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NIL IDEALS OF POWER SERIES RINGS

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Abstract

Some characterizations of nil radical and nil semisimple power series rings are given. The upper nil radical of a power series ring in an uncountable set of non-commutative indeterminates is completely described.

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1. Introduction

In this paper we investigate nil ideals of an associative power series ring $R\{X\}$ in a set X of non-commutative indeterminates commuting with coefficients from R. We begin in Section 2 with a characterization of nil one-sided ideals of $R\{X\}$ for X of cardinality $\geqslant 2$, showing that such ideals are contained in $N(R)\{X\}$, where $N(R)$ is the sum of all nilpotent ideals of R. It implies immediately that semiprime power series rings in non-commutative indeterminates are nil semisimple. This result and some of its applications have been obtained by the author (1980) for infinite X. A stronger result is proved in Section 3 for uncountable X. It asserts that a series s belongs to a nil ideal of $R\{X\}$ if and only if the ideal of R generated by the coefficients of s is nilpotent. In the final section we present a result characterizing nil power series rings of one indeterminate.

All results of the paper are stated and proved for right ideals. By analogous arguments or using the fact that if L is a nil left ideal of A and $a \in L$ then the right ideal of A generated by a is nil, one can transfer all obtained results to left ideals.

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The following simple observation will be very useful in our investigations.

LEMMA 1. *The ideal $\langle A \rangle$ of R generated by a subset A is nilpotent if and only if for any denumerable subset B of R the set AB is nilpotent.*

PROOF. If the ideal $\langle A \rangle$ is not nilpotent then for any $n = 1, 2, \ldots$ there exist elements $a_{n1}, \ldots, a_{nn} \in A$, $b_{n_1}, \ldots, b_{nn} \in R$ such that $a_{n1} b_{n1} \cdots a_{nn} b_{nn} \neq 0$. Of course $B = \{b_{ij}\}$ is such a denumerable subset of R that the set AB is not nilpotent. This proves the part "only if". The part "if" is clear.

Throughout Sections 2 and 3, $P(X)$ will be the free (non-abelian) semigroup with unity generated by the set X of cardinality $\geqslant 2$. If $p \in P(X)$ and $x \in X$ then $l(p)$ and $l_x(p)$ will denote the degree and x-degree of p respectively. The elements of $R\{X\}$ are the formal series $a = \sum a_p p$, where $a_p \in R$, $p \in P(X)$. For any $a \in R\{X\}$ we denote $\operatorname{supp} a = \{p \in P(X) \mid a_p \neq 0\}$.

2. Semisimplicity of power series rings

We begin with an auxiliary lemma concerning the semigroup $P(X)$.

LEMMA 2. *Let $x, y \in X$, $x \neq y$ and $Y = \{xyx, xy^2x, \ldots\} \subseteq P(X)$. Then*
a) *if $p, q \in P(X)$, $r, t \in Y$ and $prq = t$ then $p = q = 1$;*
b) *if for $1 \leqslant i \leqslant n$, $p_i, q_i \in P(X)$, $r_i, t_i \in Y$, $l(p_i) \geqslant l(q_i)$ and $p_1 r_1 \cdots p_n r_n = q_1 t_1 \cdots q_n t_n$ then $p_i = q_i$, $r_i = t_i$ for $1 \leqslant i \leqslant n$.*

PROOF. The part a) follows immediately from the definition of Y. To prove b) we proceed by induction on n. If $n = 1$ then $p_1 r_1 = q_1 t_1$. But $l(p_1) \geqslant l(q_1)$, so $p_1 = q_1 s$ for some $s \in P(X)$. Thus $s r_1 = t_1$ and by a) $s = 1$. Hence $r_1 = t_1$ and $p_1 = q_1$.

Let us assume now that b) is valid for $n \leqslant k$ and $p_1 r_1 \cdots p_{k+1} r_{k+1} = q_1 t_1 \cdots q_{k+1} t_{k+1}$. Let $p_1' = p_1 r_1 p_2$ and $q_1' = q_1 t_1 q_2$. We will show that $l(p_1') \geqslant l(q_1')$. Since $l(p_1) \geqslant l(q_1)$ then $p_1 = q_1$ for some $s \in P(X)$. Thus

$$(1) \qquad s r_1 p_2 \cdots r_{k+1} = t_1 q_2 \cdots q_{k+1} t_{k+1}.$$

By the assumption $l(q_2) \leqslant l(p_2)$, so if $l(p_1') < l(q_1')$ then $l(s r_1) < l(t_1)$. Now (1) implies that $s r_1 w = t_1$ for some $w \in P(X)$. Hence by a) $s = w = 1$ and, in consequence, $r_1 = t_1$. This contradiction shows that $l(p_1') \geqslant l(q_1')$. Thus, using the induction argument, we obtain $r_i = t_i$ for $2 \leqslant i \leqslant k+1$, $p_i = q_i$ for $3 \leqslant i \leqslant k+1$ and $p_1 r_1 p_2 = q_1 t_1 q_2$. The last equality and $l(p_1) \geqslant l(q_1)$, $l(p_2) \geqslant l(q_2)$ give

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$p_1 = q_1 s$ and $p_2 = wq_2$ for some $s, w \in P(X)$. Therefore $sr_1w = t_1$ and by a) $s = w = 1$. This proves the lemma.

THEOREM 1. *Let I be a nil right ideal of $R\{X\}$. If $a = \Sigma a_p p \in I$ then the ideal of R generated by $A = \{a_p \mid p \in$ supp a has the minimal degree in* supp $a\}$ *is nilpotent.*

PROOF. Let $b_1, b_2, \ldots \in R$ and $x, y \in X$, $x \neq y$. Using Lemma 2 we obtain that if p_i for $1 \leqslant i \leqslant n$ have the minimal degree in supp a then the coefficient at $p_1 xy^{k_1}x \cdots p_n xy^{k_n}x$ in the series $(a(b_1 xyx + b_2 xy^2 x + \cdots))^n$ is equal to $a_{p_1} b_{k_1} \cdots a_{p_n} b_{n_n}$. So if $(a(b_1 xyx + b_2 xy^2 x + \cdots))^n = 0$ then $(A \cdot \{b_i \mid i = 1, 2, \ldots\})^n = 0$ and Lemma 1 ends the proof.

For any ring R let $N(R)$ denote the sum of all nilpotent ideals of R. Similarly as for infinite X (Puczyłowski (1980), Corollary 3) we obtain

COROLLARY 1. *If I is a nil right ideal of $R\{X\}$ and $a \in I$ then for any integer $k \geqslant 0$ the ideal of R generated by $A_k = \{a_p \mid p \in$ supp a, $l(p) = k\}$ is nilpotent. In particular $I \subseteq N(R)\{X\}$.*

Let K be the nil radical class and S the lower strong radical determined by K. It is well known (Divinsky, Krempa and Sulinski (1971)) that a ring R is S-semisimple if and only if R contains no non-zero nil right ideals. Thus by Theorem 1 we obtain immediately

COROLLARY 2. *For any ring R the following conditions are equivalent:*

(i) *R is semiprime;*

(ii) *$R\{X\}$ is semiprime;*

(iii) $K(R\{X\}) = 0$;

(iv) $S(R\{X\}) = 0$.

3. Special cases

In this section we investigate nil ideals of $R\{X\}$ in some special cases. We start from a result on nil right ideals of bounded index.

LEMMA 3. *If $x, y \in X$, $x \neq y$, $p_1, \ldots, p_n, q_1, \ldots, q_n \in P(X)$, $\max(l(p_1), \ldots, l(p_n)) = k$ and*

$$p_1 yx^{n(k+1)} \cdots p_n yx^{n(k+1)} = q_1 yx^{n(k+1)} \cdots q_n yx^{n(k+1)} \tag{2}$$

then $p_1 = q_1, \ldots, p_n = q_n$.

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PROOF. The equality (2) implies that for some $m \in P(X)$, $q_1 = p_1 m$ or $p_1 = q_1 m$. If $p_1 = q_1 m$ then $myx^{n(k+1)} \cdots p_n yx^{n(k+1)} = yx^{n(k+1)} \cdots q_n yx^{n(k+1)}$. But if $m \neq 1$ then $m = yx^{n(k+1)}r$ for some $r \in P(X)$. This is impossible as then $l(p_1) > n(k+1) > k$. If $q_1 = p_1 m$ then $yx^{n(k+1)} \cdots p_n yx^{n(k+1)} = myx^{n(k+1)} \cdots q_n yx^{n(k+1)}$. If $m \neq 1$ then $m = yx^{n(k+1)}r$ for some $r \in P(X)$. This implies that $l_x(q_1) \geqslant n(k+1)$ and, in consequence, $l_x(q_1 yx^{n(k+1)} \cdots q_n yx^{n(k+1)}) \geqslant n(n+1)(k+1)$. On the other hand $l_x(p_1 yx^{n(k+1)} \cdots p_n yx^{n(k+1)}) \leqslant l(p_1) + \cdots + l(p_n) + n^2(k+1) \leqslant kn + n^2(k+1) < n(n+1)(k+1)$, a contradiction.

THEOREM 2. *If I is a nil right ideal of $R\{X\}$ satisfying the identity $a^n = 0$ and $\langle A \rangle$ is the ideal of R generated by $A = \{a_p \mid p \in \operatorname{supp} a,\ a \in I\}$ then $\langle A \rangle^{2n} = 0$ and, when $1 \in R$, $\langle A \rangle^n = 0$.*

PROOF. Let $x, y \in X$, $x \neq y$, $a, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in R$. The definition of A implies that there exist $s_1, \ldots, s_n \in I$ and $q_1 \in \operatorname{supp} s_1, \ldots, q_n \in \operatorname{supp} s_n$ such that a_i is the coefficient at q_i in s_i for $i = 1, \ldots, n$. Since for $i \neq j$, $1 \leqslant i, j \leqslant n$, $\operatorname{supp} s_i b_i xy^i x \cap \operatorname{supp} s_j b_j xy^j x = \varnothing$ then $a_i b_i$ is the coefficient at $p_i = q_j xy^i x$ in $s = s_1 b_1 xyx + \cdots + s_n b_n xy^n x$. Now if $k = \max(l(p_1), \ldots, l(p_n))$ then by Lemma 3 $a_1 b_2 \cdots a_n b_n$ is the coefficient at $p_1 yx^{n(k+1)} \cdots p_n yx^{n(k+1)}$ in $(syx^{n(k+1)})^n$. But $syx^{n(k+1)} \in I$, so $(syx^{n(k+1)})^n = 0$. This shows that for any $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in R$, $a_1 b_1 \cdots a_n b_n = 0$. Hence if K is the right ideal of R generated by AR then $K^n = 0$. But if J is the right ideal of R generated by A then $J = K$ whenever $1 \in R$ and $J^2 \subseteq K$ otherwise. This and the fact that indexes of nilpotency of J and $\langle A \rangle$ are equal end the proof.

In particular Theorem 2 gives immediately

COROLLARY 3. *An ideal I of $R\{X\}$ is nilpotent if and only if the ideal of R generated by coefficients of all elements of I is nilpotent.*

Now we will describe nil right ideals of $R\{X\}$ for uncountable X.

LEMMA 4. *Let $a = \sum a_p p \in R\{X\}$. If $p_1, \ldots, p_m$ are such elements of* supp *a that for some $x \in X$, $(ax)^m = 0$ and $p_1, \ldots, p_m \in P(X \setminus \{x\})$, then $a_{p_1} \cdots a_{p_m} = 0$.*

PROOF. Let us observe first that if $q_1, \ldots, q_m \in P(X)$ the equality $p_1 x \cdots p_m x = q_1 x \cdots q_m x$ implies $p_1 = q_1, \ldots, p_m = q_m$. Indeed, since $l_x(p_1) = \cdots = l_x(p_m) = 0$ then $l_x(q_1) + \cdots + l_x(q_m) + m = l_x(q_1 x \cdots q_m x) = l_x(p_1 x \cdots p_m x) = m$. Hence $l_x(q_1) = \cdots = l_x(q_m) = 0$. Now the equality $p_1 x \cdots p_m x = q_1 x \cdots q_m x$

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implies that for some $r \in P$, $p_1 = q_1 r$ or $q_1 = p_1 r$. But then $rxp_2x \cdots p_m x = xq_2x \cdots q_m x$ or $xp_2x \cdots p_m x = rxq_2x \cdots q_m x$. In both cases the condition $l_x(r) = 0$ implies that $r = 1$, so $p_1 = q_1$. Now $p_2x \cdots p_m x = q_2 x \cdots q_m x$ and, analogously, we obtain that $p_2 = q_2, \ldots, p_2 = q_2, \ldots, p_m = q_m$. We conclude from the foregoing that the coefficient at $p_1 x \cdots p_m x$ in the series $(ax)^m$ is equal to $a_{p_1} \cdots a_{p_m}$, so $a_{p_1} \cdots a_{p_m} = 0$.

THEOREM 3. *If the set X is uncountable and I is a nil right ideal of $R\{X\}$ then the ideal of R generated by coefficients of an element of I is nilpotent.*

PROOF. Let $x, y \in X$, $x \neq y$ and $b_1, b_2, \ldots \in R$. It can be easily seen that if $a = \sum a_p p \in I$ and $p \in \operatorname{supp} a$ then the coefficient at $pxy^k x$ in $a(b_1 xyx + p_2 xy^2 x + \cdots)$ is equal to $a_p b_k$. The fact that I is a nil right ideal of $R\{X\}$ implies that $\bigcup X_n = X$, where $X_n = \{z \in X \mid (a(b_1 xyz + b_2 xy^2 x + \ldots)z)^n = 0\}$. Thus for some m the set X_m is uncountable. In particular for any $p_1, \ldots, p_m \in \operatorname{supp} a$ and any natural numbers $k_1, \ldots, k_m$ there exists $z \in X_m$ such that $p_1 xy^{k_1}x, \ldots, p_m xy^{k_m}x \in P(X \setminus \{z\})$. Hence from Lemma 4, $a_{p_1} b_{k_1} \cdots a_{p_m} b_{k_m} = 0$. Now Lemma 1 ends the proof.

COROLLARY 4. *If the set X is uncountable then $S(R\{X\}) = K(R\{X\}) = N(R\{X\})$.*

PROOF. Of course $S(R\{X\}) \supset K(R\{X\}) \supset N(R\{X\})$. By Theorem 3 the sum W of all nil right ideals of $R\{X\}$ is equal to $N(R\{X\})$ so $W = N(R\{X\}) = K(R\{X\})$. Now if $I/K(R\{X\})$ is a nil right ideal of $R\{X\}/K(R\{X\})$ then I is a nil right ideal of $R\{X\}$, so $I \subseteq W = K(R\{X\})$. Thus $R\{X\}/K(R\{X\})$ is S-semisimple. In consequence $S(R\{X\}) = K(R\{X\})$.

4. The case of one indeterminate

It is known (Puczyłowski (1980), Corollary 2) that if X is a set of cardinality $\geqslant 2$ then $R\{X\}$ is nil if and only if R is nilpotent or, equivalently, $R\{X\}$ is nilpotent. This is not true for power series rings $R\{x\}$ of one indeterminate x. Namely, let P be the polynomial ring of commutative indeterminates $x_1, x_2, \ldots$ over a finite field of p elements and let I be the ideal of P generated by $x_1^p, x_2^p, \ldots$. Since P is a commutative algebra over a field of characteristic p then for any $\sum a_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k} \in P$, $(\sum a_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k})^p = \sum a_{i_1 \cdots i_k}^p x_{i_1}^p \cdots x_{i_k}^p \in I$. Now if $\sum a_i x^i \in (P/I)\{x\}$ then $(\sum a_i x^i)^p = \sum a_i^p x^{pi} = 0$. Hence $(P/I)\{x\}$ is nil and P/I is not nilpotent as for any n, $x_1 \cdots x_n \notin I$.

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Gardner and Stewart (1976) have observed that the class $\bar{K} = \{R \mid R\{x\} \text{ is nil}\}$ is not radical. Now we will describe the class $\bar{K}$ more exactly.

THEOREM 4. *Any ring of the class* $\bar{K}$ *is nil of bounded index.*

PROOF. If a ring R is not nil of bounded index then for any $n = 1, 2, \ldots$ there exists $a_n \in R$, $a_n^n \neq 0$. Let $\{k_n\}$ be the sequence of integers defined by induction as follows

$$k_1 = 1, \qquad k_{n+1} = nk_n + 1.$$

Certainly if $l > n$ then $k_l \geqslant k_{n+1} = nk_n + 1$. Thus if $k_{i_1} + \cdots + k_{i_n} = nk_n$ then $i_r \leqslant n$ for $1 \leqslant r \leqslant n$. But since $k_{i_r} \leqslant k_n$ and $k_{i_r} = k_n$ if and only if $i_r = n$, then the equality $k_{i_1} + \cdots + k_{i_n} = nk_n$ implies $i_1 = \cdots = i_n = n$. Using this fact we obtain that the coefficient at x^{nk_n} in the series $(\sum a_i x^{k_i})^n$ is equal to $a_n^n \neq 0$. Thus the series $\sum a_i x^{k_i}$ is not nilpotent.

REMARK. The Nagata-Higman Theorem (Jacobson (1964), page 274) and Theorem 4 imply that members of $\bar{K}$ which are algebras over a field of characteristic zero are nilpotent.

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