

and then the induced long exact cohomology sequence of right derived functors of p-tor_R. The rest of the proof is standard.

For the functor $\text{tor}_R: R-\text{Mod} \to R-\text{Mod}$ we have:

COROLLARY 3. *Let Q be the field of fractions of R. Then the right derived functors of* tor_R *are:*

$$R^q(\text{tor}_R)(M) = \begin{cases} M \underset{R}{\otimes} Q/R & \text{if } q = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Proceed as in Corollary 1 and observe:

$$\text{tor}_R(M) = \operatorname{Tor}_1^R(M, Q/R).$$

Also one must check: $\operatorname{Tor}_2^R(-, Q/R)=0$. Consider the short exact sequence:

$$0 \to R \to Q \to Q/R \to 0$$

which induces:

$$\cdots \to \operatorname{Tor}_2^R(-, Q) \to \operatorname{Tor}_2^R(-, Q/R) \to \operatorname{Tor}_1^R(-, R) \to \cdots$$

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since Q and R are flat as R-modules, the extreme terms are zero. The rest of the proof follows Corollary 1.

COROLLARY 4. *Let $f: M \to N$ be an epimorphism of R-modules. Then;*

$$\mathrm{tor}_R(f): \mathrm{tor}_R(M) \to \mathrm{tor}_R(N)$$

is epic if:

$$\ker(f) \otimes_R Q/R = 0.$$

Proof. Proceed as in Corollary 2.

REFERENCE

1. J. Rotman, *Notes on Homological Algebra*, Van Nostrand Reinhold, 1970.

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https://doi.org/10.4153/CMB-1976-008-5 Published online by Cambridge University Press