Proceedings of the Edinburgh Mathematical Society (2001) **44**, 285–294 ©

ASPHERICAL LABELLED ORIENTED TREES AND KNOTS

GÜNTHER HUCK[1] AND STEPHAN ROSEBROCK[2]

[1] *Department of Mathematics, Northern Arizona University, Flagstaff, AZ 86011, USA* (gph@math.nau.edu)

[2] *Fakultät für Mathematik, University of Education, Bismarckstr. 10, D-76133 Karlsruhe, Germany* (rosebrock@PH-karlsruhe.de)

(Received 5 May 1999)

Abstract The question of whether ribbon-disc complements—or, equivalently, standard 2-complexes over labelled oriented trees—are aspherical is of great importance for Whitehead's asphericity conjecture and, if solved affirmatively, would imply a combinatorial proof of the asphericity of knot complements. We present here two classes of diagrammatically reducible labelled oriented trees.

Keywords: 2-complexes; asphericity; knot groups; labelled oriented trees

AMS 2000 *Mathematics subject classification:* Primary 57M20; 20F06; 20F05; 20F32

1. Introduction and main results

Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a finite presentation where each relator is of the form $x_i = x_j x_k x_j^{-1}$. Such a presentation is sometimes called a *labelled oriented graph* (LOG), because it may be represented by a graph T_P in the following way. For each generator x_i of P define a vertex labelled i (or x_i), and for each relator $x_i = x_j x_k x_j^{-1}$ define an edge oriented from the vertex i to the vertex k labelled by j. If T_P is a tree, then P or T_P is called a *labelled oriented tree* (LOT) (see [**4**] and [**5**]).

It is well known that the standard 2-complexes K_P modelled on LOT presentations P are spines of complements of ribbon disks in four-space and that every ribbon-disk complement has such a spine (see [**5**]). Therefore, the conjecture that all 2-complexes K_P modelled on LOTs are aspherical is commonly called the 'ribbon-disk conjecture'. In two-dimensional homotopy theory this conjecture represents a subcase of the Whitehead conjecture stating that all subcomplexes of aspherical 2-complexes are aspherical. This follows from the observation that, by adding one extra relation of the form $x_i = 1$ (where x_i is any one of the generators) to a LOT presentation, one obtains a contractible and hence aspherical 2-complex. Moreover, it was pointed out by Howie in [**4**] that the ribbon-disk conjecture together with the Andrews–Curtis conjecture implies the Whitehead conjecture for all finite 2-complexes. Since Wirtinger spines of classical knots are modelled on LOT presentations, a combinatorial proof of the ribbon-disk conjecture would also comprise a combinatorial proof of the asphericity of knot complements.

We present two classes of aspherical LOTs in this paper. The statement of the results relies on the following definitions.

We call a LOG P in our context *reducible* if there exists a generator in P that occurs exactly once among the set of relators. For T_P this means there is a vertex of valence one, which does not occur as an edge label. A LOG which is not reducible is called *reduced* (note the change of terminology from [**6**]).

A LOG is called *compressed* if every relator contains three different generators. If a LOT is not compressed or reduced, it may be transformed by simple homotopy operations into a compressed and reduced LOT.

A LOG P is called *injective* if each vertex of T_P occurs at most once as an edge label.

Let $\mathcal{P}$ be the class of all LOGs P, where the corresponding graph T_P does not contain a cycle, i.e. where T_P is a forest. This class certainly contains all LOTs.

Our main result is the following theorem.

Theorem 1.1. *Let $P \in \mathcal{P}$ be compressed and injective. If T_P does not contain a subtree that is a reducible LOT, then K_P is diagrammatically reducible.*

A 2-complex K is called *diagrammatically reducible* (DR) if every spherical diagram over K can be reduced by a folding operation (for details see the next section). Following [**1**], a 2-complex K is called *diagrammatically aspherical* (DA) if each spherical diagram over P can be converted by diamond moves to one that can be reduced by a folding operation (as in the definition of DR). DR implies DA, which in turn implies aspherical. It is known that not all standard 2-complexes modelled on compressed LOTs are DA (see [**10**]). However, the known examples of LOTs that are not DR are all non-injective.

The Wirtinger presentation P read from the projection of a tame knot (with one relation suppressed) is a LOT presentation and its 2-complex K_P is a spine of the knot complement. We will call it the 'Wirtinger spine'. Weinbaum gives a combinatorial proof of the asphericity of complements of alternating knots for the 'Dehn complex', which is a different spine of a knot complement [**11**] (see also [**7**]). As a corollary to Theorem 1.1 we obtain Weinbaum's result for the Wirtinger spine in a different way.

Theorem 1.2. *If a knot in the 3-sphere admits a tame alternating projection, then its complement has a Wirtinger spine coming from a LOT that is DA.*

Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a LOG. To change the orientation of the edge corresponding to one of its relations $r_t : x_i = x_j x_k x_j^{-1}$ will mean to replace r_t by $r'_t : x_k = x_j x_i x_j^{-1}$. This is the same as changing the orientation of the corresponding edge in T_P. An *orientation* of a LOG P is a LOG that arises from P by changing the orientations of a (possibly empty) subset of edges of T_P.

Theorem 1.3. *For any LOG $P \in \mathcal{P}$ there is an orientation Q of P such that K_Q is diagrammatically reducible.*

In terms of ribbon-disk complements, such a change of orientation of a LOT has the following effect. The ribbon disk is modified by locally changing some of the ribbon intersections, as shown in Figure 1.

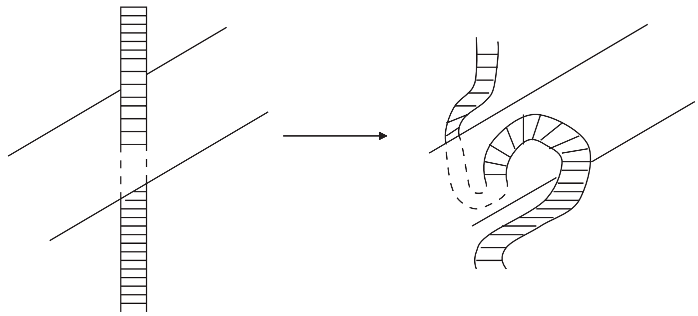

Figure 1. Reversing the orientation of an edge.

2. Some graphs

All graphs are assumed to be finite. Multiple edges are allowed. For a graph G let $V(G)$ be the set of vertices and $E(G)$ the set of edges of G. We sometimes write $G = (V, E)$, if $E = E(G)$ and $V = V(G)$. For any finite set S let $|S|$ be the number of elements of S.

Let P be any finite presentation. The *Whitehead graph* W_P is the boundary of a regular neighbourhood of the only vertex of K_P with the induced cell decomposition. It is a non-oriented graph consisting of two vertices x_i^+, x_i^- for each generator x_i of P, which correspond to the beginning and the end of the oriented loop labelled x_i in K_P, respectively, and t edges for each relator of length t in P. These edges are the intersections of the corners of the 2-cell with the boundary of the regular neighbourhood, briefly called the 'corners'. If P is a LOG, then each relation $x_i x_j x_k^{-1} x_j^{-1} = 1$ contributes four edges (or corners) to $W_P : (x_i^-, x_j^+)$, (x_k^-, x_j^-), (x_k^+, x_j^-) and (x_i^+, x_j^+).

The *left graph* L_P is the full subgraph of W_P with vertex set $V(L_P) = \{x_i^+ \mid \forall i\}$. The *right graph* R_P is the full subgraph of W_P with vertex set $V(R_P) = \{x_i^- \mid \forall i\}$. If x_i^+ (x_i^-) is a vertex of $V(L_P)$ ($V(R_P)$), we will often write i or x_i for it. If $x_i = x_j x_k x_j^{-1}$ is a relator of P, then this relator induces an edge between i and j in L_P and between j and k in R_P.

We consider piecewise linear maps $f : C \to K_P$, where C is a cellular decomposition of the 2-sphere and P is a LOG. If open cells are mapped homeomorphically to open cells, then f is called a *spherical diagram over* P. f is called *reducible* if there is a pair of 2-cells in C having a boundary edge t in common and being mapped onto the same 2-cell in K_P by folding over t. A 2-complex K is called *diagrammatically reducible* (DR) if each spherical diagram over K is reducible (for details see [**3**] or [**10**]).

The following result is well known (see, for example, [**5**]).

Lemma 2.1. *Let P be a LOG. If L_P or R_P is a forest, then K_P is DR.*

Proof. Let $f : C \to K_P$ be a spherical diagram over a LOG presentation P. There is a covering space $\overline{K}_P$ of K_P induced by the homomorphism that identifies all generators. Its vertices may be enumerated by the integers. The lift of f to $\overline{K}_P$ has to have a minimum. The boundary of a regular neighbourhood of the pre-image of this minimum in C, which is a circle, maps to L_P. Since L_P is a forest, f is reducible.

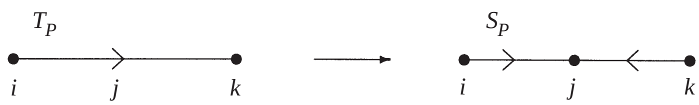

Figure 2. From T_P to S_P.

Related arguments apply to the maximum of the lift of f to $\overline{K}_P$ and R_P. □

Lemma 2.2. *Let P be a compressed injective LOG. If L_P and R_P are forests, then any orientation Q of P is DR.*

Proof. A standard 2-complex K_Y is said to satisfy the *weight test* if the edges of the Whitehead graph W_Y can be given real-valued weights satisfying the following conditions:

(1) the sum of weights of every reduced circuit (i.e. closed path without backtracking) in W_Y is $\geqslant 2$; and

(2) for every 2-cell D of K_Y whose boundary consists of d edges, the sum of the weights of the edges of W_Y that correspond to the corners of D is less than or equal to $d-2$.

A 2-complex that satisfies the weight test is DR (see [**3**] or [**6**]).

The standard 2-complex of the compressed injective LOG P satisfies the weight test by giving the edges of W_P the following weights: the edges of L_P and R_P get weight 0, all the other edges get weight 1. The lemma now follows from the following result.

Theorem 2.3. *[see [**6**]] Let P be a compressed injective LOG which satisfies the weight test. Then any LOG which is an orientation of P also satisfies the weight test.*

□

The graph S_P, defined below, will be instrumental in the proof of Theorem 1.1. Let P be a LOG. The directed graph S_P is defined as follows: $V(S_P) = V(T_P)$ and each edge from i to k labelled by j of T_P gives rise to two oriented edges in S_P, one going from i to j and the other from k to j (see Figure 2).

It is easy to see that if Q is an orientation of P, then $S_P = S_Q$. If $r_t : x_i = x_j x_k x_j^{-1}$ is a relator of P, then we have seen that this relator induces an edge between i and j in L_P and between j and k in R_P. In S_P there are also edges between i and j and between j and k induced from r_t. So L_P and R_P are subgraphs of S_P. Furthermore it is easy to see that if we identify x_i^+ of L_P with x_i^- of R_P we get S_P without the orientation of the edges of S_P.

3. Proof of Theorem 1.1

For a compressed injective LOG in $\mathcal{P}$ which does not contain a reducible sub-LOT, we will find an orientation where the corresponding left and right graph are forests. Then Lemma 2.2 implies Theorem 1.1.

Let G be a directed graph such that for every vertex v the number of edges ending in v (i.e. being oriented towards v) is even. An *admissible partition* of $E(G)$ is a partition into two sets E_1 and E_2 such that for every vertex $v \in V(G)$ the following holds: the number of edges of E_1 ending in v is equal to the number of edges of E_2 ending in v.

For any injective LOG P, it is clear that S_P is a directed graph such that zero or two edges end at every vertex.

Lemma 3.1. *Let P be an injective LOG. Assume there is an admissible partition of the edge-set of S_P into two sets E_1 and E_2. Then there is an orientation Q of P, such that $L_Q = (V(S_P), E_1)$ and $R_Q = (V(S_P), E_2)$.*

Proof. Let P be an injective LOG and E_1 and E_2 be an admissible partition of S_P. Let j be a vertex of S_P where two edges, e_1 and e_2, end. Let $e_1 = (i, j) \in E_1$ and $e_2 = (k, j) \in E_2$. e_1 and e_2 correspond to the two halves of an edge e of T_P with endpoints i and k and label j. We now choose the orientation of e in T_Q to be from i to k. Hence, $e_1 = (i, j)$ will belong to L_Q and $e_2 = (k, j)$ to R_Q. If we do this for every vertex j of S_P where two edges end, we obtain the desired orientation Q of P. □

So in order to prove Theorem 1.1 all we have to do is find an admissible partition of S_P into sets E_1 and E_2 such that $G_l = (V(S_P), E_1)$ and $G_r = (V(S_P), E_2)$ are forests. Then we have found an orientation Q of P for which $L_Q = G_l$ and $R_Q = G_r$ are forests, which implies, by Lemma 2.2, that K_P is DR.

Lemma 3.2. *Let $P \in \mathcal{P}$ be injective. Assume there is a $k \in \mathbb{N}$ and a subgraph $H \subset S_P$ with $|V(H)| = k$ and $|E(H)| \geqslant 2k - 1$. Then T_P contains a subtree that is a reducible LOT.*

Proof. Let $V(H) = \{x_1, \dots, x_k\}$. Since P is injective, maximally two edges of S_P can end at the same vertex. Therefore H could maximally have $2k$ edges; however, it is easy to see that $E(H) = 2k$ would imply that T_P has a subgraph with k vertices and k edges, contradicting the fact that T_P has no cycles. Hence, $|E(H)| = 2k - 1$ and the set of $2k - 1$ edges of H partitions into $k - 1$ pairs ending (without loss of generality) in the $k - 1$ vertices $x_1, \dots, x_{k-1}$ and one additional edge ending in x_k. Each of the $k - 1$ pairs of edges corresponds to an edge in T_P with label in $\{x_1, \dots, x_{k-1}\}$ and endpoints in $\{x_1, \dots, x_k\}$. Since T_P is a forest, these $k - 1$ edges in T_P connecting k vertices must form a tree T' which is a sub-LOT. The additional edge of H that ends in x_k corresponds to an edge e in T_P with label x_k and exactly one endpoint in $\{x_1, \dots, x_k\}$. Therefore, $T' \cup e$ is a tree that corresponds to a reducible sub-LOT of T_P. □

Theorem 3.3. *Let G be a directed graph such that zero or two edges end at every vertex. Then there is an admissible partition of G into two forests if and only if for all subgraphs H of G it follows that $|E(H)| < 2|V(H)| - 1$.*

Proof. Assume there is a subgraph $H \subset G$ with $|V(H)| = k$ and $|E(H)| \geqslant 2k-1$. Any partition of G into two forests induces a partition of H into two forests. Since $|V(H)| = k$ each of them has at most $k - 1$ edges in contradiction to $|E(H)| \geqslant 2k - 1$.

https://doi.org/10.1017/S0013091599000474 Published online by Cambridge University Press

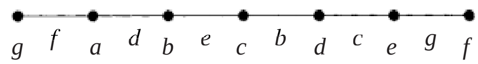

Figure 3. A compressed injective LOT that does not satisfy the weight test.

Now assume that for all subgraphs H of G it follows that $|E(H)| < 2|V(H)| - 1$. For $X \subset V(G)$, let $\overline{X} = V(G) - X$. There are vertices $\{x_1, \dots, x_j\} \subset V(G)$ where no edges end, called the *roots* of G. We use a result of Edmonds [**2**] (see also [**8**]) which can be rephrased in our context as follows.

> Let G be a directed graph such that zero or two edges end at every vertex. If for any non-empty $X \subset V(G)$ which does not contain any of the roots of G, there are at least two edges having their head in X and tail in $\overline{X}$ then there is an admissible partition of G into two forests.

So it remains to show that for any non-empty $X \subset V(G)$ which does not contain the roots of G, there are at least two edges having their head in X and tail in $\overline{X}$. Suppose there is an $X \subset V(G)$, $|X| = k$, with none of the root vertices belonging to X such that at most one edge of G has its head in X and tail in $\overline{X}$. Since X contains no roots, there are exactly $2k$ edges having their head in X. At most one of them has its tail in $\overline{X}$. Hence there are $2k - 1$ edges with heads and tails in X which form a subgraph $H \subset G$ with $X = V(H)$ and $|E(H)| = 2|V(H)| - 1$, in contradiction to our assumption above. □

Now we have all the parts of the proof of Theorem 1.1. Since T_P does not contain a subtree which is a reducible LOT we know by Lemma 3.2 that for all subgraphs H of S_P it follows that $|E(H)| < 2|V(H)| - 1$. Theorem 3.3 then gives us an admissible partition of S_P into two forests. By Lemma 3.1 these forests are the left and right graph of some orientation of P, and by Lemma 2.2 we know that K_P is diagrammatically reducible. □

It is still an open question whether there is an injective non-DR LOT. However, the following example shows that the proof of Theorem 1.1 cannot be generalized to all injective LOTs.

Example 3.4. The injective compressed LOT of Figure 3 with any orientation of its edges does not fulfil the weight test (this can be seen with the program Graph [**9**]). It contains a reducible sub-LOT.

4. Proof of Theorem 1.2

The following result is well known. It is a special case of Theorem 4.3 of [**1**].

Proposition 4.1. *Let Q and Y be LOGs. Let T_P be defined by identifying a vertex of T_Q with a vertex of T_Y. If K_Q and K_Y are DA, then K_P is DA.*

It is easy to construct non-injective aspherical LOTs or injective aspherical LOTs which contain a reducible sub-LOT from Theorem 1.1 and Proposition 4.1. Identify two aspherical LOTs along a generator.

Figure 4. Constructing aspherical non-injective LOTs.

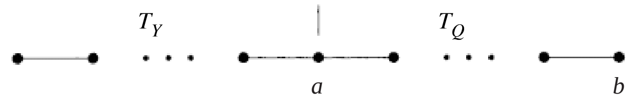

Figure 5. The LOT T_P.

Example 4.2. See Figure 4. In the picture, take any orientation of the edges. Here, the generator b of Q is identified with d of U. K_Q and K_U are DA by Theorem 1.1. So the resulting LOT is DA by Proposition 4.1.

For the proof of Theorem 1.2 we consider, without loss of generality, only alternating knot projections which have a minimal number of crossings for the given knot, in particular the alternating projections will have no small loops. Suppose there is such an alternating knot projection such that a Wirtinger spine obtained from this projection is not DA. Further suppose that, among all such alternating knot projections with non-DA Wirtinger spines, we choose an example with least number of crossings. Let U be its Wirtinger presentation and let $P_t = U - \{r_t\}$, where r_t is one of the relators, be a presentation for the spine which is not DA. U is an injective LOG with T_U a circle and, since the alternating projection has no small loops, U is compressed. Hence P_t is a compressed injective LOT. If K_{P_t} is not DA, then it is not DR, and by Theorem 1.1 P_t must contain a reducible sub-LOT Q.

Let a be the generator of Q which does not occur as a conjugator of a relator of Q, and let b be the other vertex of valence one in T_Q. Note that Q must be a proper sub-LOT of P_t, otherwise, if $Q = P_t$, the omitted relator r_t would contain only the two generators a and b, making U non-compressed.

Let T_P be the LOT obtained from T_U by omitting the edge of $T_U - T_Q$ with b in its boundary. Then T_P is an injective LOT built from two injective LOTs T_Q and $T_Y = T_P - T_Q$ joined by the generator a (see Figure 5).

Let $\kappa \subset \mathbb{R}^3$ be the knot corresponding to U. T_Q and $T_U - T_Q$ have only the two generators a and b in common. This implies the existence of an embedded 2-sphere $S^2 \subset \mathbb{R}^3$ having only two points in common with κ which come from the strings corresponding to the generators a and b. So T_Y and T_Q are Wirtinger presentations of tame alternating knots in the 3-sphere.

Since the knot corresponding to U was chosen with a minimal number of crossings, K_Q and K_Y are DA. Proposition 4.1 then gives the desired result. □

5. Proof of Theorem 1.3

The following proposition together with Lemma 2.1 implies Theorem 1.3.

https://doi.org/10.1017/S0013091599000474 Published online by Cambridge University Press

Proposition 5.1. *For any LOG $P \in \mathcal{P}$ there exists an orientation Q of P, such that L_Q is a forest.*

Proof. Let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle \in \mathcal{P}$. For each edge of T_P going from x_i to x_k labelled by x_j there is one edge in L_P between x_i and x_j. Changing the orientation of the edge in T_P leads to replacing the corresponding edge in L_P by one between x_k and x_j.

So, defining an orientation Q of the LOG P (and thereby determining the graph L_Q) is equivalent to choosing an endpoint of each edge $e \in T_P$ which has to be connected with the vertex of the label of e in L_Q.

For a generator x_i of a given $P \in \mathcal{P}$, we define $\nu(i)$ to be the number of occurrences of x_i as an edge label in T_P. Now define

$$\nu(P) = \min_{1 \leqslant i \leqslant n} \{m - \nu(i)\},$$

where P has m relators and n generators.

It is easy to see that $P \in \mathcal{P}$ with $\nu(P) = 0$ satisfies Proposition 5.1. Let x_i be the vertex with $\nu(i) = m$. Every edge label is x_i. Note that T_P is a forest. To define the orientation Q of P that makes L_Q a forest, we choose in every component of T_P one vertex as a root and orient all edges of the component towards the root. In addition we require that the root of the component which contains the vertex x_i is x_i. Then every edge of L_Q will have x_i as one endpoint, no edge is a loop, and for every $k \neq i$ there is at most one edge in L_Q having x_k as endpoint. Hence, L_Q consists of a tree of diameter two having x_i as its centre and, possibly, some isolated vertices.

The proof is by induction over ν. We assume that every LOG $P \in \mathcal{P}$ with $\nu(P) = q$ $(0 \leqslant q < m - 1)$ may be oriented, such that its left graph has no cycles.

Consider all presentations $P \in \mathcal{P}$ such that $\nu(P) = q + 1$ and every orientation of the edges of T_P induces at least one cycle in L_P. Among all those, let $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be one with a minimal number of relators.

Let x_i be one of the generators that occurs most as an edge label in T_P, so $q + 1 = m - \nu(i)$. Take any edge e, which has an x_j with $i \neq j$ as label. Such an edge has to exist, since $\nu(P) > 0$. Replace x_j by x_i as the label of e and call the resulting presentation P'. Then, by induction, we can find an orientation of $T_{P'}$ such that $L_{P'}$ is a forest.

Transfer this orientation to T_P (we still call the resulting presentation P although its edges have a different orientation, which leads to a different presentation). Now we claim that L_P consists of exactly one component, say K_1, with exactly one cycle and other components, $K_2, \dots, K_k$ $(k \geqslant 2)$, which are trees. This is true because if we take the edge $l_e \in L_P$ induced from e, disconnect it from x_j, and glue it to x_i we must get the forest $L_{P'}$. The edge l_e is part of the cycle in K_1. Let K_2 be the component which contains the vertex x_i. Since x_i appears at least once as an edge label in T_P, K_2 has at least one edge l_d coming from an edge $d \in T_P$ carrying x_i as a label (see Figure 6).

Let T_Q be the LOG obtained from T_P by omitting the edge d of T_P. We claim that every orientation of T_Q leads to a cycle in L_Q. If this is true, we have a contradiction, since $\nu(Q) = q$ or $q+1$, and in the latter case Q has one relator less than P, contradicting

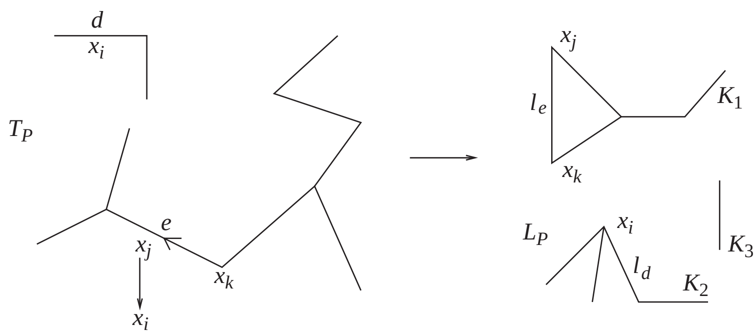

Figure 6. The LOG T_P and its left graph.

the assumption that P was a counterexample for $\nu(P) = q + 1$ with minimal number of relators.

So assume we have an orientation of T_Q where L_Q has no cycles (by abuse of notation we still call the resulting presentation Q). Putting back the edge d into T_Q resulting in a LOG T_R (by keeping the orientation of T_Q and taking the orientation which d had in T_P), we get a cycle in L_R. This is because every orientation of T_P induces a cycle and T_R is nothing else than a new orientation of T_P.

L_P has exactly one cycle c and the component of L_P containing this cycle was called K_1. Let A be the set of vertices of K_1. On the other hand, L_R has exactly one cycle which contains the edge l_d and l_d has neither of its endpoints in A. Consider the set of edges in T_P whose orientation is changed by going from T_P to T_R and let F be the corresponding set of transformations of edges that change L_P to L_R. Every such transformation disconnects one end of an edge in the left graph from the vertex and reconnects it to another vertex. Since the cycle c is not present in L_R there must be at least one transformation f_1 of an edge of c in the set F changing L_P to L_{P_1}. Assume first that L_{P_1} no longer has a cycle whose vertices are in A. The only way this could have happened is that the transformation f_1 took an edge of c and changed one of its endpoints to a vertex outside of A, thereby connecting two components of L_P. Then L_{P_1} would be without cycles, contradicting the hypotheses that any orientation of T_P induces cycles in the left graph.

Hence L_{P_1} must again have exactly one cycle c_1 and the vertices of c_1 are still in A. Since such a cycle c_1 is not present in L_R, again, there must be another transformation f_2 in F that changes an edge of c_1 resulting in L_{P_2}. By the same argument as above, L_{P_2} must contain exactly one cycle with its vertices in A. This process would have to continue forever, leading to a contradiction since there are only finitely many transformations in the set F. $\square$

Remarks

(1) The above proof is not constructive. On the other hand, for a given LOT it is easy to test all orientations for cycles in the left graph. This gives a constructive method for finding a DR orientation.

https://doi.org/10.1017/S0013091599000474 Published online by Cambridge University Press

(2) If we have a LOG where the left graph has no cycles, we can switch all orientations of its edges. We end up with a LOG where the right graph has no cycles, which also implies the diagrammatic reducibility of the corresponding 2-complex. So we can easily strengthen the result of Theorem 1.3 to:

For any LOG $P \in \mathcal{P}$ there are two different orientations Q and Y of P such that K_Q and K_Y are diagrammatically reducible.

(3) It is easy to construct a LOG P with the Euler-characteristic of a LOT, satisfying $\nu(P) = 0$ and containing a cycle in T_P, such that every orientation of its edges leads to a cycle in L_P.

Acknowledgements. We thank Wolfgang Metzler for helpful discussions and Bruce Richter for telling us about the result of Edmonds (cited in the proof of Theorem 3.3).

References

1. I. M. Chiswell, D. J. Collins and J. Huebschmann, Aspherical group presentations, *Math. Z.* **178** (1981), 1–36.
2. J. Edmonds, Edge-disjoint branchings, in *Combinatorial algorithms* (ed. R. Rustin), pp. 91–96 (Academic, New York, 1973).
3. S. Gersten, *Reducible diagrams and equations over groups*, pp. 15–73, Essays in Group Theory, Mathematical Science Research Institute Publications, no. 8 (Springer, 1987).
4. J. Howie, Some remarks on a problem of J. H. C. Whitehead, *Topology* **22** (1983), 475–485.
5. J. Howie, On the asphericity of ribbon disk complements, *Trans Am. Math. Soc.* (1) **289** (1985), 281–302.
6. G. Huck and S. Rosebrock, Weight tests and hyperbolic groups, in *Combinatorial and geometric group theory* (ed. J. Howie, A. Duncan and N. Gilbert), London Mathematical Society Lecture Note Series, vol. 204, pp. 174–183 (Cambridge University Press, 1995).
7. R. Lyndon and P. Schupp, *Combinatorial group theory* (Springer, 1977).
8. C. Mao-Cheng, Arc-disjoint arborescences of digraphs, *J. Graph Theory* **7** (1983), 235–240.
9. F. Petry and S. Rosebrock, Graph, a program for the weight test, http://www.ph-karlsruhe.de/~rost0001/web/mathsw/sw.html (1993).
10. S. Rosebrock, A reduced spherical diagram into a ribbon-disk complement and related examples, in *Topology and combinatorial group theory* (ed. P. Latiolais), Lecture Notes in Mathematics, vol. 1440, pp. 175–185 (Springer, 1990).
11. C. M. Weinbaum, The word and conjugacy problem for the knot group of any tame prime alternating knot, *Proc. Am. Math. Soc.* **22** (1971), 22–26.