EMBEDDING A SEMIGROUP OF TRANSFORMATIONS

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Let $X$ be an arbitrary set and $\theta$ a transformation of $X$. One may use $\theta$ to induce an associative operation in $\mathcal{F}_X$, the set of all mappings of $X$ to itself as follows:

$$\alpha \ast \beta = \alpha \theta \beta \quad (\alpha, \beta \in \mathcal{F}_X).$$

We denote the resulting semigroup by $(\mathcal{F}_X; \theta)$. Magill (1967) introduced this structure and it has been studied by Sullivan and by myself.

Sullivan asks when $(\mathcal{F}_X; \theta)$ can be embedded in $(\mathcal{F}_X, \circ)$, the full transformation semigroup under composition. He shows that if $X$ is finite this can be done if and only if $\theta$ is a permutation of $X$ and that any embedding (an isomorphism perforce) is of the form

$$\alpha \rightarrow g^{-1} \theta g \quad (\alpha \in \mathcal{F}_X)$$

where $g$ is a permutation of $X$. The infinite case is left open. The purpose of this note is to prove the following.

**Theorem 1.** If $X$ is infinite then any $(\mathcal{F}_X, \theta)$ may be embedded in $(\mathcal{F}_X, \circ)$.

**Proof.** Let $X = X_E \cup X_0$ where $X_0$ and $X_0$ are disjoint and of the same cardinality, necessarily that of $X$. Select bijections $g$ and $h$ such that

$$h: X \rightarrow X_E \quad \text{and} \quad g: X \rightarrow X_0.$$

For $\alpha$ in $\mathcal{F}_X$ we define $\alpha \phi$ as follows

$$x \alpha \phi = x h^{-1} \alpha g \quad (x \in X_E)$$

$$= x g^{-1} \theta \alpha g \quad (x \in X_0).$$

It is clear that the first part of the definition guarantees that $\alpha \rightarrow \alpha \phi$ is one to one. Observe that $\alpha \phi: X \rightarrow X_E = X_0$. It follows that for $\alpha$ and $\beta$ in $\mathcal{F}_X$ we have that

$$\alpha \phi \beta \phi = \alpha \phi g^{-1} \theta \beta g. \quad \text{Thus if } x \text{ is in } X_E,$$
Embedding a semigroup

\[ xx\beta \phi = xh^{-1}xg \cdot g^{-1} \theta \beta g = xh^{-1}x\theta \beta g = x(x \ast \beta)\phi \]

while if \( x \) is in \( X_0 \),

\[ xx\phi \beta \phi = xg^{-1}x\theta xg \cdot g^{-1} \theta \beta g = xg^{-1}x\theta \beta g = (x \ast \beta)\phi. \]

It follows that \((x \ast \beta)\phi = x\phi \beta \phi\), as required.

The classification of the embeddings of \((T_x; \theta)\) in \((T_x, \circ)\) is extremely difficult. Partial results have been obtained. We shall describe our most pleasant result in this direction.

We call a transformation semigroup \( S \subseteq T_x \) irreducible if the set

\[ xS = \{xx; x \in S\} \]

coincides with \( X \) for each \( x \) in \( X \). Further, we shall say than an embedding \( \phi \) of \((T_x; \theta)\) in \((T_x, 0)\) is irreducible if \( T_x \phi \) is irreducible.

**Theorem 2.** Any irreducible embedding of \((T_x; \theta)\) in \((T_x, \circ)\) is of the form

\[ x \rightarrow g^{-1} \theta xg \quad (x \in T_x) \]

for some fixed permutation \( g \) of \( X \).

**Proof.** Assume \( \phi \) is such an embedding and let \( \kappa = \kappa_x \) denote the constant function in \( T_x \) with range \( x \). We choose \( y \) in the range of \( \kappa \phi \) and consider \( y(\kappa \phi)^{-1} \).

If \( z \) is any member of this latter set then for any \( x \) in \( T_x \)

\[ (zx \phi) \kappa \phi = z(\kappa \phi) \phi = z \kappa \phi = y. \]

This shows that \( y(\kappa \phi)^{-1} \) is invariant under \( T_x \phi \) and hence, by irreducibility, coincides with \( X \). This shows that \( \phi \) maps constants to constants from which follows

\[ \kappa_x \phi = \kappa_x g \quad (x \in X) \]

where \( g \) is an injective transformation of \( X \). But then for each \( x \)

\[ \kappa_x \theta xg = (\kappa_x \ast x)\phi = \kappa_x \phi x\phi \]

which implies \( \theta xg = gx \phi \). Thus \( X g x \phi \subseteq X g \), and this is contrary to irreducibility unless \( g \) is onto. In this case \( g \) permutes \( X \) and \( \alpha \phi = g^{-1} \theta xg \), as required.

It is clear that \( \phi \) above is an embedding if and only if \( \theta \) is onto \( X \). Hence we have the following:

**Corollary.** It is possible to irreducibly embed \((T_x; \theta)\) in \((T_x, \circ)\) if and only if \( \theta \) is onto.
References


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