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ON HEREDITARILY LINDELÖF SPACES

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This paper considers the question of when a space with the property that each discrete subspace is countable is hereditarily Lindelöf. The question is answered affirmatively for the class of R_0 P spaces and for the class of hereditarily meta-Lindelöf spaces. A characterization of hereditarily Lindelöf spaces in terms of countable subspaces is given.

1. Introduction

For some time topologists have been interested in the question: if each discrete subspace of a compact Hausdorff space is countable, is the space hereditarily Lindelöf? In general, the answer is negative [8]. In this paper we answer this question in the affirmative for two classes of spaces different from the compact Hausdorff spaces, namely for the class of R_0 P spaces and the class of hereditarily meta-Lindelöf spaces. A topological space (X, T) is R_0 if whenever a point belongs to an open set its closure is contained in that open set; that is, $x \in G \in T$ implies $\mathrm{cl}\{x\} \subset G$. A space X is a P space if each G_δ subset of X is open. The topological properties of P spaces have been studied by Misra [5].

It was shown by Nedev [6] that a symmetrizable space X is hereditarily Lindelöf if and only if each discrete subspace of X is countable, and this result was extended to a larger family of F spaces by Harley and Stephenson [4]. We prove that, in general, a space X is

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hereditarily Lindelöf if and only if each anti-Lindelöf subspace of X is countable. A space is anti-Lindelöf if its only Lindelöf subspaces are countable. Furthermore, we show that if X is R_0 and a P space and each of its discrete subspaces is countable then X is hereditarily Lindelöf. We also show that the R_0 P hypothesis can be replaced by a hereditarily meta-Lindelöf condition.

It can be argued that the question we are concerned with could be more naturally posed in the category of Lindelöf Hausdorff P spaces than in the category of compact Hausdorff spaces. For example, we have the result of Misra [5, Proposition 4.2 (f)] that the Lindelöf Hausdorff P topology on a set is maximal Lindelöf and minimal Hausdorff P. It follows immediately from Theorem 2 below that the question has an affirmative answer in the category of Lindelöf Hausdorff P spaces.

Our proofs make use of the concept of topological anti-properties, especially the anti-Lindelöf property, introduced by Bankston [2], whose set-theoretic and notational conventions we follow. In particular, the cardinality of a set Y is denoted by $|Y|$ and the symbol $\square$ denotes the end of a proof. In Section 2 we give the relevant simple properties of the class of anti-Lindelöf spaces. A more detailed discussion is available in [2] and [7]. In Section 3 we prove our results by showing that the anti-(anti-Lindelöf) spaces are precisely the hereditarily Lindelöf spaces.

2. Anti-Lindelöf spaces

In general, if K is a class of topological spaces, the spectrum of K, denoted by $\text{spec}(K)$, is the class of cardinal numbers κ such that any topology on a set of power κ lies in K. For example, any topology on a countable set must be Lindelöf, and any uncountable set having the discrete topology is not Lindelöf. Thus $\text{spec}(\text{Lindelöf}) = \Omega$. Anti-$K$ is defined to be the class of spaces X such that whenever Y is a subspace of X then $Y \in K$ if and only if $|Y| \in \text{spec}(K)$.

Anti-K spaces, where K is a class of spaces defined by one of several covering properties, have been discussed in [2] and [7], and where K is defined by one of the separation properties, in [7]. Here we restrict our attention to the anti-Lindelöf spaces which were first considered in [2, Section 4].

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Any uncountable set with the cocountable topology is not anti-Lindelöf because it is Lindelöf. Any countable space is anti-Lindelöf as well as Lindelöf. The following result provides a non-trivial example of an anti-Lindelöf space.

PROPOSITION. *Let X be a topological space partitioned into countable subsets by an uncountable open cover. Then X is anti-Lindelöf.*

Proof. Let C be such a cover for X and E be an uncountable subset of X. Then E meets uncountably many members of C, and these give an uncountable open cover for E which has no countable subcover, so that E is not Lindelöf. □

COROLLARY. *The ordinal space $[0, \Omega)$ is anti-Lindelöf.*

Proof. For each $i \in [0, \Omega)$ we define G_i as follows:

$$G_0 = [0, \omega+1),\ G_1 = (\omega, 2\omega+1),\ G_2 = (2\omega, 3\omega+1),\ \ldots,$$
$$G_i = (i\omega, (i+1)\omega+1),\ \ldots .$$

Then $C = \{G_i : i \in [0, \Omega)\}$ is an uncountable open cover which partitions $[0, \Omega)$ into countable subsets, so that $[0, \Omega)$ is anti-Lindelöf, by the previous proposition. □

3. Hereditarily Lindelöf spaces

We show that a double application of the anti-$(\cdot)$ operation to the class of Lindelöf spaces yields the class of hereditarily Lindelöf spaces. First we need a lemma, similar in conclusion to Theorem 3 of Stephenson [*10*].

LEMMA 1. *Every non-Lindelöf space X has an uncountable anti-Lindelöf subspace. Furthermore, if X is R_0, and a P space, it has an uncountable discrete subspace.*

Proof. Let C be an open cover of X which has no countable subcover. Let $x_0 \in X$. Then there is a $G_0 \in C$ such that $x_0 \in G_0$. Let $x_1 \in X-G_0$ and $G_1 \in C$ be such that $x_1 \in G_1$. By transfinite induction we get, for each $i \in [0, \Omega)$, an $x_i \in G_i - \cup\{G_j : j < i\}$. Let $E = \{x_i : i \in [0, \Omega)\}$ and F be any uncountable subset of E. Then

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$\{G_i : x_i \in F\}$ is an open cover of F which has no countable subcover. Hence F is not Lindelöf and thus E is anti-Lindelöf.

Next if X is an R_0 P space, we let $V_0 = G_0$, $V_1 = G_1 \cap (X\text{-cl}\{x_0\})$, and $V_\alpha = G_\alpha \cap (X\text{-}\cup\{\text{cl}\{x_i\} : i < \alpha\})$ for each $\alpha \in [0, \Omega)$. Then the set E is uncountable and discrete, since $V_\alpha \cap E = \{x_\alpha\}$. □

THEOREM 1. *X is hereditarily Lindelöf if and only if X is anti-(anti-Lindelöf).*

Proof. It is clear that spec(anti-Lindelöf) = Ω, for the indiscrete topology on any uncountable set is Lindelöf and hence not anti-Lindelöf.

Let X be hereditarily Lindelöf, and suppose X is not anti-(anti-Lindelöf). Then there is a subspace Y of X such that Y is anti-Lindelöf but $|Y| \notin$ spec(anti-Lindelöf). Hence Y is uncountable and therefore not Lindelöf, contradicting X is hereditarily Lindelöf.

Conversely, let X be anti-(anti-Lindelöf). If X is not hereditarily Lindelöf there is a non-Lindelöf subspace W of X. By Lemma 1, W has an uncountable anti-Lindelöf subspace E. But $|E| \notin$ spec(anti-Lindelöf), contradicting the fact that X is anti-(anti-Lindelöf). □

Restating Theorem 1 we have that X is hereditarily Lindelöf if and only if each anti-Lindelöf subspace of X is countable. While any discrete space is anti-Lindelöf the converse is false. Nevertheless, the second part of Lemma 1 enables us to obtain the following result.

THEOREM 2. *If X is an R_0 P space and each discrete subspace of X is countable then X is hereditarily Lindelöf.*

Proof. If X is not hereditarily Lindelöf there is a non-Lindelöf subspace Y of X. By Lemma 1, Y has an uncountable discrete subspace, contradicting the hypothesis. □

In the following result, the R_0 P hypothesis of Lemma 1 is replaced by meta-Lindelöfness.

LEMMA 2. *If X is meta-Lindelöf and non-Lindelöf it has an*

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uncountable discrete subspace.

Proof. Let C be an open cover of X which has no countable subcover. Let $\mathcal{D}$ be a point-countable open refinement of C. Then $\mathcal{D}$ has no countable subcover. Let $x_0 \in X$ and let $x_0 \in G_0 \in \mathcal{D}$. Then there is an $x_1 \in G_1 - \cup\{G \in \mathcal{D} : x_0 \in G\}$, since the latter collection is countable. Now, by transfinite induction, we get for each $i \in [0, \Omega)$, an $x_i \in G_i - \cup\{G \in \mathcal{D} : x_j \in G,$ for some $j < i\}$. Clearly, the set $\{x_i : i \in [0, \Omega)\}$ is uncountable and discrete. □

THEOREM 3. *If X is hereditarily meta-Lindelöf and each discrete subspace of X is countable then X is hereditarily Lindelöf.*

Proof. If X is not hereditarily Lindelöf there is a non-Lindelöf subspace Y of X, and since Y is meta-Lindelöf, Lemma 2 yields an uncountable discrete subspace of Y, and hence of X. □

REMARKS. The proof of Lemma 2 above is an adaptation of an argument of Boyte [3]. For T_1 spaces, the assertion in Theorem 3 above is a consequence of Corollary 1 in [1].

EXAMPLE. Let X be an uncountable set, p be a fixed point in X, and T be the Fortissimo topology [9, p. 53] on X. Thus a set $G \subset X$ is open if and only if $X - G$ is countable or contains p. Then (X, T) is an R_0 P space which is hereditarily metacompact, but not an F space. Firstly, X is T_1 and hence R_0. Secondly, let $G = \cap\{G_n : n \in N,$ the positive integers$\}$, where each G_n is open. Then $X - G = \cup\{X-G_n : n \in N\}$, and if $p \in X-G$, G is open. Otherwise, each $X - G_n$ is countable, and so is $X - G$. Again G is open, and X is a P space. Thirdly, let E be any subset of X and $C = \{G_i : i \in I\}$ be an open cover of E. If $p \in E$, then $p \in G_j$ for some $j \in I$, and take $V_p = G_j$. If $x \in E-V_p$ take $V_x = \{x\}$. Then the collection $\{V_x : x \in E\}$ is an open locally finite refinement of C. Thus X is hereditarily paracompact. Finally, let $G = X - \{p\}$. Then G is an open dense subset of X. Let S be any sequence in G. Then $X - S$ is an open neighbourhood of p not meeting S, so that S does not converge to

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p . Thus X is not a sequential space, and hence not an F space [4, Theorem 2.8]. Thus (X, T) is an R_0 P space which is hereditarily metacompact, but not an F space.

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