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# ALMOST BAIRE CLASS ONE MULTIFUNCTIONS

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In this paper we employ quasi-continuous multifunctions and introduce almost Baire class 1 multifunctions in order to generalize a theorem of Kuratowski and also to answer a question posed by him concerning Baire class 1 multifunctions. We also show that certain multifunctions can be decomposed into mutually singular multifunctions.

## 1. Notation and preliminary results

Throughout this paper $X$ and $Y$ will be fixed non-empty topological spaces. Let $P(Y)$ , $N(Y)$ , $C(Y)$ and $K(Y)$ be the classes of all subsets, all non-empty subsets, all non-empty closed subsets and all non-empty closed compact subsets, respectively, of $Y$. A function $\Gamma: X \to P(Y)$ is called a *multifunction* and the set $D(\Gamma) = \{x \in X | \Gamma(x) \neq \emptyset\}$ is the *effective domain* of $\Gamma$ . For $A \subseteq X$ , let $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$ ; the set $\Gamma(X)$ is called the *range* of $\Gamma$ . For $B \subseteq Y$ , let $\overset{+}{\Gamma}(B) = \{x \in X | \Gamma(x) \subseteq B\}$ and $\Gamma^{-}(B) = \{x \in X | \Gamma(x) \cap B \neq \emptyset\}$ . The closure, interior and the boundary of a set $A \subseteq X$ will be denoted by $\bar{A}$ , $\operatorname{Int}(A)$ and $\operatorname{Fr}(A)$ , respectively, where $\operatorname{Fr}(A) = \bar{A} \cap \overline{X \setminus A} = \bar{A} \setminus \operatorname{Int}(A)$ . Obviously, $\operatorname{Fr}(A) = \operatorname{Fr}(X \setminus A)$ .

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Let $A \subseteq X$; recall that $A$ is of the *first Baire category* (*meagre*) in $X$ if $A$ is a countable union of nowhere dense subsets of $X$, $A$ is of the *second Baire category* (*non-meagre*) in $X$ if $A$ is not of the first Baire category in $X$ and that $A$ is *residual* in $X$ if $X\backslash A$ is of the first Baire category in $X$. A non-empty class of subsets of $X$ is an *ideal* (a $\sigma$-*ideal*) if it is hereditary and additive ($\sigma$-additive), see [6], pp.6, 12. Let $I$ denote the ideal of all nowhere dense sets and $J$ the $\sigma$-ideal of all first Baire category sets in $X$. Let $\Delta$ denote the symmetric difference operator on $P(X)$.

DEFINITION 1.1. Let $A$ and $B$ be subsets of $X$.

(1) ([6, p.11]). $A$ is said to be *congruent to* $B$ *modulo* $I$, denoted by $A \sim B \bmod I$, if $A \,\Delta\, B \in I$, or equivalently, if $A\backslash B \in I$ and $B\backslash A \in I$.

(2) ([6, p.87]). $A$ is said to be *open* (*closed*) *modulo* $J$ if there is an open (closed) set $G \subseteq X$ such that $A \sim G \bmod J$.

(3) $A$ is said to be $G_\delta$ *modulo* $J$, denoted by $A \overset{\delta}{\sim} G \bmod J$, if there exists a $G_\delta$-set $G \subseteq X$ such that $A \sim G \bmod J$.

(4) $A$ is said to be $F_\sigma$ *modulo* $J$, denoted by $A \overset{\sigma}{\sim} F \bmod J$, if there exists an $F_\sigma$-set $F \subseteq X$ such that $A \sim F \bmod J$.

(5) ([2, p.388]). If $A = G \,\Delta\, P$, where $G$ is open in $X$ and $P \in J$, then $A$ is said to be a *Baire set* in $X$.

Let

$\mathcal{A} = \{A \subseteq X | A$ is open modulo $J\}$, $\mathcal{B} = \{A \subseteq X | A$ is a Baire set in $X\}$, $\mathcal{C} = \{A \subseteq X | A$ is closed modulo $J\}$, $\mathcal{F} = \{A \subseteq X | A$ is $F_\sigma$ modulo $J\}$ and $\mathcal{G} = \{A \subseteq X | A$ is $G_\delta$ modulo $J\}$.

LEMMA 1.2. *(1)* $A \in \mathcal{A}$ *if and only if* $A$ *is of the form* $A = (G\backslash P) \cup R$, *where* $G$ *is open and* $P, R \in J$.

*(2) If* $\mathcal{T}$ *is the topology of* $X$, *then* $\mathcal{B}$ *is the* $\sigma$-*algebra generated by the class* $\mathcal{T} \cup J$.

*(3)* $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{F} = \mathcal{G}$.

Proof. (1): See [6, p.87]; (2): see [11, p.19].

(3): For $\mathcal{A} = \mathcal{C}$ , see [6, p.69] and for $\mathcal{A} = \mathcal{B}$ , see [2, p.388]. It is obvious that $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A} \subseteq \mathcal{G}$ . Let $A \in \mathcal{G}$ . Then $A \overset{\delta}{\sim} G$ mod $\mathcal{J}$ , and since $G \in \mathcal{A}$ ($\mathcal{A}$ being a $\sigma$-algebra), it follows that $G \sim 0$ mod $\mathcal{J}$ , where $0$ is open in $X$ . By [6, p.11, VIII (1)], it follows that $A \sim 0$ mod $\mathcal{J}$ , thus $A \in \mathcal{A}$ . We deduce that $\mathcal{G} \subseteq \mathcal{A}$ and consequently $\mathcal{A} = \mathcal{G}$. Similarly, $\mathcal{A} = \mathcal{F}$ . This completes the proof.

The role played by Baire sets in topology is analogous to that of measurable sets in analysis.

DEFINITION 1.3. Let $A$ be a subset of $X$ .

*(1)* ([9, p.36]). $A$ is said to be *semi-open* if there exists an open set $U$ in $X$ such that $U \subseteq A \subseteq \bar{U}$ .

*(2)* ([3, p.99]). $A$ is said to be *semi-closed* if $X \backslash A$ is semi-open.

REMARK 1.4. (1) Every open and every closed subset of $X$ is semi-open.

(2) From (1) above we deduce that every open and every closed subset of $X$ is semi-closed.

(3) A non-empty semi-open subset of $X$ contains a non-empty open set.

PROPOSITION 1.5. *Let $A$ be a subset of $X$.*

*(1) $A$ is semi-open if and only if $A \subseteq \overline{\mathrm{Int}(A)}$.*

*(2) If $A$ is open then $\bar{A} \backslash A$ is nowhere dense.*

*(3) $A$ is semi-closed if and only if there exists a closed set $C \subseteq X$ such that $\mathrm{Int}(C) \subseteq A \subseteq C$ .*

*(4) If $A$ is open and $B$ is semi-open in $X$ , then $A \cap B$ is semi-open.*

*(5) $A$ is semi-open if and only if $\bar{A} = \overline{\mathrm{Int}(A)}$.*

*(6) $A$ is semi-open if and only if $A = (\mathrm{Int}(A)) \cup B$ , where $B \subseteq Fr(A)$ .*

*(7) If $A$ is semi-open (or semi-closed), then $Fr(A)$ is nowhere dense.*

*(8) If $A$ is semi-closed then $A = 0 \cup B$, where $0$ is open , $B$ is nowhere dense and $0 \cap B = \emptyset$*

Proof. (1) and (2): see [9, pp.36, 37]; (3) and (4): see [3, pp.100, 101]; (5) and (6): see [4, p.69].

(7): If $A$ is semi-open, then $Fr(A) = \bar{A} \backslash \mathrm{Int}(A) = \overline{\mathrm{Int}(A)} \backslash \mathrm{Int}(A)$ from (5). It follows from (2) that $Fr(A)$ is nowhere dense. If $A$ is semi-closed,

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then since $Fr(A) = Fr(X\backslash A)$, it follows that $Fr(A)$ is nowhere dense.

(8): If $A$ is semi-closed there exists, by (3), a closed set $C \subseteq X$ such that $\text{Int}(C) \subseteq A \subseteq C$. Then $A = (\text{Int}(C)) \cup (A\backslash\text{Int}(C))$. Put $O = \text{Int}(C)$ and $B = A\backslash\text{Int}(C)$. Since $B \subseteq C\backslash\text{Int}(C)$, it is nowhere dense.

DEFINITION 1.6. A multifunction $\Gamma: X \to N(Y)$ is said to be *upper-quasi-continuous*, briefly u-q-c (*lower-quasi-continuous*, briefly l-q-c) at a point $x_o \in X$ if for any open set $V$ in $Y$ satisfying $\Gamma(x_o) \subseteq V$ $(\Gamma(x_o) \cap V \neq \emptyset)$ and for any open neighbourhood $U$ of $x_o$ in $X$, there exists a non-empty open set $G \subseteq U$ such that $G \subseteq \Gamma^+(V)$ $(G \subseteq \Gamma^-(V))$.

A multifunction $\Gamma: X \to N(Y)$ is u-q-c (l-q-c) on $X$ if it is u-q-c (l-q-c) at every point of $X$ and $\Gamma$ is quasi-continuous at a point $x_o \in X$ (on $X$) if it is both u-q-c and l-q-c at $x_o \in X$ (on $X$).

PROPOSITION 1.7. *A multifunction* $\Gamma: X \to N(Y)$ *is u-q-c (l-q-c) on* $X$ *if and only if for any open set* $V \subseteq Y$ *the set* $\Gamma^+(V)$ $(\Gamma^-(V))$ *is semi-open in* $X$.

Proof. We prove the case of u-q-c $\Gamma$, the other case is similar. Let $\Gamma$ be u-q-c and let $V$ be any open set in $Y$. If $\Gamma^+(V) = \emptyset$, then $\Gamma^+(V)$ is semi-open. So, let $\Gamma^+(V) \neq \emptyset$, $x_o \in \Gamma^+(V)$ and $U$ be any open neighbourhood of $x_o$ in $X$. There exists a non-empty open set $G \subseteq U$ such that $G \subseteq \Gamma^+(V)$. Then $G = \text{Int}(G) \subseteq \text{Int}(\Gamma^+(V))$, hence $U \cap \text{Int}(\Gamma^+(V)) \neq \emptyset$. Then $x_o \in \overline{\text{Int}(\Gamma^+(V))}$, thus $\Gamma^+(V) \subseteq \overline{\text{Int}(\Gamma^+(V))}$ and $\Gamma^+(V)$ is semi-open by 1.5(1). Conversely, let $\Gamma^+(V)$ be semi-open in $X$ for every open set $V$ in $Y$. Let $x_o \in X$, $V$ open in $Y$ such that $\Gamma(x_o) \subseteq V$ and let $U$ be any open neighbourhood of $x_o$. The set $U \cap \Gamma^+(V)$ is non-empty and is, from 1.5(4), semi-open. From 1.4(3) we have a non-empty open set $G \subseteq U \cap \Gamma^+(V)$, hence $G \subseteq \Gamma^+(V)$. This proves the upper-quasi-continuity of $\Gamma$ at $x_o$.

DEFINITION 1.8. ([1]). A multifunction $\Gamma: X \to N(Y)$ is said to be *upper-semi-continuous*, briefly u-s-c (*lower-semi-continuous*, briefly l-s-c) on $X$ if for any open set $V \subseteq Y$ the set $\Gamma^+(V)$ $(\Gamma^-(V))$ is open in

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$X$, and $\Gamma$ is semi-continuous on $X$ if it is both u-s-c and l-s-c on $X$.

It is clear that every semi-continuous multifunction on $X$ is also quasi-continuous on $X$. To see that the converse is not true, consider the multifunction $\Gamma: R \to R^2$, defined by $\Gamma(x) = \{(0,0)\}$ if $x \geq 0$ and $\Gamma(x) = \{(\frac{1}{|x|}, -1)\}$ if $x < 0$.

## 2. Main results

We employ the *Vietoris* (or *exponential* topology) as developed by Michael [10]. The collection $\mathcal{D}(Y)$ of all classes of the form

(i) $[0_1, 0_2, \ldots, 0_n] = \{A \in C(Y) | A \subseteq \bigcup_{i=1}^{n} 0_i, A \cap 0_i \neq \emptyset;\ i = 1,2,\ldots,n\}$,

with $0_1, 0_2, \ldots, 0_n$ all open in $Y$, is a base for the Vietoris topology on $C(Y)$. A subbase for this topology is the collection $S(Y)$ consisting of all classes having one of the following forms:

(ii) $0^+ = \{A \in C(Y) | A \subseteq 0\}$, $0^- = \{A \in C(Y) | A \cap 0 \neq \emptyset\}$,

with $0$ open in $Y$. If $B \in \mathcal{D}(Y)$, then by (i) and (ii) above,

(iii) $B = [0_1, 0_2, \ldots, 0_n] = 0^+ \cap (\bigcap_{i=1}^{n} 0_i^-)$, where $0 = \bigcup_{i=1}^{n} 0_i$.

Consider $K(Y)$, with the relative Vietoris topology, as a subspace of $C(Y)$.

LEMMA 2.1. *Let $(Y,d)$ be a metric space and $\Gamma: X \to K(Y)$ be l-q-c on $X$. Then $\Gamma^-(C) \overset{\delta}{\sim} G \bmod J$ for every set $C \in C(Y)$.*

Proof. Let $C \in C(Y)$ and put $\rho(y,C) = \inf_{c \in C} d(y,c)$ where $y \in Y$. For the open sets $0_n = \{y \in Y | \rho(y,C) < \frac{1}{n}\}$, where $n = 1,2,3,\ldots$, it follows that $C = \{y \in Y | \rho(y,C) = 0\} \subseteq 0_n$ for each $n$; consequently, $\Gamma^-(C) \subseteq \bigcap_{n=1}^{\infty} \Gamma^-(0_n)$. To establish the converse inclusion, let $x \in \bigcap_{n=1}^{\infty} \Gamma^-(0_n)$ and $y_n \in \Gamma(x) \cap 0_n$, where $n = 1,2,3,\ldots$. Then $\rho(y_n, C) < \frac{1}{n}$ where $n = 1,2,3,\ldots$. Since $\Gamma(x) \in K(Y)$, there exists a subsequence $(y_{k_n})$ of $(y_n)$ such that $y_{k_n} \to y_0 \in \Gamma(x)$ as $n \to \infty$. Since $\lim_{n\to\infty} \rho(y_{k_n}, C) = \rho(y_0, C) = 0$, it follows that $\Gamma(x) \cap C \neq \emptyset$, so $x \in \Gamma^-(C)$ ; consequently, $\bigcap_{n=1}^{\infty} \Gamma^-(O_n) \subseteq \Gamma^-(C)$ . Now, $\Gamma^-(C) = \bigcap_{n=1}^{\infty} \Gamma^-(O_n)$, each set $\Gamma^-(O_n)$ is semi-open and by 1.5(6), $\Gamma^-(O_n) = (\mathrm{Int}(\Gamma^-(O_n))) \cup B_n$ , with $(\mathrm{Int}(\Gamma^-(O_n))) \cap B_n = \emptyset$ and $B_n \subseteq Fr(\Gamma^-(O_n))$. By 1.5(7), $B_n$ is nowhere dense for every $n$ . Put $A_n = \mathrm{Int}(\Gamma^-(O_n))$ for every $n$ . Then $\Gamma^-(C) = \bigcap_{n=1}^{\infty}(A_n \cup B_n)$, which can be written in the form $\Gamma^-(C) = (\bigcap_{n=1}^{\infty} A_n) \cup B$ , where $B$ is of the first Baire category in $X$ . The result follows by putting $G = \bigcap_{n=1}^{\infty} A_n$ .

COROLLARY 2.2. *Let $(Y,d)$ be a metric space and $\Gamma: X \to K(Y)$ be l-s-c on $X$ . Then $\Gamma^-(C)$ is a $G_\delta$-set for each closed set $C \in C(Y)$ .*

Proof. This follows from the fact that each set $\Gamma^-(O_n)$ in the proof of 2.1 is open in $X$ .

Kuratowski [7], p.70 shows that if $X$ and $Y$ are metric, with $Y$ in addition compact, and if $\Gamma: X \to C(Y)$ is semi-continuous, then $\Gamma$ is of Baire class 1 (that is, inverse images of open sets are $F_\sigma$-sets). We accept the following more lenient definition.

DEFINITION 2.3. (1) Let $\Gamma: X \to N(Y)$ be a multifunction. Then $\Gamma$ is said to be of *Baire class 1* if $\Gamma^{-1}(O)$ is an $F_\sigma$-set in $X$ for each open $O \subseteq K(Y)$. Furthermore, $\Gamma$ is said to be *almost of Baire class 1* if $\Gamma^{-1}(O) \overset{\sigma}{\sim} F \bmod J$ for each open $O \subseteq K(Y)$.

(2) If $A \subseteq B \subseteq X$ , then $A$ is said to be an *$F_\sigma$-set* (a *$G_\delta$-set*) relative to $B$ if $A$ is the intersection of $B$ with an $F_\sigma$-set (a $G_\delta$-set) in $X$ .

(3) Let $\Gamma: X \to P(Y)$ be a multifunction. Then $\Gamma$ is said to be of *Baire class 1 relative to a set* $T \subseteq D(\Gamma)$ if $\Gamma^{-1}(O)$ is an $F_\sigma$-set relative to $T$ for each open $O \subseteq K(Y)$ . Furthermore, $\Gamma$ is said to be *empty almost everywhere* on $X$ if there exists a residual set $A \subseteq X$ such that $\Gamma(A) = \emptyset$ .

The usage of the term "almost everywhere" in 2.3(3) above is motivated by the fact that in numerous problems of topology the notion of a set of the first Baire category is analogous to that of a set of measure zero in analysis.

PROPOSITION 2.4. *Let* $X$ *and* $Y$ *be such that every closed subset of each of them is a* $G_\delta$*-set. Suppose further that* $Y$ *is a* $T_1$ *and second countable space. If* $\Gamma: X \to N(Y)$ *is u-q-c on* $X$ *, then* $\Gamma$ *is almost of Baire class 1.*

Proof. Let $O$ be open in $K(Y)$ and let $\mathcal{D}(Y)$ be a countable base for $K(Y)$ , see [*10*, p.162]. Put $O = \bigcup_{n=1}^{\infty} B_n$ , where $B_n \in \mathcal{D}(Y)$ for each $n$ . Then $\Gamma^{-1}(O) = \bigcup_{n=1}^{\infty} \Gamma^{-1}(B_n)$ . By (iii), $B_n = (O^{(n)})^+ \cap (\bigcap_{i=1}^{p_n} (O_i^{(n)})^-)$ , where $n = 1,2,3,\ldots,$ with $O^{(n)} = \bigcup_{i=1}^{p_n} O_i^{(n)}$ , and $O_i^{(n)}$ is open in $Y$ , where $i = 1,2,3,\ldots,p_n$ ; $n = 1,2,3,\ldots$ . Then

(iv) $\Gamma^{-1}(O) = \bigcup_{n=1}^{\infty}[\Gamma^{-1}((O^{(n)})^+) \cap (\bigcap_{i=1}^{p_n}\Gamma^{-1}((O_i^{(n)})^-)]$ .

Since $\Gamma^{-1}((O^{(n)})^+) = \Gamma^+(O^{(n)})$ , $O^{(n)}$ is open in $Y$ and $\Gamma$ is u-q-c on $X$ , it follows that $\Gamma^{-1}((O^{(n)})^+)$ is semi-open in $X$ . Hence, by 1.5(6) and (7),

(v) $\Gamma^{-1}((O^{(n)})^+) = [\mathrm{Int}(\Gamma^+(O^{(n)}))] \cup B^{(n)}$

where $B^{(n)} \subseteq Fr(\Gamma^+(O^{(n)}))$, $B^{(n)}$ is nowhere dense and $[\mathrm{Int}(\Gamma^+(O^{(n)}))] \cap B^{(n)} = \emptyset$ . By assumption, each $O_i^{(n)}$ is an $F_\sigma$-set in $Y$ ; put $O_i^{(n)} = \bigcup_{k=1}^{\infty} C_{k,i}^{(n)}$ , where each $C_{k,i}^{(n)}$ is closed in $Y$ . Then

(vi) $\Gamma^{-1}((O_i^{(n)})^-) = \Gamma^-(O_i^{(n)}) = \bigcup_{k=1}^{\infty} \Gamma^-(C_{k,i}^{(n)})$ .

Since $\Gamma^-(C_{k,i}^{(n)})$ is semi-closed, it follows from 1.5(8) that

(vii) $\Gamma^-(C_{k,i}^{(n)}) = U_{k,i}^{(n)} \cup D_{k,i}^{(n)}$ ,

where $U_{k,i}^{(n)}$ is open in $X$, $D_{k,i}^{(n)}$ is nowhere dense and $U_{k,i}^{(n)} \cap D_{k,i}^{(n)} = \emptyset$ . It follows from (iv), (v), (vi) and (vii) that

$$\Gamma^{-1}(O) = \bigcup_{n=1}^{\infty} \Big[[\{\mathrm{Int}\ \Gamma^+(O^{(n)})\} \cup B^{(n)}] \cap [\bigcap_{i=1}^{p_n} \bigcup_{k=1}^{\infty}(U_{k,i}^{(n)} \cup D_{k,i}^{(n)})]\Big] .$$

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Since each open subset of $X$ is an $F_\sigma$-set and $J$ is an $\sigma$-ideal, one easily sees that $\Gamma^{-1}(0)$ can be written in the form $\Gamma^{-1}(0) = F \cup M$, where $F$ is an $F_\sigma$-set and $M \in J$. Clearly, $\Gamma^{-1}(0) \overset{g}{\approx} F \bmod J$. This is the desired result.

The term "mutually singular" in the next definition is motivated by the usage of this term in measure theory; for mutually singular integrals, see [*13*, p.242].

DEFINITION 2.5. (1) The multifunctions $\Gamma_1: X \to P(Y)$ and $\Gamma_2: X \to P(Y)$ are said to be *mutually singular*, denoted by $\Gamma_1 \perp \Gamma_2$, whenever there exist two disjoint subsets $A$ and $B$ of $X$ such that $X = A \cup B$ and $\Gamma_1(A) = \Gamma_2(X\backslash A) = \emptyset$.

(2) The multifunction $\Gamma: X \to P(Y)$ is said to be *decomposable* if it can be written in the form $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1: X \to P(Y)$, $\Gamma_2: X \to P(Y)$ and $\Gamma_1 \perp \Gamma_2$.

PROPOSITION 2.6. *Let $X$ and $Y$ satisfy the hypotheses of proposition 2.4.*

*(1) If $\Gamma: X \to N(Y)$ is u-s-c on $X$, then $\Gamma$ is of Baire class 1.*

*(2) If $\Gamma: X \to K(Y)$ is u-q-c on $X$, then $\Gamma$ can be decomposed into two mutually singular multifunctions $\Gamma_1: X \to P(Y)$ and $\Gamma_2: X \to P(Y)$ such that $\Gamma_1$ is of Baire class 1 relative to a residual subset of $X$ and $\Gamma_2$ is empty almost everywhere on $X$.*

Proof. (1): We refer to the proof of proposition 2.4. Since $\Gamma^{-1}((O^{(n)})^+)$ is open, each set $B^{(n)}$ in (v) is empty. Since $\Gamma^-(C^{(n)}_{k,i})$ is closed, each set $\Gamma^{-1}((O^{(n)}_i)^-)$ in (vi) is an $F_\sigma$-set. It follows from (iv) that $\Gamma^{-1}(0)$ is an $F_\sigma$-set, which proves the result.

(2): Let $A = \{x \in X | \Gamma$ is not u-s-c at $x\}$. Then $A$ is of the first Baire category, see [*4*], p.72, theorem 15. Consequently, $B = X\backslash A$ is residual. Define $\Gamma_1: X \to P(Y)$ and $\Gamma_2: X \to P(Y)$ respectively

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by

$$\Gamma_1(x) = \begin{cases} \emptyset & \text{if } x \in A \\ \Gamma(x) & \text{if } x \in B \end{cases} \quad \text{and} \quad \Gamma_2(x) = \begin{cases} \Gamma(x) & \text{if } x \in A \\ \emptyset & \text{if } x \in B. \end{cases}$$

Then $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \perp \Gamma_2$ . Since the restriction $\Gamma_* = \Gamma_1|B$ is u.s.c. on $B$ , it follows from (1) that $\Gamma_*$ is of Baire class 1 on $B$ . Consequently, if $O$ is non-empty in $K(Y)$ , then $\Gamma_1^{-1}(O) = \Gamma_*^{-1}(O)$ $= \bigcup_{n=1}^{\infty} F_n^{(B)} = B \cap (\bigcup_{n=1}^{\infty} F_n^{(X)})$ , where $F_n^{(B)}$ is closed in $B$ and $F_n^{(X)}$ closed in $X$ . If $O$ is empty in $K(Y)$ , then obviously $\Gamma_1^{-1}(O)$ is an $F_\sigma$-set relative to $B$ . In either case, $\Gamma_1$ is of Baire class 1 relative to $B$ . It is clear that $\Gamma_2$ is empty almost everywhere on $X$ .

PROPOSITION 2.7. *Let $X$ be such that every closed subset is a $G_\delta$-set and let $Y$ be a separable metric space. If $\Gamma: X \to K(Y)$ is l-q-c on $X$ , then $\Gamma$ is almost of Baire class 1.*

Proof. The first part of the proof of Proposition 2.4 can be carried over to the present situation:

(viii) $\Gamma^{-1}(O) = \bigcup_{n=1}^{\infty}[\Gamma^{-1}((O^{(n)})^+) \cap (\bigcap_{i=1}^{p_n}\Gamma^{-1}((O_i^{(n)})^-)]$ ,

with $O$ open in $K(Y)$ and $O_i^{(n)}$ open in $Y$ . Now, $\Gamma^{-1}((O^{(n)})^+)$ $= \Gamma^+(O^{(n)}) = X\backslash\Gamma^-(Y\backslash O^{(n)})$. Since $Y\backslash O^{(n)}$ is closed in $Y$ and $\Gamma$ is l-q-c on $X$ , it follows from 2.1 that $\Gamma^-(Y\backslash O^{(n)}) \overset{\delta}{\sim} G^{(n)}$ mod $J$ . This yields $X\backslash\Gamma^-(Y\backslash O^{(n)}) \overset{\sigma}{\sim} (X\backslash G^{(n)})$ mod $J$ , consequently

(ix) $\Gamma^{-1}((O^{(n)})^+) \overset{\sigma}{\sim} (X\backslash G^{(n)})$ mod $J$ .

Also, $\Gamma^{-1}((O_i^{(n)})^-) = \Gamma^-(O_i^{(n)})$ , which is semi-open in $X$ . Hence by 1.5(6) and (7),

(x) $\Gamma^{-1}((O_i^{(n)})^-) = [\mathrm{Int}(\Gamma^-(O_i^{(n)})] \cup B_i^{(n)}$ ,

where $B_i^{(n)} \subseteq Fr(\Gamma^-(O_i^{(n)})$, $[\mathrm{Int}(\Gamma^-(O_i^{(n)})] \cap B_i^{(n)} = \emptyset$ and $B_i^{(n)}$ is nowhere dense in $X$. Since $\mathrm{Int}(\Gamma^{-}(O_i^{(n)}))$ is an $F_\sigma$-set in $X$ and the class of all nowhere dense subsets of $X$ is an ideal, it follows that $\bigcap_{i=1}^{p_n} \Gamma^{-1}((O_i^{(n)})^{-})$ can be written in the form $\bigcap_{i=1}^{p_n} \Gamma^{-1}((O_i^{(n)})^{-}) = F^{(n)} \cup B^{(n)}$, where $F^{(n)}$ is an $F_\sigma$-set and $B^{(n)}$ is nowhere dense. Consequently,

(xi) $\bigcap_{i=1}^{p_n} \Gamma^{-1}((O_i^{(n)})^{-}) \overset{\sigma}{\sim} F^{(n)} \mod J$.

It follows from (viii) - (xi) above and from [6], p.12, §2, VIII formulae (2) and (4) that

$$\Gamma^{-1}(O) \overset{\sigma}{\sim} F \mod J,$$

where $F = \bigcup_{n=1}^{\infty}[(X \backslash G^{(n)}) \cap F^{(n)}]$. This completes the proof.

COROLLARY 2.8. *Let $X$ and $Y$ satisfy the hypotheses of proposition 2.7.*

*(1) If $\Gamma: X \to K(Y)$ is l-s-c on $X$, then $\Gamma$ is of Baire class 1.*

*(2) If $\Gamma: X \to K(Y)$ is l-q-c on $X$, then $\Gamma$ can be decomposed into two mutually singular multifunctions $\Gamma_1: X \to P(Y)$ and $\Gamma_2: X \to P(Y)$ such that $\Gamma_1$ is of Baire class 1 relative to a residual subset of $X$ and $\Gamma_2$ is empty almost everywhere on $X$.*

Proof. (1): We refer to the proof of proposition 2.7. It follows from 2.2 that each set $\Gamma^{-1}((O^{(n)})^{+})$ in (ix) is an $F_\sigma$-set in $X$. Also, since each set $\Gamma^{-1}((O_i^{(n)})^{-})$ in (x) is open, it is $F_\sigma$ by hypothesis. Hence by (viii), $\Gamma^{-1}(O)$ is an $F_\sigma$-subset of $X$, which proves the result.

(2): The proof is similar to that of 2.6.(2), with the exception that we use [4, p.72, Theorem 16] to find a first Baire category set A.

If we combine the corresponding results in Propositions 2.4 and 2.7, and in proposition 2.6 and corollary 2.8, then we have the following theorem.

THEOREM 2.9. *Let $X$ be such that every closed subset is a $G_\delta$-set and let $Y$ be a separable metric space.*

*(1) If* $\Gamma: X \to K(Y)$ *is quasi-continuous on* $X$ *, then* $\Gamma$ *is almost of Baire class 1 and* $\Gamma$ *can be decomposed as described in Proposition 2.6(2) and in Corollary 2.8(2).*

*(2) If* $\Gamma: X \to K(Y)$ *is semi-continuous on* $X$ *, then* $\Gamma$ *is of Baire class 1.*

REMARK 2.10. (1) If we employ Kenderov's ([5, p.150]) less stringent definition of semi-continuity, in which $\Gamma(x)$ is allowed to be empty for some $x \in X$ , then Prop. 2.6(2) and Coroll. 2.8(2), and consequently Theorem 2.9(1), can be generalized correspondingly to the effect that both multifunctions $\Gamma_1$ and $\Gamma_2$ in the decomposition of $\Gamma$ are u.s.c. on $X$ .

(2) It was stated just below Corollary 2.2 that Kuratowski [7, p.70] shows that if $X$ and $Y$ are metric and $Y$ also compact, and if $\Gamma: X \to C(Y)$ is semi-continuous on $X$ , then $\Gamma$ is of Baire class 1. In [8, p.47] Kuratowski posed the following question:

Can metrizability and compactness of the spaces $X$ and $Y$ respectively be replaced by weaker assumptions? Surely, Theorem 2.9 shows that this can be done. For an example of a non-metrizable space $X$ satisfying the requirements of Theorem 2.9, let $X$ be the set of all natural numbers and equip $X$ with the finite complement topology, see [12, p.49].

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