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THE PALM-DUALITY FOR RANDOM SUBSETS OF d-DIMENSIONAL GRIDS

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Abstract

The Palm version of a stationary random subset of a d-dimensional grid is contructed using the two-step change-of-origin and change-of-measure method. An elementary proof is given of the fact that the Palm version is characterized by point-stationarity (distributional invariance under bijective shifts of the origin from a point of the set to another point of the set).

Keywords: Palm theory; point processes, stationary random field

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1. Introduction

Let $d > 1$ be an integer and G be a d-dimensional grid, that is, a locally finite additive subgroup of $\mathbb{R}^d$. The standard example is $G = \mathbb{Z}^d$. Note that $\mathbb{Q}^d$ is not a grid.

This paper presents a particular approach to the Palm theory of stationary simple point processes on G or, in other words, the Palm theory of stationary random subsets of G. It is the discrete counterpart of Thorisson (1999), (2000, Chapter 9), where the duality between a stationary point process in $\mathbb{R}^d$ and its 'point-stationary' Palm version is established using a two-step change-of-measure and change-of-origin method. Recall that the Palm version is obtained by conditioning the point process on having a point at the origin. Palm theory is of importance in fields such as queueing theory and stochastic geometry; for extensive expositions see Matthes *et al.* (1978), Neveu (1977), Daley and Vere-Jones (1988), and Kallenberg (2002).

Palm theory admits a considerable simplification in the present discrete context, in particular, in the treatment of point-stationarity. Point-stationarity formalizes the intuitive idea of a random set for which the behavior relative to a given point of the set is independent of the point selected as the origin; the point at the origin is a *typical point*. Note that this is different from stationarity which means that the behavior relative to a given nonrandom location is independent of the location selected as the origin; the origin is a *typical location*. A simple example of a stationary set in one dimension is the set of heads obtained by doubly infinite independent and identically distributed coin tosses. Whereas, if we condition on there being a head (a point) at the origin then the set of heads turns from being stationary to being point-stationary, since shifting to the nth head on the right (or left) does not change its distribution. Note that, in this example, choosing the nth head on the right is *reversible*; we can go back to the point we came from by choosing the nth head on the left. Also, note that the shift is *bijective*.

In Thorisson (1999), point-stationarity was defined to be 'distributional invariance under bijective point-shifts *against any independent stationary background*', and proved to be the

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characterizing property of the Palm version of a stationary point process in $\mathbb{R}^d$. The fact that the Palm version is distributionally invariant under bijective point-shifts is actually implicit in Mecke (1975), as pointed out by Heveling and Last (2005). For further background, see Thorisson (2000) and Heveling and Last (2005).

For some years now there has been considerable research activity related to removing the independent stationary background from the above definition of point-stationarity; see Ferrari *et al.* (2004), Holroyd and Peres (2003), and Timar (2004). Finally, Heveling and Last (2005) proved that this could be done. They proved that the Palm version of a stationary point process in $\mathbb{R}^d$ is characterized by distributional invariance under bijective point-shifts without the external randomization.

In this paper we shall show that, in the context of random subsets of G, the above point-stationarity problem is almost trivial. It should be possible to extend these results to general discrete groups, but the approach here relies heavily on the the structure of $\mathbb{R}^d$, in particular the proof of Theorem 4.1.

The plan of the paper is as follows. In Section 2, we consider the Palm version and introduce the change-of-measure and change-of-origin. In Section 3, we show that the Palm version is point-stationary in the sense of being distributionally invariant under shifts induced by reversible point maps. In Section 4, we show that a random subset point-stationary in this sense becomes stationary after the inverse change of measure and origin.

It should be noted that here we only treat the standard Palm duality where the point-stationary dual is the stationary set conditioned on having a point at the origin, and not the other Palm duality where the point-stationary dual is the stationary set seen from a typical point. It should also be noted that it is straightforward to extend the treatment here to random fields associated with the random set.

2. The Palm version of a stationary random set

A random set of points in G is easiest to represent as a collection of 0–1 valued random variables

$$X = (X_s : s \in G)$$

defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The random field X can also be viewed as a random element in $(H, \mathcal{H})$, where $H = \{0, 1\}^G$ and $\mathcal{H}$ is the product σ-algebra making all the projection maps taking $(x_s : s \in G)$ in H to x_i in $\{0, 1\}$, $i \in G$, measurable.

Define the shift maps θ_i, shifting the origin of X to a new location $i \in G$, by

$$\theta_i x := (x_{i+s} : s \in G), \qquad x \in H.$$

The random field X is *stationary* if

$$\theta_i X \stackrel{\mathrm{D}}{=} X, \qquad i \in G \qquad \text{(the origin is a typical } \textit{location}\text{)},$$

where '$\stackrel{\mathrm{D}}{=}$' denotes identity in distribution.

Call an element i of G a *location* and (with $x \in H$ given) call i a *point* only if $x_i = 1$. Also, call a d-dimensional random variable T taking values in G a *point* if $X_T = 1$ identically. Assume throughout the paper that there is at least one point. Associate to each point a subset of G – called a *cell* – in some shift-invariant way so that the cells are disjoint and cover G. For instance, we can take the cells to be the *Voronoi cells*. These are defined by associating to each point the locations in G that are closer to that point than to any other point. If there is a choice between two or more points, choose the one with the lowest lexicographic order.

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Put

$$C = C(X) := \text{the cell containing the origin } 0,$$
$$\Pi = \Pi(X) := \text{the point of } C.$$

We are now ready for the following change-of-origin:

$$X^\circ := \theta_\Pi X = (X_{\Pi+s} : s \in G),$$
$$C^\circ := C - \Pi = \text{the cell } C \text{ seen from its point},$$
$$S = S(X) := -\Pi = \text{the location of the original origin in } C^\circ.$$

Note that

$$|C^\circ| = |C| = \text{the number of elements in } C.$$

Theorem 2.1. *If X is stationary and has at least one point then, for all nonnegative $\mathcal{H}$-measurable functions f and all $i \in G$,*

$$\mathrm{E}[\mathbf{1}_{\{S=i\}}\, f(X^\circ)] = \mathrm{E}[\mathbf{1}_{\{i\in C^\circ\}}\, \mathbf{1}_{\{X_0=1\}}\, f(X)], \tag{2.1}$$
$$\mathrm{E}[f(X^\circ)] = \mathrm{E}[|C|\, \mathbf{1}_{\{X_0=1\}}\, f(X)], \tag{2.2}$$
$$\mathrm{E}[\mathbf{1}_{\{S=i\}}\, f(X^\circ)] = \mathrm{E}\left[\frac{\mathbf{1}_{\{i\in C^\circ\}}\, f(X^\circ)}{|C^\circ|}\right]. \tag{2.3}$$

Proof. Fix a nonnegative $\mathcal{H}$-measurable function f and an $i \in G$. Let g_i be the function defined on H by

$$g_i(X) = \mathbf{1}_{\{S=i\}}\, f(\theta_{-i} X)$$

and note that (since $S = i$ implies $\theta_{-i} X = X^\circ$)

$$g_i(X) = \mathbf{1}_{\{S=i\}}\, f(X^\circ),$$

and that (since $\{S(\theta_i X) = i\} = \{i \in C^\circ,\ X_0 = 1\}$)

$$g_i(\theta_i X) = \mathbf{1}_{\{i\in C^\circ\}}\, \mathbf{1}_{\{X_0=1\}}\, f(X).$$

Since $\theta_i X \stackrel{\mathrm{D}}{=} X$, this yields (2.1). Summing over $i \in G$ in (2.1), yields (2.2). Note that taking $f = 1$ in (2.2) yields $1 = \mathrm{E}[|C|\, \mathbf{1}_{\{X_0=1\}}]$ and, thus, $|C|$ is almost surely finite on $\{X_0 = 1\}$. This (and the fact that $|C^\circ| = |C|$) allows us to apply (2.2) with $f(X^\circ)$ replaced by $\mathbf{1}_{\{i\in C^\circ\}}\, f(X^\circ)/|C^\circ|$ to obtain

$$\mathrm{E}\left[\frac{\mathbf{1}_{\{i\in C^\circ\}}\, f(X^\circ)}{|C^\circ|}\right] = \mathrm{E}[\mathbf{1}_{\{i\in C^\circ\}}\, \mathbf{1}_{\{X_0=1\}}\, f(X)].$$

Compare this with (2.1) to obtain (2.3).

Corollary 2.1. *If X is stationary and has at least one point then* $\mathrm{P}(|C| < \infty) = 1$ *and the conditional distribution of S given X° is uniform on $|C^\circ|$, i.e.*

$$\mathrm{P}(S = i \mid X^\circ) = \frac{1}{|C^\circ|}, \qquad i \in C^\circ. \tag{2.4}$$

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Proof. Put $f(X^\circ) = \mathbf{1}_{\{|C|=\infty\}}$ in (2.3) to obtain

$$\mathrm{E}[\mathbf{1}_{\{S=i\}}\,\mathbf{1}_{\{|C|=\infty\}}] = 0,$$

and sum over $i \in G$ to obtain $\mathrm{P}(|C| = \infty) = 0$. Furthermore, (2.4) follows from (2.3) by the definition of conditional probabilities.

We are now ready for the change-of-measure. Define a new probability measure P° on $(\Omega, \mathcal{F})$ by

$$\mathrm{dP}^\circ = \frac{1}{|C|\,\mathrm{E}[1/|C|]}\,\mathrm{dP}. \tag{2.5}$$

Corollary 2.2. *If X is stationary and has at least one point, then*

$$\mathrm{P}(X_0 = 1) = \mathrm{E}\left[\frac{1}{|C|}\right] = \frac{1}{\mathrm{E}^\circ[|C^\circ|]}, \tag{2.6}$$

and X° under P° *is the* Palm version *of X, that is,*

$$\mathrm{P}(X \in A \mid X_0 = 1) = \mathrm{P}^\circ(X^\circ \in A), \qquad A \in \mathcal{H}. \tag{2.7}$$

Proof. Put $f(X^\circ) = f(X) = 1/|C|$ in (2.2) to obtain the first identity in (2.6), and use (2.5) to obtain the second. In order to establish (2.7), replace $f(X)$ in (2.2) by $f(X)/|C|$ and $f(X^\circ)$ by $f(X^\circ)/|C|$ to obtain

$$\mathrm{E}[\mathbf{1}_{\{X_0=1\}}\, f(X)] = \mathrm{E}\left[\frac{f(X^\circ)}{|C|}\right],$$

and divide by $\mathrm{P}(X_0 = 1) = \mathrm{E}[1/|C|]$ to obtain the first identity in

$$\mathrm{E}[f(X) \mid X_0 = 1] = \frac{\mathrm{E}[f(X^\circ)/|C|]}{\mathrm{E}[1/|C|]} = \mathrm{E}^\circ[f(X^\circ)],$$

while the second identity follows from (2.5). Take $f = \mathbf{1}_A$ to obtain (2.7).

3. The Palm version is point-stationary

Put

$$H^\circ = \{x \in H : x_0 = 1\} \quad \text{and} \quad \mathcal{H}^\circ = \mathcal{H} \cap H^\circ.$$

Call an $\mathcal{H}^\circ$-measurable map π from H° to G a *point-map* if

$$x_{\pi(x)} = 1, \qquad x \in H^\circ.$$

Define the associated *point-shift* θ_π from H° to H° by

$$\theta_\pi x = \theta_{\pi(x)} x, \qquad x \in H^\circ.$$

Call π and θ_π *reversible* if there is a point-map ν such that

$$\nu(\theta_\pi x) = -\pi(x), \qquad x \in H^\circ.$$

Note that then θ_π is a bijection with inverse θ_ν, i.e.

$$\pi(\theta_\nu x) = -\nu(x), \qquad x \in H^\circ,$$

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and note that

$$\{X_i = 1,\ i + \pi(\theta_i X) = 0\} = \{X_0 = 1,\ \nu(X^\circ) = i\}. \tag{3.1}$$

Also, note that with $x \in H^\circ$ fixed the mapping from the set of points $\{s : x_s = 1\}$ to itself, taking i to $i + \pi(\theta_i x)$ is a bijection if and only if π is reversible.

Definition 3.1. Say that a random field X° is *point-stationary* if $X^\circ_0 = 1$ identically and, for all reversible π,

$$\theta_\pi X^\circ \stackrel{\mathrm{D}}{=} X^\circ \quad \text{(the origin is a typical } \textit{point}\text{)}.$$

Theorem 3.1. *If X is stationary under* P *and has at least one point, then $X^\circ := \theta_\Pi X$ is point-stationary (in the sense of Definition 3.1) under* P° *defined in (2.5).*

Proof. Take π reversible, f nonnegative $\mathcal{H}^\circ$-measurable, and $i \in G$. Note that if g is the function defined by

$$g(X) = \mathbf{1}_{\{X_0=1\}}\, \mathbf{1}_{\{\pi(X^\circ)=-i\}}\, f(\theta_\pi X),$$

then

$$g(\theta_i X) = \mathbf{1}_{\{X_i=1\}}\, \mathbf{1}_{\{i+\pi(\theta_i X)=0\}}\, f(\theta_\pi \theta_i X).$$

Since X and $\theta_i X$ have the same distribution under P, this yields the first identity in

$$\begin{aligned}\mathrm{E}[\mathbf{1}_{\{X_0=1\}}\, \mathbf{1}_{\{\pi(X^\circ)=-i\}}\, f(\theta_\pi X)] &= \mathrm{E}[\mathbf{1}_{\{X_i=1\}}\, \mathbf{1}_{\{i+\pi(\theta_i X)=0\}}\, f(\theta_\pi \theta_i X)] \\ &= \mathrm{E}[\mathbf{1}_{\{X_0=1\}}\, \mathbf{1}_{\{\nu(X^\circ)=i\}}\, f(X)],\end{aligned}$$

while the second follows from (3.1) and the fact that

$$f(\theta_\pi \theta_i X) = f(X) \quad \text{on } \{X_i = 1,\ i + \pi(\theta_i X) = 0\}.$$

Sum over $i \in G$ to obtain

$$\mathrm{E}[\mathbf{1}_{\{X_0=1\}}\, f(\theta_\pi X)] = \mathrm{E}[\mathbf{1}_{\{X_0=1\}}\, f(X)].$$

Divide by $\mathrm{P}(X_0 = 1)$ on both sides and compare with (2.7) to obtain

$$\mathrm{E}^\circ[f(\theta_\pi X^\circ)] = \mathrm{E}^\circ[f(X^\circ)],$$

that is, the distribution of $\theta_\pi X^\circ$ under P° does not depend on π.

4. From point-stationarity back to stationarity

In this section we shall reverse Theorem 3.1 to show that a point-stationary random subset is the Palm version of a stationary one provided the expected number of elements in the cell containing the origin is finite. The following theorem is the key result.

Theorem 4.1. *Let $X^\circ = (X^\circ_s : s \in G)$ be a family of zero-one variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P}^\circ)$ with $X^\circ_0 = 1$ identically. Let C° be the cell containing the origin. If X° is point-stationary (in the sense of Definition 3.1) then*

$$\mathrm{E}^\circ\Bigg[\sum_{s\in i+C^\circ} f(\theta_s X^\circ)\Bigg] = \mathrm{E}^\circ\Bigg[\sum_{s\in C^\circ} f(\theta_s X^\circ)\Bigg], \tag{4.1}$$

for all nonnegative $\mathcal{H}^\circ$-measurable f and all $i \in G$.

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Proof. Let K be the subset of G consisting of the vectors $\boldsymbol{k} = (k_1, \ldots, k_d)$ such that $\boldsymbol{k}$ is not in nG for any integer $n \geq 2$ and such that the first nonzero k_j is positive. Since G is locally finite, the set K is nonempty and G is the disjoint union of $\{0\}$ and the half-lines $-\mathbb{N}\boldsymbol{k}$ and $\mathbb{N}\boldsymbol{k}$, $\boldsymbol{k} \in K$, where $\mathbb{N} = \{1, 2, \ldots\}$. With $\boldsymbol{k} \in K$, order the elements of $\mathbb{Z}\boldsymbol{k}$ in the natural way so that, with m and n integers, $m\boldsymbol{k} < n\boldsymbol{k}$ denotes that $m < n$.

Define reversible point-maps $\pi_n^{\boldsymbol{k}}$, $n \in \mathbb{Z}$, by ordering the points on the two-sided lines $\mathbb{Z}\boldsymbol{k}$ as follows. For fixed $\boldsymbol{k} \in K$ and $x \in H^\circ$, put

$$\pi_0^{\boldsymbol{k}}(x) = 0,$$

$$a = \sum_{s \in -\mathbb{N}\boldsymbol{k}} x_s = \text{number of points on the half-line } -\mathbb{N}\boldsymbol{k},$$

$$b = \sum_{s \in \mathbb{N}\boldsymbol{k}} x_s = \text{number of points on the half-line } \mathbb{N}\boldsymbol{k},$$

$$c = a + b + 1 = \text{number of points on the two-sided line } \mathbb{Z}\boldsymbol{k}.$$

If $a = b = \infty$, define recursively, for $n \in \mathbb{N}$,

$$\pi_n^{\boldsymbol{k}}(x) = \inf\{s \in \mathbb{N}\boldsymbol{k} \colon s > \pi_{n-1}^{\boldsymbol{k}}(x) \text{ and } x_s = 1\},$$

$$\pi_{-n}^{\boldsymbol{k}}(x) = \sup\{s \in -\mathbb{N}\boldsymbol{k} \colon s < \pi_{-n+1}^{\boldsymbol{k}}(x) \text{ and } x_s = 1\}.$$

If $a < \infty$ and $b = \infty$, put

$$p_0^{\boldsymbol{k}}(x) := \inf\{s \in \mathbb{Z}\boldsymbol{k} \colon x_s = 1\} = \text{the lowest point on the line } \mathbb{Z}\boldsymbol{k}, \tag{4.2}$$

and recursively, for $n \in \mathbb{N}$, define

$$p_n^{\boldsymbol{k}}(x) = \inf\{s \in \mathbb{Z}\boldsymbol{k} \colon s > p_{n-1}^{\boldsymbol{k}}(x) \text{ and } x_s = 1\}, \tag{4.3}$$

and then put, for $n \in \mathbb{N}$,

$$q_0^{\boldsymbol{k}}(x) = p_0^{\boldsymbol{k}}(x), \qquad q_n^{\boldsymbol{k}}(x) = p_{2n}^{\boldsymbol{k}}(x), \quad \text{and} \quad q_{-n}^{\boldsymbol{k}}(x) = p_{2n-1}^{\boldsymbol{k}}(x),$$

to obtain a two-sided sequence as follows: let α be such that $q_\alpha^{\boldsymbol{k}}(x) = 0$ and put

$$\pi_n^{\boldsymbol{k}}(x) = q_{\alpha+n}^{\boldsymbol{k}}(x), \qquad n \in \mathbb{Z}.$$

If $a = \infty$ and $b < \infty$, proceed in an analogous way to the case when $a < \infty$ and $b = \infty$.

Finally, if $a < \infty$ and $b < \infty$, define $p_n^{\boldsymbol{k}}(x)$, $0 \leq n < c$, by (4.2) and (4.3), and then put

$$\pi_n^{\boldsymbol{k}}(x) = p_{(a+n) \,(\mathrm{mod}\, c)}^{\boldsymbol{k}}(x), \qquad n \in \mathbb{Z}.$$

Note that in this last case the two-sided sequence $\pi_n^{\boldsymbol{k}}(x)$, $n \in \mathbb{Z}$, repeats itself with period c. Note also that in all the above cases, $\pi_{-n}^{\boldsymbol{k}}$ is the reverse of $\pi_n^{\boldsymbol{k}}$.

Now, take $\boldsymbol{k} \in K$ and $n \in \mathbb{Z}$ and put

$$C_n^{\boldsymbol{k}} = \text{the cell of } X^\circ \text{ containing } \pi_n^{\boldsymbol{k}}(X^\circ),$$

$$A^{\boldsymbol{k}} = \bigcup_{n \neq 0} C_n^{\boldsymbol{k}},$$

$$A = \bigcup_{\boldsymbol{k} \in K} A^{\boldsymbol{k}} = G \setminus C^\circ.$$

Take $1 \le m \le \infty$, $i \in G$, and f nonnegative $\mathcal{H}^\circ$-measurable. Let

$$I_m^{\boldsymbol{k}} \text{ be the indicator of the event } \Big\{ \sum_{s \in \mathbb{Z}\boldsymbol{k}} X_s^\circ = m \Big\},$$

and note that if g is the function defined by

$$g(X^\circ) = I_m^{\boldsymbol{k}} \sum_{s \in (i+C^\circ) \cap C_n^{\boldsymbol{k}}} f(\theta_s X^\circ),$$

then

$$g(\theta_{\pi_{-n}^{\boldsymbol{k}}} X^\circ) = I_m^{\boldsymbol{k}} \sum_{s \in (i+C_{-n}^{\boldsymbol{k}}) \cap C^\circ} f(\theta_s X^\circ).$$

Since X° and $\theta_{\pi_{-n}^{\boldsymbol{k}}} X^\circ$ have the same distribution under P°, we obtain

$$\mathrm{E}^\circ\Big[I_m^{\boldsymbol{k}} \sum_{s \in (i+C^\circ) \cap C_n^{\boldsymbol{k}}} f(\theta_s X^\circ) \Big] = \mathrm{E}^\circ\Big[I_m^{\boldsymbol{k}} \sum_{s \in (i+C_{-n}^{\boldsymbol{k}}) \cap C^\circ} f(\theta_s X^\circ) \Big].$$

Sum over $n \neq 0$ if $m = \infty$ and over $0 < n < m$ if $m < \infty$, to obtain

$$\mathrm{E}^\circ\Big[I_m^{\boldsymbol{k}} \sum_{s \in (i+C^\circ) \cap A^{\boldsymbol{k}}} f(\theta_s X^\circ) \Big] = \mathrm{E}^\circ\Big[I_m^{\boldsymbol{k}} \sum_{s \in (i+A^{\boldsymbol{k}}) \cap C^\circ} f(\theta_s X^\circ) \Big].$$

Next, sum over $1 \le m \le \infty$ to get rid of $I_m^{\boldsymbol{k}}$ on both sides, and then over $\boldsymbol{k} \in K$ to obtain

$$\mathrm{E}^\circ\Big[\sum_{s \in (i+C^\circ) \cap A} f(\theta_s X^\circ) \Big] = \mathrm{E}^\circ\Big[\sum_{s \in (i+A) \cap C^\circ} f(\theta_s X^\circ) \Big].$$

Finally, add $\mathrm{E}^\circ[\sum_{s \in (i+C^\circ) \cap C^\circ} f(\theta_s X^\circ)]$ to both sides to obtain (4.1).

Corollary 4.1. *Let* $\mathrm{E}^\circ[|C^\circ|] < \infty$ *and let the conditional distribution of* S *given* X° *be uniform on* C°*. If* X° *is point-stationary (in the sense of Definition 3.1) then*

$$X := \theta_S X^\circ \quad \textit{(the inverse change-of-origin)}, \tag{4.4}$$

is stationary under the probability measure P *defined on* $(\Omega, \mathcal{F})$ *by*

$$\mathrm{dP} := \frac{|C^\circ|}{\mathrm{E}^\circ[|C^\circ|]}\, \mathrm{dP}^\circ \quad \textit{(the inverse change-of-measure)}. \tag{4.5}$$

Proof. Take f nonnegative $\mathcal{H}^\circ$-measurable and $i \in G$ and apply (4.4) and (4.5) to obtain

$$\mathrm{E}[f(\theta_i X)] = \frac{\mathrm{E}^\circ[|C^\circ| f(\theta_{i+S} X^\circ)]}{\mathrm{E}^\circ[|C^\circ|]}.$$

Since S is uniform on C° we obtain

$$\mathrm{E}[f(\theta_i X)] = \frac{\mathrm{E}^\circ[\sum_{s \in i+C^\circ} f(\theta_s X^\circ)]}{\mathrm{E}^\circ[|C^\circ|]}.$$

Apply (4.1) to obtain

$$\mathrm{E}[f(\theta_i X)] = \frac{\mathrm{E}^\circ[\sum_{s \in C^\circ} f(\theta_s X^\circ)]}{\mathrm{E}^\circ[|C^\circ|]}.$$

This means that the distribution of $\theta_i X$ under P does not depend on i.

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