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Abstract

We study the Rankin–Selberg integral for a pair of representations of $\mathrm{SO}_{2l} \times \mathrm{GL}_n$, where SO_{2l} is defined over a local non-Archimedean field and is either split or quasi-split. The integrals span a fractional ideal, and its unique generator, which contains any pole which appears in the integrals, is called the greatest common divisor (gcd) of the integrals. We describe the properties of the gcd and establish upper and lower bounds for the poles. In the tempered case we can relate it to the L-function of the representations defined by Shahidi. Results of this work may lead to a gcd definition for the L-function.

1. Introduction

Let F be a local non-Archimedian field. We study the Rankin–Selberg integral for a finite type generic representation π of $\mathrm{SO}_{2l}(F)$ and an irreducible generic representation τ of $\mathrm{GL}_n(F)$. The group $\mathrm{SO}_{2l}(F)$ will be either split or quasi-split, i.e. split over a quadratic extension of F. Jacquet *et al.* [JPS83] constructed this integral for $\mathrm{GL}_k(F) \times \mathrm{GL}_n(F)$. Our construction follows the method devised by Gelbart and Piatetski-Shapiro [GPR87] for $G(F) \times \mathrm{GL}_n(F)$, where $G(F)$ is a split classical group of rank n. This method has been extended by Ginzburg [Gin90] to the split group $G = \mathrm{SO}_l$ with $\lfloor l/2 \rfloor \geqslant n$ and later by Ginzburg *et al.* [GPR97] to an orthogonal group G, corresponding to a quadratic form with an arbitrary Witt index, where the representation of G is not necessarily generic.

Henceforth references to the field are omitted from the notation. Let $V(\tau, s)$ be the space of the representation of SO_{2n+1} parabolically induced from $\tau|\det|^{s-\frac{1}{2}}$, $s \in \mathbb{C}$. Let $M(\tau, s): V(\tau, s) \to V(\tau^*, 1-s)$ be the standard intertwining operator, where τ^* is isomorphic to the contragredient representation, and $M^*(\tau, s)$ be the standard normalized intertwining operator. An element $f_s \in V(\tau, s)$ is called a standard section if its restriction to a certain fixed maximal compact subgroup is independent of s. Denote by $\xi(\tau, \mathrm{std}, s)$ the space of standard sections. Let $\xi(\tau, \mathrm{hol}, s) = \mathbb{C}[q^{-s}, q^{s}] \otimes \xi(\tau, \mathrm{std}, s)$ be the space of holomorphic sections (where q is the cardinality of the residue field of F). Following the method of Piatetski-Shapiro and Rallis [PR87, PR86] (see below) we define the set of 'good sections',

$$\xi(\tau, \mathrm{good}, s) = \xi(\tau, \mathrm{hol}, s) \cup M^*(\tau^*, 1-s)\xi(\tau^*, \mathrm{hol}, 1-s).$$

The Rankin–Selberg integrals $\Psi(W, f_s, s)$, where W is a Whittaker function for π and $f_s \in \xi(\tau, \mathrm{good}, s)$, satisfy a functional equation which was used in [Kap10b] to define the γ-factor $\gamma(\pi \times \tau, \psi, s)$ (where ψ is a fixed additive character of the field). In [Kap10b] we proved

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that $\gamma(\pi \times \tau, \psi, s)$ is multiplicative in both variables. Here we extend this study and define the greatest common divisor (gcd) and the ϵ-factor. The integrals span a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$, and its unique generator, in the form $P(q^{-s})^{-1}$ for $P \in \mathbb{C}[X]$ such that $P(0) = 1$, is what we call the gcd of the integrals, $\gcd(\pi \times \tau, s)$. The functional equation defining $\epsilon(\pi \times \tau, \psi, s)$, an exponential, is

$$\frac{\Psi(W, M^*(\tau, s)f_s, 1-s)}{\gcd(\pi \times \tau^*, 1-s)} = \epsilon(\pi \times \tau, \psi, s)\frac{\Psi(W, f_s, s)}{\gcd(\pi \times \tau, s)}$$

(the actual equation is slightly different, see (3.6)). The primary focus of this work is the gcd and its properties. Motivated by the work of Jacquet *et al.* [JPS83], we establish several key properties that may lead to a definition of the L-function of π and τ as $\gcd(\pi \times \tau, s)$. Such a definition provides another point of view on the L-function. It is expected to have many applications, since the poles of the integrals indicate relations between the representations. This work may also have applications in analyzing the poles of the global L-function; in fact this was the original motivation for the definition of good sections (see below).

The present definition of the L-function of $\pi \times \tau$ is due to the work of Shahidi on his method of local coefficients (e.g. [Sha81, Sha90]). In the tempered case we essentially relate the gcd to this L-function. We need the following assumption.

Conjecture 1.1. The factor $\gamma(\pi \times \tau, \psi, s)$ is identical, up to a normalization factor in $\mathbb{C}[q^{-s}, q^s]^*$, with Shahidi's γ-factor.

This conjecture is actually a theorem in the split case according to the results of [Kap10a, Kap10b, Kap12, Sou95]. In the quasi-split case some more work on the Archimedean integrals is needed. Soudry [Sou93, Sou95, Sou00] proved the conjecture for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. The normalization factor depends on the groups, on ψ and on certain central characters (and is independent of s).

Theorem 1.1. *Let π and τ be tempered representations for which Conjecture 1.1 holds. Let $L(\pi \times \tau, s)$ be the L-function attached to π and τ by Shahidi. Then $L(\pi \times \tau, s)^{-1}$ divides $\gcd(\pi \times \tau, s)^{-1}$ and $\gcd(\pi \times \tau, s) \in L(\pi \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^s]$, where $M_\tau(s)$ (defined in § 3.2) contains the poles of $M^*(\tau, s)$ and $M^*(\tau^*, 1-s)$, $M_\tau(s)^{-1} \in \mathbb{C}[q^{-s}, q^s]$. Moreover, if the operators $L(\tau, S^2, 2s-1)^{-1}M(\tau, s)$ and $L(\tau^*, S^2, 1-2s)^{-1}M(\tau^*, 1-s)$ are holomorphic,*

$$\gcd(\pi \times \tau, s) = L(\pi \times \tau, s).$$

According to a result of Casselman and Shahidi [CS98, Theorem 5.1], $L(\tau, S^2, 2s-1)^{-1}$ $M(\tau, s)$ is holomorphic for an irreducible supercuspidal τ, and is expected to be holomorphic for a tempered τ. Therefore, Theorem 1.1 states that in the tempered case, under a reasonable assumption on the operators (holding in the supercuspidal case), the gcd definition gives the L-function defined by Shahidi. Without the assumption, the gcd and this L-function are equal up to the poles of $M_\tau(s)$. For the general case (regardless of Conjecture 1.1) we have the following upper and lower bounds.

Theorem 1.2. *Let $\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_k}}^{\mathrm{GL}_n}(\tau_1 \otimes \cdots \otimes \tau_k)$ be an irreducible representation, where $P_{n_1,\ldots,n_k}$ is a parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$. Then*

$$\gcd(\pi \times \tau, s) \in \left(\prod_{i=1}^{k} \gcd(\pi \times \tau_i, s)\right) M_{\tau_1 \otimes \cdots \otimes \tau_k}(s)\mathbb{C}[q^{-s}, q^s].$$

Theorem 1.3. *Let $\pi = \mathrm{Ind}_{P_k}^{\mathrm{SO}_{2l}}(\sigma \otimes \pi')$ be a finite type representation, where P_k is a parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_k \times \mathrm{SO}_{2(l-k)}$. Let $\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_a}}^{\mathrm{GL}_n}(\tau_1 \otimes \cdots \otimes \tau_a)$ be irreducible ($a \geqslant 1$). Then*

$$\gcd(\pi \times \tau, s) \in L(\sigma \times \tau, s)\left(\prod_{i=1}^{a} \gcd(\pi' \times \tau_i, s)\right) L(\sigma^* \times \tau, s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s)\mathbb{C}[q^{-s}, q^{s}]. \tag{1.1}$$

Here $L(\sigma \times \tau, s)$ and $L(\sigma^* \times \tau, s)$ are the L-factors of [JPS83]. In the case of $k = l \leqslant n$, relation (1.1) holds under the assumptions that σ is irreducible and τ is of Langlands' type. Note that, for $k = l$, by definition $\gcd(\pi' \times \tau_i, s) \equiv 1$.

Theorem 1.4. *Let π be irreducible and τ be tempered such that $L(\tau, S^2, 2s-1)^{-1}M(\tau, s)$ and $L(\tau^*, S^2, 1-2s)^{-1}M(\tau^*, 1-s)$ are holomorphic. Write $\pi = \mathrm{Ind}_{P_k}^{\mathrm{SO}_{2l}}(\sigma \otimes \pi')$ as a standard module (in particular π' is tempered: see § 6.2). Then, if $k < l$ or $k = l > n$,*

$$L(\sigma \times \tau, s) \gcd(\pi' \times \tau, s) L(\sigma^* \times \tau, s) \in \gcd(\pi \times \tau, s)\mathbb{C}[q^{-s}, q^{s}].$$

The factor $M_{\tau_1 \otimes \cdots \otimes \tau_k}(s)$ (defined in § 3.2) designates an upper bound to the poles of $M^*(\tau, s)$ and $M^*(\tau^*, 1-s)$. We also prove more detailed versions of Theorems 1.2 and 1.3 (see Corollaries 8.1 and 9.1, respectively). The following is an immediate corollary of Theorems 1.1, 1.3 and 1.4.

Corollary 1.1. *Let π and τ be as in Theorem 1.4 and assume that Conjecture 1.1 holds for π' and τ. Then $L(\pi \times \tau, s)^{-1}$ divides $\gcd(\pi \times \tau, s)^{-1}$ and $\gcd(\pi \times \tau, s) \in L(\pi \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^{s}]$.*

The Rankin–Selberg method for studying Langlands' automorphic L-functions is to find integral representations for these functions. We briefly describe the steps of the method, following Cogdell [Cog06]. The global integral admits a factorization into an Euler product of local factors, called the local Rankin–Selberg integrals. In order to relate the global integral to the L-function, one computes the local integrals with unramified data and shows that they produce local L-functions. This is roughly sufficient to determine the analytic properties of the restricted L-function. In order to study the global L-function, the local integrals at the finite ramified places as well as the Archimedean places must be studied. The local analysis typically involves a functional equation and local factors, namely L, γ and ϵ-factors.

In their pioneering work Jacquet *et al.* [JPS83] defined and analyzed the local Rankin–Selberg integrals at the finite places, for a pair of generic representations of $\mathrm{GL}_k \times \mathrm{GL}_n$. They defined the L-factor as the gcd of the integrals and computed it inductively, up to representations of Langlands' type. This work is an attempt to carry out the local analysis for $\mathrm{SO}_{2l} \times \mathrm{GL}_n$. Our approach is somewhat similar to [JPS83] and oftentimes we adapt their arguments.

The main difference between our construction and the setting of $\mathrm{GL}_k \times \mathrm{GL}_n$ is the intertwining operator in the functional equation. The first attempt to define a gcd in our setting is to consider the fractional ideal spanned by $\Psi(W, f_s, s)$ with $f_s \in \xi(\tau, \mathrm{hol}, s)$. However, this definition does not fit well in the functional equation. If only holomorphic sections were used, the ratio $\gcd(\pi \times \tau^*, 1-s)^{-1}\Psi(W, M^*(\tau, s)f_s, 1-s)$ might not be a polynomial, because $M^*(\tau, s)f_s$ is not necessarily a holomorphic section. In turn the ϵ-factor might not be an exponential.

Moreover, there is an intrinsic problem with considering only holomorphic sections. Assume that π and τ are irreducible supercuspidal. Then one can show that for $f_s \in \xi(\tau, \mathrm{hol}, s)$, $\Psi(W, f_s, s)$ is holomorphic (see Corollary 4.2), while the Langlands' L-function $L(\pi \times \tau, s)$ may have a pole.

The notion of good sections was introduced by Piatetski-Shapiro and Rallis [PR86, PR87], in the construction of a (global) Rankin triple L-function, in order to analyze the poles of the restricted L-function defined over the set of finite primes. Ikeda [Ike92, Ike99] extended their work and studied the gcd in the unramified and Archimedean cases. The method of good sections has also been applied by Harris *et al.* [HKS96], in the context of the local theta correspondence between unitary groups, in order to define ϵ-factors.

Although the actual definition of good sections varies to some extent among the studies mentioned above, there are essential properties of such a set that are common to all. Firstly, it is stable under the normalized intertwining operator, i.e. the operator is a bijection of good sections. Secondly, it contains the holomorphic sections, required in order for the integrals to span a fractional ideal.

The idea of using good sections instead of just holomorphic addresses the lack of symmetry in the functional equation. Since $M^*(\tau, s)\xi(\tau, \mathrm{good}, s) = \xi(\tau^*, \mathrm{good}, 1 - s)$, both sides of the equation are polynomials. The gcd may contain poles originating from the intertwining operator. It is then quickly seen that $\epsilon(\pi \times \tau, \psi, s) \in \mathbb{C}[q^{-s}, q^s]^*$. Regarding the scenario of supercuspidal representations mentioned above, $\Psi(W, f_s, s)$ for a good section f_s might not be holomorphic.

One of the basic properties required of the integrals is meromorphic continuation to functions in $\mathbb{C}(q^{-s})$. Therefore the largest reasonable set of sections to consider would be rational sections, i.e. $\mathbb{C}(q^{-s}) \otimes \xi(\tau, \mathrm{std}, s)$. We are looking for poles of the integrals rather than poles introduced by sections, and hence we try to be conservative in our usage of rational sections. Theorem 1.1 and Corollary 1.1 yield that the gcd captures the notion of the L-function, for a certain range of parameters. The next case to consider would be an arbitrary irreducible generic τ.

We explain the role of the factors $M_\tau(s), M_{\tau_1 \otimes \cdots \otimes \tau_k}(s)$ in our results. The bound $\gcd(\pi \times \tau, s) \in L(\pi \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^s]$ of Theorem 1.1 implies that $\gcd(\pi \times \tau, s)$ may include poles from both $M^*(\tau^*, 1 - s)$ and $M^*(\tau, s)$, which do not appear in $L(\pi \times \tau, s)$. Theorem 5.1 of Casselman and Shahidi [CS98] is used for showing that $\gcd(\pi \times \tau, s)$ cannot contain poles from $M^*(\tau, s)$. In this way it 'separates' $\gcd(\pi \times \tau, s)$ from $\gcd(\pi \times \tau^*, 1 - s)$, resembling the expectation that the quotient $L(\pi \times \tau^*, 1 - s)L(\pi \times \tau, s)^{-1}$ would be reduced (see e.g. [Cog06]). The factor $M_{\tau_1 \otimes \cdots \otimes \tau_k}(s)$ in the bounds reflects a certain multiplicative property of the poles of the intertwining operators.

The last paragraph also illustrates a (possible) shortcoming of the method of good sections. Namely, it is difficult to control the additional poles of the intertwining operators (see § 8.4). Note that (even in the tempered case) the relation between the analytic properties of the intertwining operators and the local coefficients is still only partially understood.

The present work is among the few attempts so far to provide a gcd definition to the L-function, and the first attempt to carry over the work of Jacquet *et al.* [JPS83] to the Rankin–Selberg convolutions of $G \times \mathrm{GL}_n$, for a classical group G. Such a study was suggested by Gelbart and Piatetski-Shapiro [GPR87, p. 136]. Our technique and results readily adapt to the integrals studied by Ginzburg [Gin90] and Soudry [Sou93], due to the similar nature and technical closeness of the constructions.

Shahidi's definition of L-functions for irreducible generic representations [Sha90] begins with the tempered case, then uses Langlands' classification. It would be interesting to show that in the general case, the gcd also factorizes according to this classification.

The rest of this paper is organized as follows. In § 2 we supply basic definitions. In § 3 we describe the integrals and the intertwining operators, and define the local factors. The basic tools used in the study of the integrals are developed in § 4. Section 5 contains the proof of a weak lower bound on the gcd, which is used to prove Theorem 1.4. In § 6 we prove Theorems 1.1 and 1.4. Section 7 provides the basic framework for proving Theorems 1.2 and 1.3, whose proof occupies §§ 8 and 9 (respectively).

2. Preliminaries

2.1 The groups

Let F be a local non-Archimedean field of characteristic zero. Denote by $\mathcal{O}$ the ring of integers of F; $\mathcal{P}=\varpi\mathcal{O}$ is the maximal ideal and $|\varpi|^{-1}=q=|\mathcal{O}/\mathcal{P}|$. For $k\geqslant 1$ let $J_k\in \mathrm{GL}_k(F)$ be the matrix with 1 on the anti-diagonal and 0 elsewhere. Let $\rho\in F^*$. If $\rho\in F^2$, define $J_{2l,\rho}=J_{2l}$ and set $\rho=\beta^2$. Otherwise write $J_{2l,\rho}=\mathrm{diag}(I_{l-1},\left(\begin{smallmatrix}0&1\\-\rho&0\end{smallmatrix}\right),I_{l-1})\cdot J_{2l}$. We use ρ to define the special even orthogonal group $\mathrm{SO}_{2l}(F)$. Let

$$G_l(F)=\mathrm{SO}_{2l}(F)=\{g\in \mathrm{SL}_{2l}(F):{}^tgJ_{2l,\rho}g=J_{2l,\rho}\},$$

regarded as an algebraic group over F. When $\rho=\beta^2$ it is split, and we may assume that $|\beta|=1$. Otherwise it is quasi-split, i.e. non-split over F and split over a quadratic extension of F, and we may assume $|\rho|\geqslant 1$. Also let $\gamma=\frac{1}{2}\rho$. Throughout, references to the field are omitted.

Fix the Borel subgroup $B_{G_l}=T_{G_l}\ltimes U_{G_l}$, where U_{G_l} is the subgroup of upper triangular unipotent matrices in G_l. When G_l is quasi-split,

$$T_{G_l}=\left\{\mathrm{diag}\left(t_1,\ldots,t_{l-1},\begin{pmatrix}a&b\rho\\b&a\end{pmatrix},t_{l-1}^{-1},\ldots,t_1^{-1}\right):a^2-b^2\rho=1\right\}.$$

Let $P_k=L_k\ltimes V_k$ be the standard maximal parabolic subgroup with $L_k=\{\mathrm{diag}(x,y,J_k({}^tx^{-1})J_k):x\in\mathrm{GL}_k,y\in G_{l-k}\}\cong \mathrm{GL}_k\times G_{l-k}$, $V_k<U_{G_l}$. When G_l is split, we have an additional standard maximal parabolic subgroup ${}^{\kappa}P_l=\kappa^{-1}P_l\kappa$ where $\kappa=\mathrm{diag}(I_{l-1},J_2,I_{l-1})$. For any parabolic subgroup $P<G_l$, $\overline{P}$ is the parabolic subgroup opposite to P containing the Levi part of P and δ_P is the modulus character of P.

For any $l'\leqslant l$, $G_{l'}$ is embedded in $L_{l-l'}\cong \mathrm{GL}_{l-l'}\times G_{l'}$, thereby viewed as a subgroup of G_l. In addition, let K_{G_l} be a special good maximal compact open subgroup (e.g. in the split case $K_{G_l}=G_l(\mathcal{O})$). For $k>0$, $\mathcal{N}_{G_l,k}=(I_{2l}+M_{2l\times 2l}(\mathcal{P}^k))\cap G_l$ is a 'small' compact open (subgroup) neighborhood of the identity.

The special odd orthogonal group is

$$H_n=\mathrm{SO}_{2n+1}=\{g\in\mathrm{SL}_{2n+1}:{}^tgJ_{2n+1}g=J_{2n+1}\}.$$

We use a notation similar to the above for H_n (e.g. B_{H_n}, $K_{H_n}=H_n(\mathcal{O})$). Let $Q_k=M_k\ltimes U_k$ be the standard parabolic subgroup with a Levi part $M_k\cong\mathrm{GL}_k\times H_{n-k}$. For $n'\leqslant n$, $H_{n'}$ is embedded in H_n through $M_{n-n'}$.

In the group GL_k, fix the Borel subgroup $B_{\mathrm{GL}_k}=A_k\ltimes Z_k$, where A_k is the diagonal subgroup and Z_k is the subgroup of upper triangular unipotent matrices. For $k_1,\ldots,k_m\geqslant 1$ a partition of k ($m>1$), let $P_{k_1,\ldots,k_m}=A_{k_1,\ldots,k_m}\ltimes Z_{k_1,\ldots,k_m}$ be the standard parabolic subgroup of GL_k which corresponds to the partition $(k_1,\ldots,k_m)$. Its Levi part $A_{k_1,\ldots,k_m}$ is isomorphic to $\mathrm{GL}_{k_1}\times\cdots\times\mathrm{GL}_{k_m}$. Also denote by Y_k the mirabolic subgroup of GL_k, i.e. the subgroup of $m\in\mathrm{GL}_k$ with the last row $(0,\ldots,0,1)$. For an element $x\in\mathrm{GL}_k$ we define $x^*=J_k({}^tx^{-1})J_k$.

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Oftentimes GL_k will be regarded as a subgroup of G_l (respectively H_n), embedded in $L_{k'}$ (respectively $M_{k'}$) for $k' \geqslant k$. Since the embedding of A_n in M_n is just T_{H_n}, we identify T_{H_n} with A_n. Regarding G_l, T_{G_l} is identified with $G_1 \times A_{l-1}$.

In general, for a subgroup Y we write ${}^gY = g^{-1}Yg$ and ${}^gy = g^{-1}yg$.

2.2 Embedding G_l in H_n, $l \leqslant n$

Let F^k be the k-dimensional column space. Denote by $(\ ,\ )$ the symmetric form defined on F^{2n+1} by J_{2n+1} (i.e., $(u,v) = {}^tuJ_{2n+1}v$). Let $\mathcal{E}_{H_n} = (e_1, \dots, e_n, e_{n+1}, e_{-n}, \dots, e_{-1})$ be the standard basis of F^{2n+1}, such that the Gram matrix of $\mathcal{E}_{H_n}$ with respect to $(\ ,\ )$ is J_{2n+1} (e.g. $e_{-i} = e_{2n+2-i}$). Set $e_\gamma = e_n + \gamma e_{-n}$ (recall that $\gamma = \frac{1}{2}\rho$). The image of G_l in H_n is $\mathrm{SO}(V)$ where V is the orthogonal complement of

$$\mathrm{Span}_F\{e_1, \dots, e_{n-l}, e_\gamma, e_{-(n-l)}, \dots, e_{-1}\}.$$

We select a basis $\mathcal{E}_{G_l}$ of V with a Gram matrix $J_{2l,\rho}$. To write $g \in G_l$ in coordinates relative to $\mathcal{E}_{H_n}$, we form $\mathcal{E}'_{G_l}$ by adding e_γ to $\mathcal{E}_{G_l}$ as the $(l+1)$th vector, extend g by defining $ge_\gamma = e_\gamma$ and compute, for M the transition matrix from $\mathcal{E}'_{G_l}$ to $\mathcal{E}_{H_l}$,

$$[g]_{\mathcal{E}_{H_n}} = \mathrm{diag}(I_{n-l}, M^{-1}[g]_{\mathcal{E}'_{G_l}}M, I_{n-l}).$$

If G_l is split,

$$M = \mathrm{diag}\left(I_{l-1}, \begin{pmatrix} \frac{1}{4} & \frac{1}{2\beta} & -\frac{1}{2\beta^2} \\ \frac{1}{2} & 0 & \frac{1}{\beta^2} \\ -\frac{1}{2}\beta^2 & \beta & 1 \end{pmatrix}, I_{l-1}\right).$$

Otherwise

$$M = \mathrm{diag}\left(I_{l-1}, \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2}\gamma^{-1} \\ \frac{1}{2} & 0 & -\frac{1}{2}\gamma^{-1} \end{pmatrix}, I_{l-1}\right).$$

Remark 2.1. The vector e_γ is defined using $\frac{1}{2}\rho$ (instead of ρ), so that, when $\rho \in F^2$ (hence G_l is split), the Witt index of the orthogonal complement of $\mathrm{Span}_F\{e_\gamma\}$ would be n. This is necessary for embedding split G_l in H_l.

Remark 2.2. In the construction of the integral for $l < n$, e_γ is used to define a certain character ψ_γ of a unipotent subgroup N_{n-l}. The embedding $G_l < H_n$ is such that G_l normalizes N_{n-l} and stabilizes ψ_γ. See § 4.2.

One property of this embedding is that, although $A_{l-1} < T_{H_n}$, T_{G_l} is not a subgroup of T_{H_n}. We write explicitly the form of an element from G_1 under the embedding in H_1, in the split case.

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Of course this embedding carries over to H_n by the embedding $H_1 < H_n$:

$$x = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}+\frac{1}{4}(b+b^{-1}) & \frac{1}{2\beta}(b-b^{-1}) & \frac{2}{\beta^2}\left(\frac{1}{2}-\frac{1}{4}(b+b^{-1})\right) \\ \frac{1}{4}\beta(b-b^{-1}) & \frac{1}{2}(b+b^{-1}) & -\frac{1}{2\beta}(b-b^{-1}) \\ \frac{1}{2}\beta^2\left(\frac{1}{2}-\frac{1}{4}(b+b^{-1})\right) & -\frac{1}{4}\beta(b-b^{-1}) & \frac{1}{2}+\frac{1}{4}(b+b^{-1}) \end{pmatrix}. \tag{2.1}$$

It will be useful to denote $\lfloor x \rfloor = b$ if $|b| \leqslant 1$ and $\lfloor x \rfloor = b^{-1}$ otherwise. Also let $[x] = \max(|b|, |b|^{-1})$. The following lemma shows how to write a torus element of G_l approximately as a torus element of H_n. Put

$$\jmath(a,b) = \begin{pmatrix} a & b & -\frac{1}{2}b^2a^{-1} \\ & 1 & -ba^{-1} \\ & & a^{-1} \end{pmatrix} \in H_1 \quad (a \in F^*, b \in F).$$

LEMMA 2.1. *For any $k_0 > 0$ there are $h_1, h_2 \in H_1$ and $k \geqslant k_0$ such that for all $x = \mathrm{diag}(b, b^{-1})$, with $[x] > q^k$, in H_1 we have $x \in B_{H_1}h_1\mathcal{N}_{H_1,k_0}$ if $\lfloor x \rfloor = b$, and $x \in B_{H_1}h_2\mathcal{N}_{H_1,k_0}$ if $\lfloor x \rfloor = b^{-1}$. Specifically, $x \in m_xu_xh_i\mathcal{N}_{H_1,k_0}$ with $m_x = \jmath(\lfloor x \rfloor, 0)$, $u_x = \jmath(1, c\lfloor x \rfloor^{-1})$ where $c = -2\beta^{-1}$ if $\lfloor x \rfloor = b$ and $c = 2\beta^{-1}$ if $\lfloor x \rfloor = b^{-1}$.*

Proof of Lemma 2.1. Let $\epsilon = \mathrm{diag}(\beta^{-1}, 1, \beta)$, let $k' \geqslant k_0$ be such that ${}^{\epsilon^{-1}}\mathcal{N}_{H_1,k'} < \mathcal{N}_{H_1,k_0}$, and take $k'' \geqslant k'$ such that $q^{-2k''} \leqslant |2|q^{-k'}$.

In H_1, x is given by (2.1). Write $x' = {}^{\epsilon}x$. We will exhibit $m' \in B_{H_1}$ and h' such that $m'x' \in h'\mathcal{N}_{H_1,k'}$, whence $m = {}^{\epsilon^{-1}}m'$ and $h = {}^{\epsilon^{-1}}h'$ satisfy $mx \in h\mathcal{N}_{H_1,k_0}$. Let $m' = \jmath(t, tv)$ and, for $\xi \in F$, $u(\xi) = \mathrm{diag}(1, -1, 1) \cdot J_3 \cdot \jmath(1, -\xi) \in H_1$.

We select $k \geqslant k'$ for which $q^{-k} < |4|q^{-k'}$. Suppose that $\lfloor x \rfloor = b$ and assume $|b| < q^{-k}$. Take $v = 2 + 4b(1 + 2\varpi^{k''})$, $t^{-1} = -8b(1 + \varpi^{k''})$ and $h' = u(2)$. By matrix multiplication we see that

$$m'x' \in h'\mathcal{N}_{H_1,k'} = \begin{pmatrix} \mathcal{P}^{k'} & \mathcal{P}^{k'} & 1+\mathcal{P}^{k'} \\ \mathcal{P}^{k'} & -1+\mathcal{P}^{k'} & -2+\mathcal{P}^{k'} \\ 1+\mathcal{P}^{k'} & -2+\mathcal{P}^{k'} & -2+\mathcal{P}^{k'} \end{pmatrix}.$$

Note that if one assumes $|2| = 1$, to verify this computation it is enough to check the last two rows of $m'x'$, and then use the fact that $m'x' \in H_1$.

Returning to x, we have obtained $x \in m^{-1}h\mathcal{N}_{H_1,k_0}$ with $m^{-1} = \jmath(-8b(1 + \varpi^{k''}), -2\beta^{-1}(1 + 2b + 4b\varpi^{k''}))$, $h = \jmath(\beta^{-2}, 0)u(2\beta^{-1})$. Then $m^{-1} = m_xu_xz$ with $z = \jmath(-8(1 + \varpi^{k''}), -4\beta^{-1}(1 + 2\varpi^{k''}))$. Let $h_1 = zh$. Then h_1 depends only on k'', k', k_0. Also $x \in m_xu_xh_1\mathcal{N}_{H_1,k_0}$.

Regarding $\lfloor x \rfloor = b^{-1}$, take $v = -2 - 4b^{-1}(1 + 2\varpi^{k''})$, $t^{-1} = -8b^{-1}(1 + \varpi^{k''})$ and $h' = u(-2)$. $\square$

In the quasi-split case, it will usually be sufficient to know that the image of G_1 in H_1 is a compact subgroup which is covered, for any $k > 0$, by a finite union $\bigcup x_i\mathcal{N}_{H_1,k}$ ($x_i \in H_1$).

2.3 Embedding H_n in G_l, $n < l$

Here (,) is defined using $J_{2l,\rho}$. Let $\mathcal{E}_{G_l} = (e_1, \ldots, e_l, e_{-l}, \ldots, e_{-1})$ be the standard basis of F^{2l} such that the Gram matrix of $\mathcal{E}_{G_l}$ with respect to (,) is $J_{2l,\rho}$. Let $e_\gamma = \frac{1}{4}e_l - \gamma e_{-l}$ when G_l is split and $e_\gamma = \frac{1}{2}e_{-l}$ otherwise. The group H_n is embedded in G_l as $\mathrm{SO}(V)$, where V is the orthogonal

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complement of

$$\mathrm{Span}_F\{e_1,\ldots,e_{l-n-1},e_\gamma,e_{-(l-n-1)},\ldots,e_{-1}\}.$$

If $\mathcal{E}_{H_n}$ is a basis of V with a Gram matrix J_{2n+1}, $\mathcal{E}'_{H_n}$ is obtained from $\mathcal{E}_{H_n}$ by adding e_γ as the $(n+1)$th vector. Extend $h\in H_n$ by defining $he_\gamma=e_\gamma$; then

$$[h]_{\mathcal{E}_{G_l}}=\mathrm{diag}(I_{l-n-1},M^{-1}[h]_{\mathcal{E}'_{H_n}}M,I_{l-n-1}).$$

Here $M=\mathrm{diag}(I_n,\left(\begin{smallmatrix}2&-\beta^{-2}\\ \beta&(2\beta)^{-1}\end{smallmatrix}\right),I_n)$ in the split case and $\mathrm{diag}(I_n,\left(\begin{smallmatrix}0&2\\ 1&0\end{smallmatrix}\right),I_n)$ otherwise.

2.4 Representations

Representations will always be smooth, admissible, of finite type and generic, i.e. admit unique Whittaker models. Tempered representations are assumed to be irreducible by definition. We fix a non-trivial additive character ψ of F and construct a character of Z_k by $z\mapsto\psi(\sum_{i=1}^{k-1}z_{i,i+1})$ (ψ is non-degenerate for $k>1$). If a representation π has a Whittaker model with respect to a character χ, the model is denoted by $\mathcal{W}(\pi,\chi)$. If π has a central character, it is denoted by ω_π. A representation of GL_k is always assumed to have a central character.

For $s\in\mathbb{C}$ and $g\in\mathrm{GL}_n$ denote $\alpha^s(g)=|\det g|^{s-\frac{1}{2}}$. For a representation τ of GL_n, denote by τ^* the representation (of GL_n) defined on the space of τ by $\tau^*(g)=\tau(g^*)$ ($g^*=J_n({}^tg^{-1})J_n$). When τ is irreducible, $\mathcal{W}(\widetilde{\tau},\psi)=\mathcal{W}(\tau^*,\psi)$, where $\widetilde{\tau}$ is the contragredient representation.

Let $\lambda(x)f$ (respectively $x\cdot f$) be the left-translation (respectively right-translation) of f by x.

2.5 Sections

We recall the definition of a holomorphic section of a parabolically induced representation, parameterized by an unramified character of the Levi part. For a thorough treatment of this subject refer to Waldspurger [Wal03, § 4] and Muić [Mui08].

Let τ be a representation of GL_n on the space U. Set $K=K_{H_n}$. Consider the induced representation $\mathrm{Ind}_{Q_n\cap K}^{K}(\tau)$ whose space we denote by $V(\tau)=V_{Q_n\cap K}^{K}(\tau)$. For any fixed $s\in\mathbb{C}$ we have the representation $\mathrm{Ind}_{Q_n}^{H_n}(\tau\alpha^s)$ (normalized induction) in the space $V(\tau,s)=V_{Q_n}^{H_n}(\tau,s)$. Any $f\in V(\tau)$ can be extended to an element of $V(\tau,s)$, according to the Iwasawa decomposition. This produces an isomorphism between $V(\tau)$ and $V(\tau,s)$ as K-representation spaces. The image of $f\in V(\tau)$ in $V(\tau,s)$ is denoted by f_s.

Consider the following family of functions. For $k\in K$, $N<K$ a compact open subgroup and $v\in U$ which is invariant by $({}^{k^{-1}}N)\cap Q_n$, define $\mathrm{ch}_{kN,v}\in V(\tau)$ by

$$\mathrm{ch}_{kN,v}(k')=\begin{cases}\tau(a)v & k'=akn,\, a\in Q_n\cap K,\, n\in N,\\ 0 & \text{otherwise.}\end{cases}$$

These functions span $V(\tau)$ (see [BZ76, 2.24]). Then $\mathrm{ch}_{kN,v}$ extends to $\mathrm{ch}_{kN,v,s}\in V(\tau,s)$.

We will usually consider s as a parameter. A function $f(s,h):\mathbb{C}\times H_n\to U$ such that for all s the mapping $h\mapsto f(s,h)$ belongs to $V(\tau,s)$ is called a section.

A section f is called standard if for any fixed $k\in K$ the function $s\mapsto f(s,k)$ is independent of s (i.e., it is a constant function). Let $\xi_{Q_n}^{H_n}(\tau,\mathrm{std},s)$ be the space of standard sections. The elements of this space are precisely the functions $(s,h)\mapsto f_s(h)$ where $f\in V(\tau)$. The subgroup K acts on this space by $k\cdot f(s,h)=f(s,hk)$. For $f\in\xi_{Q_n}^{H_n}(\tau,\mathrm{std},s)$ there is a compact open subgroup $N<K$ such that $f(s,hy)=f(s,h)$ for all $s\in\mathbb{C}$, $h\in H_n$ and $y\in N$. Furthermore, there is a subset $B\subset H_n$ such that for all s the support of the function $h\mapsto f(s,h)$ equals B. The subset B is invariant for multiplication on the right by N and on the left by Q_n.

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We usually pick a section f and study the function $h \mapsto f(s,h)$ as s varies. Therefore we introduce the following convention. We write, *a priori*, f_s instead of f and think of s as a parameter. So for any $f_s \in \xi_{Q_n}^{H_n}(\tau, \mathrm{std}, s)$ there is a compact open subgroup $N < K$ independent of s, such that f_s is right-invariant by N, and the support of f_s in H_n is independent of s. In order to obtain a concrete function on H_n, we must fix s. Then we say that for a fixed s, $f_s \in V(\tau, s)$.

The space of holomorphic sections is $\xi_{Q_n}^{H_n}(\tau, \mathrm{hol}, s) = \mathbb{C}[q^{-s}, q^s] \otimes_{\mathbb{C}} \xi_{Q_n}^{H_n}(\tau, \mathrm{std}, s)$. We use the abbreviated notation $\xi(\tau, \cdot, s) = \xi_{Q_n}^{H_n}(\tau, \cdot, s)$. A holomorphic section takes the form $f_s = \sum_{i=1}^m P_i(q^{-s}, q^s) f_s^{(i)}$ where $P_i \in \mathbb{C}[q^{-s}, q^s]$, $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$. The following claim shows that $\xi(\tau, \mathrm{hol}, s)$ is an H_n-space, where H_n acts by right-translations on the second component of the tensors.

Claim 2.1. *For any $f_s \in \xi(\tau, \mathrm{hol}, s)$ and $h \in H_n$, we have $h \cdot f_s \in \xi(\tau, \mathrm{hol}, s)$.*

Proof of Claim 2.1. It is enough to prove the claim for $f_s \in \xi(\tau, \mathrm{std}, s)$. Take $N < K$ as above. Given $h \in H_n$, $h \cdot f_s$ is right-invariant on $N_h = ({}^{h^{-1}}N) \cap K$. Let $k_1, \ldots, k_m \in K$ be distinct representatives of the double coset space $Q_n \backslash H_n / N_h$. For any k_i, write $k_i h = q_i k_i' \in Q_n K$, $q_i = a_i u_i$ $(a_i \in \mathrm{GL}_n,\ u_i \in U_n)$. Then $h \cdot f_s(k_i) = P_i v_i$ where $P_i = \delta_{Q_n}^{\frac{1}{2}}(q_i) |\det a_i|^{s - \frac{1}{2}} \in \mathbb{C}[q^{-s}, q^s]$ and $v_i = \tau(a_i) f_s(k_i') \in U$. Note that v_i is invariant by $({}^{k_i^{-1}}N_h) \cap Q_n$ and independent of s because $k_i' \in K$ and f_s is standard. Hence we have $\mathrm{ch}_{k_i N_h, v_i} \in V(\tau)$ and $h \cdot f_s = \sum_{i=1}^m P_i \cdot \mathrm{ch}_{k_i N_h, v_i, s} \in \xi(\tau, \mathrm{hol}, s)$. $\square$

For an element $f_s \in \xi(\tau, \mathrm{hol}, s)$ there is a compact open subgroup $N < K$ independent of s such that f_s is right-invariant by N. Whenever we take a subgroup by which f_s is right-invariant, we implicitly mean such a subgroup. The support of f_s in H_n may depend on s.

Let $f_s = \sum_{i=1}^m P_i f_s^{(i)} \in \xi(\tau, \mathrm{hol}, s)$ with $0 \neq P_i \in \mathbb{C}[q^{-s}, q^s]$, and $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$. Pick a subgroup N such that $f_s^{(i)}$ is right-invariant by N for all i. Then if $k_1, \ldots, k_b \in K$ are distinct representatives for $Q_n \backslash H_n / N$, then $f_s = \sum_{i,j} P_i \,\mathrm{ch}_{k_j N, v_{i,j}, s}$. Let $U^{k_j} \subset U$ be the subspace invariant by $({}^{k_j^{-1}}N) \cap Q_n$. Since the function $\phi : U^{k_j} \to V(\tau)$ given by $\phi(v) = \mathrm{ch}_{k_j N, v}$ is linear, we may rewrite f_s so that, for each j, the nonzero vectors in $\{v_{1,j}, \ldots, v_{m,j}\}$ are linearly independent. Consequently f_s can be written as

$$f_s = \sum_{i=1}^{m'} P_i' \cdot \mathrm{ch}_{k_i' N, v_i', s}, \tag{2.2}$$

with $P_i' \neq 0$, $k_i' \in \{k_1, \ldots, k_b\}$ (the double cosets $Q_n k_i' N$ are not necessarily disjoint), and the data (m', k_i', N, v_i') do not depend on s. Moreover, if $\{i_1, \ldots, i_c\}$ is a set of indices satisfying $k_{i_1}' = \cdots = k_{i_c}'$, then $v_{i_1}', \ldots, v_{i_c}'$ are linearly independent.

We will mostly be dealing with either holomorphic sections or the images of such, under specific intertwining operators. This leads us to consider the space $\xi_{Q_n}^{H_n}(\tau, \mathrm{rat}, s) = \mathbb{C}(q^{-s}) \otimes_{\mathbb{C}} \xi(\tau, \mathrm{std}, s)$ of rational sections. It is also an H_n-space.

In a slightly more general context, let G be one of the groups defined in § 2.1 and $P < G$ be a parabolic subgroup with a Levi part $L \cong \mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_m} \times G'$, where G' is either the trivial group $\{1\}$ or a group of the same type as G. Assume that τ is a representation of L. An m-tuple $\underline{s} = (s_1, \ldots, s_m) \in \mathbb{C}^m$ defines an unramified character $\alpha^{\underline{s}}$ of L by $(g_1, \ldots, g_m, g') \mapsto \alpha^{s_1}(g_1) \cdot \cdots \cdot \alpha^{s_m}(g_m)$. The space $V_{P \cap K_G}^{K_G}(\tau)$ of the representation $\mathrm{Ind}_{P \cap K_G}^{K_G}(\tau)$ is isomorphic to the space $V_P^G(\tau, \underline{s})$ of $\mathrm{Ind}_P^G(\tau \alpha^{\underline{s}})$ (as K_G-spaces). The holomorphic sections are $\xi_P^G(\tau, \mathrm{hol}, \underline{s}) = \mathbb{C}[q^{\mp s_1}, \ldots, q^{\mp s_m}] \otimes_{\mathbb{C}} \xi_P^G(\tau, \mathrm{std}, \underline{s})$. Note that K_G may be chosen such that $\alpha^{\underline{s}}|_{K_G} \equiv 1$.

2.6 Equality up to units

Consider the polynomial ring $\mathbb{C}[q^{-s}, q^s]$ and its field of fractions $\mathbb{C}(q^{-s})$. We denote by $\simeq$ an equality of polynomials or rational functions which holds up to invertible factors of $\mathbb{C}[q^{-s}, q^s]$. For example, $q^{-s} \simeq 1$ and for any $P \in \mathbb{C}[q^{-s}, q^s]$ there is some $P' \in \mathbb{C}[q^{-s}]$ such that $P \simeq P'$.

3. The integrals and local factors

3.1 The integrals

We present the integrals for $G_l \times \mathrm{GL}_n$ and a pair of representations $\pi \times \tau$. Let π be a representation of G_l whose underlying space is $\mathcal{W}(\pi, \psi_\gamma^{-1})$, where ψ_γ is the generic character of U_{G_l} given by

$$\psi_\gamma(u) = \begin{cases} \psi\Big(\sum_{i=1}^{l-2} u_{i,i+1} + \frac{1}{4} u_{l-1,l} - \gamma u_{l-1,l+1}\Big) & G_l \text{ is split,} \\ \psi\Big(\sum_{i=1}^{l-2} u_{i,i+1} + \frac{1}{2} u_{l-1,l+1}\Big) & G_l \text{ is quasi-split.} \end{cases}$$

Let τ be an irreducible representation of GL_n realized in $\mathcal{W}(\tau, \psi)$. For $s \in \mathbb{C}$, form the representation $\mathrm{Ind}_{Q_n}^{H_n}(\tau\alpha^s)$ (see § 2.5). An element $f_s \in V(\tau, s) = V_{Q_n}^{H_n}(\tau, s)$ is regarded as a function on $H_n \times \mathrm{GL}_n$, where for any $h \in H_n$ the function $b \mapsto f_s(h, b)$ lies in $\mathcal{W}(\tau, \psi)$.

Remark 3.1. The assumption that τ is irreducible is not needed for the definition of the integrals *per se*. Several properties proved below will hold without this restriction. The irreducibility is needed for the intertwining operators. To avoid confusion we restrict ourselves, *a priori*, to an irreducible representation τ.

There are two possible forms for the integral according to the size of l relative to n.

Definition 3.1. Let $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$, $f_s \in V(\tau, s)$.

(i) For $l \leqslant n$ the integral is

$$\Psi(W, f_s, s) = \int_{U_{G_l}\backslash G_l} W(g) \int_{R_{l,n}} f_s(w_{l,n} r g, 1) \psi_\gamma(r)\, dr\, dg,$$

where

$$w_{l,n} = \begin{pmatrix} & \gamma I_l & & & \\ & & & & I_{n-l} \\ & & (-1)^{n-l} & & \\ I_{n-l} & & & & \\ & & & \gamma^{-1} I_l & \end{pmatrix},$$

$$R_{l,n} = \left\{ \begin{pmatrix} I_{n-l} & x & y & 0 & z \\ & I_l & 0 & 0 & 0 \\ & & 1 & 0 & y' \\ & & & I_l & x' \\ & & & & I_{n-l} \end{pmatrix} \right\} < H_n$$

and $\psi_\gamma(r) = \psi(r_{n-l,n})$ (the notation ψ_γ is used because this character is the restriction of a character which depends on γ, defined in § 4.2).

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(ii) For $l > n$,

$$\Psi(W, f_s, s) = \int_{U_{H_n}\backslash H_n} \left(\int_{R^{l,n}} W(rw^{l,n}h)\, dr \right) f_s(h, 1)\, dh.$$

Here

$$w^{l,n} = \begin{pmatrix} & I_n & & & \\ I_{l-n-1} & & & & \\ & & I_2 & & \\ & & & & I_{l-n-1} \\ & & & I_n & \end{pmatrix},$$

$$R^{l,n} = \left\{ \begin{pmatrix} I_n & & & & \\ x & I_{l-n-1} & & & \\ & & I_2 & & \\ & & & I_{l-n-1} & \\ & & & x' & I_n \end{pmatrix} \right\} < G_l.$$

For $f_s \in \xi(\tau, \mathrm{hol}, s) = \xi_{Q_n}^{H_n}(\tau, \mathrm{hol}, s)$ these integrals are absolutely convergent for $\Re(s) \gg 0$, i.e. the integrals with $|W|$, $|f_s|$ and without ψ_γ for $l \leqslant n$ are convergent. Moreover, there is some $s_0 > 0$ which depends only on the representations π and τ, such that for all $\Re(s) > s_0$ the integrals are absolutely convergent. Additionally, they have a meromorphic continuation to functions in $\mathbb{C}(q^{-s})$. These properties were proved for the Rankin–Selberg integrals of $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$ by Soudry [Sou93]. The arguments carry over simply to our case. For example, meromorphic continuation is established using Bernstein's continuation principle (see [Sou93, § 8.4] and [Ban98]). The development of the integral as a local factor of a global integral, for $l < n$, was detailed in [Kap12].

3.2 The intertwining operators

Let τ be an irreducible representation of GL_n realized in $\mathcal{W}(\tau, \psi)$, and let $M(\tau, s) : V(\tau, s) \to V(\tau^*, 1-s)$ be the standard intertwining operator. It is given (formally) by the integral

$$M(\tau, s) f_s(h, b) = \int_{U_n} f_s(w_n u h, d_n b^*)\, du \quad (h \in H_n, b \in \mathrm{GL}_n).$$

Here

$$w_n = \begin{pmatrix} & & I_n \\ & (-1)^n & \\ I_n & & \end{pmatrix}, \quad d_n = \mathrm{diag}(-1, 1, \ldots, (-1)^n) \in \mathrm{GL}_n.$$

This integral converges absolutely for $\Re(s) \gg 0$. Note that τ^* is realized in $\mathcal{W}(\tau^*, \psi)$. Denote by $M^*(\tau, s)$ the standard intertwining operator normalized by Shahidi's local coefficient $\gamma(\tau, S^2, \psi, 2s-1)$, where S^2 is the symmetric square representation, i.e.

$$M^*(\tau, s) = \gamma(\tau, S^2, \psi, 2s-1) M(\tau, s).$$

Since τ is irreducible and generic, by the definition of the local coefficient

$$M^*(\tau, s) M^*(\tau^*, 1-s) = 1. \tag{3.1}$$

We collect a few results regarding the poles of the intertwining operator. The L-group of the Levi part of Q_n is $\mathrm{GL}_n(\mathbb{C})$. The adjoint action of $\mathrm{GL}_n(\mathbb{C})$ on the Lie algebra of the L-group of U_n is S^2, which is irreducible [Sha92, p. 5]. Therefore, in this case the local coefficient $\gamma(\tau, S^2, \psi, 2s-1)$

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and Shahidi's γ-factor are equal, up to a unit in $\mathbb{C}[q^{-s}, q^s]$ (see [Sha90, Theorem 3.5]; in the language of Shahidi this is an '$m = 1$' situation). According to Shahidi [Sha90, § 7],

$$\gamma(\tau, S^2, \psi, 2s-1) \simeq \frac{L(\tau^*, S^2, 2-2s)}{L(\tau, S^2, 2s-1)}. \tag{3.2}$$

(In general the local coefficient is, essentially, a product of γ-factors so there would be additional L-functions to consider.)

Let $P \in \mathbb{C}[X]$ be a polynomial of minimal degree, with $P(0) = 1$, such that $P(q^{-s})M^*(\tau, s)$ is a holomorphic operator. We set $\ell_\tau(s) = P(q^{-s})^{-1}$. Using the rationality properties of the intertwining operator (see [Mui08, Wal03]), for any $f_s \in \xi(\tau, \mathrm{hol}, s)$, $\ell_\tau(s)^{-1}M^*(\tau, s)f_s$ belongs to $\xi(\tau^*, \mathrm{hol}, 1-s)$. Also note that $\ell_{\tau^*}(1-s)^{-1} \in \mathbb{C}[q^s]$ and $\ell_{\tau^*}(1-s)^{-1}M^*(\tau^*, 1-s)$ is holomorphic.

By a result of Shahidi, the following theorem holds ([Sha90, Proposition 7.2a], see also [CS98]).

Theorem 3.1. *For a tempered τ, $L(\tau, S^2, s)$ is holomorphic for $\Re(s) > 0$.*

The following result of Casselman and Shahidi [CS98, Theorem 5.1] will be used to bound the poles of the intertwining operator.

Theorem 3.2. *For an irreducible supercuspidal τ, $L(\tau, S^2, 2s-1)^{-1}M(\tau, s)$ is holomorphic.*

Remark 3.2. In [CS98] the result is stated for standard modules which satisfy injectivity at a certain level, in particular it is valid for standard modules induced from generic irreducible supercuspidal representations [CS98, Theorem 3.4].

For τ as in Theorem 3.2 or a tempered τ with the assumption that $L(\tau, S^2, 2s-1)^{-1}M(\tau, s)$ is holomorphic (see Theorem 1.1), $\ell_\tau(s) = L(\tau^*, S^2, 2-2s)$. For a general irreducible τ, the poles of the intertwining operator can be bounded by a product of L-functions, using multiplicativity properties: see (3.3) below and the proof of Corollary 5.3. One could let $\ell_\tau(s)$ be defined as this product of L-functions. The factors $\ell_\tau(s)$ mostly impact the upper bounds on the gcd, since they comprise the factors $M_\tau(s)$ (see below). Hence letting them satisfy the minimality property, rather than designate local components of global factors, improves the bounds. See also § 8.4.

Let $\tau_1 \otimes \tau_2^*$ be an irreducible representation of A_{n_1,n_2}. We have the standard intertwining operator

$$\begin{aligned} M(\tau_1 \otimes \tau_2^*, (s, 1-s)) &: V_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\tau_1|\det|^{n/2} \otimes \tau_2^*|\det|^{n/2}, (s, 1-s)) \\ &\to V_{P_{n_2,n_1}}^{\mathrm{GL}_n}(\tau_2^*|\det|^{n/2} \otimes \tau_1|\det|^{n/2}, (1-s, s)). \end{aligned}$$

Then $M^*(\tau_1 \otimes \tau_2^*, (s, 1-s))$ is the standard intertwining operator normalized by Shahidi's local coefficient, and by [Sha90, Theorem 3.5], with a minor abuse of notation,

$$M^*(\tau_1 \otimes \tau_2^*, (s, 1-s)) \simeq \frac{L(\tau_1^* \times \tau_2^*, 2-2s)}{L(\tau_1 \times \tau_2, 2s-1)} M(\tau_1 \otimes \tau_2^*, (s, 1-s)).$$

Define $\ell_{\tau_1 \otimes \tau_2^*}(s) \in \mathbb{C}[q^{-s}]$ similarly to $\ell_\tau(s)$, i.e. $\ell_{\tau_1 \otimes \tau_2^*}(s)^{-1}M^*(\tau_1 \otimes \tau_2^*, (s, 1-s))$ is holomorphic. The result of [CS98] now reads as the following theorem.

Theorem 3.3. *For irreducible supercuspidal τ_1 and τ_2, $L(\tau_1 \times \tau_2, 2s-1)^{-1}M(\tau_1 \otimes \tau_2^*, (s, 1-s))$ is holomorphic.*

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If $\tau = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau_2)$, according to the multiplicativity of the intertwining operators [Sha81, Theorem 2.1.1] and local coefficients [Sha81, Proposition 3.2.1],

$$M^*(\tau, s) = M^*(\tau_1, s)M^*(\tau_1 \otimes \tau_2^*, (s, 1-s))M^*(\tau_2, s). \tag{3.3}$$

Hereby we define the factors $M_\tau(s), M_{\tau_1\otimes\cdots\otimes\tau_k}(s)$ appearing in the introduction (e.g. in Theorem 1.2). Let τ_i be an irreducible representation of GL_{n_i} for $i = 1, \ldots, k$. Then

$$M_{\tau_1\otimes\cdots\otimes\tau_k}(s) = \prod_{i=1}^{k} \ell_{\tau_i}(s)\ell_{\tau_i^*}(1-s) \prod_{1\leqslant i<j\leqslant k} \ell_{\tau_i\otimes\tau_j^*}(s)\ell_{\tau_j^*\otimes\tau_i}(1-s).$$

In particular for $k = 1$, $M_{\tau_1}(s) = \ell_{\tau_1}(s)\ell_{\tau_1^*}(1-s)$. These factors are inverses of polynomials in $\mathbb{C}[q^{-s}, q^s]$. Note that, by definition, $M_{\tau_1\otimes\cdots\otimes\tau_k}(s) = M_{\tau_k^*\otimes\cdots\otimes\tau_1^*}(1-s)$. Theorems 3.2 and 3.3 enable us to calculate or bound $M_{\tau_1\otimes\cdots\otimes\tau_k}(s)$. For example, if τ_1, τ_2 are supercuspidal, then $\ell_{\tau_1\otimes\tau_2^*}(s) = L(\tau_1^* \times \tau_2^*, 2-2s)$ and

$$M_{\tau_1\otimes\tau_2}(s) = \left(\prod_{i=1}^{2} L(\tau_i^*, S^2, 2-2s)L(\tau_i, S^2, 2s)\right) L(\tau_1^* \times \tau_2^*, 2-2s)L(\tau_2 \times \tau_1, 2s).$$

In the course of proving Theorem 1.2 we will encounter poles of $M^*(\tau_i, s)$ and $M^*(\tau_i^*, 1-s)$. The crucial property of $M_{\tau_i}(s)$ is that both $M_{\tau_i}(s)^{-1}M^*(\tau_i, s)$ and $M_{\tau_i}(s)^{-1}M^*(\tau_i^*, 1-s)$ are holomorphic.

Assume that $\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_k}}^{\mathrm{GL}_n}(\tau_1 \otimes \cdots \otimes \tau_k)$ is irreducible. Then (3.3) implies that

$$\ell_\tau(s) \in \prod_{i=1}^{k} \ell_{\tau_i}(s) \prod_{1\leqslant i<j\leqslant k} \ell_{\tau_i\otimes\tau_j^*}(s)\mathbb{C}[q^{-s}, q^s]. \tag{3.4}$$

Hence $M_\tau(s) \in M_{\tau_1\otimes\cdots\otimes\tau_k}(s)\mathbb{C}[q^{-s}, q^s]$, i.e. $M_{\tau_1\otimes\cdots\otimes\tau_k}(s)$ is an upper bound for the poles of $M_\tau(s)$.

3.3 Definitions of local factors

In [Kap10b] we defined the γ-factor of $\pi \times \tau$ as the proportionality factor between $\Psi(W, f_s, s)$ and $\Psi(W, M^*(\tau, s)f_s, 1-s)$. Namely, there is a factor $\gamma(\pi \times \tau, \psi, s) \in \mathbb{C}(q^{-s})$ such that for all $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ and $f_s \in \xi(\tau, \mathrm{rat}, s)$,

$$\gamma(\pi \times \tau, \psi, s)\Psi(W, f_s, s) = c(l, \tau, \gamma, s)\Psi(W, M^*(\tau, s)f_s, 1-s). \tag{3.5}$$

Here $c(l, \tau, \gamma, s) \in \mathbb{C}[q^{-s}, q^s]^*$ equals $\omega_\tau(\gamma)^{-2}|\gamma|^{-2n(s-\frac{1}{2})}$ if $n < l$, or 1 otherwise. It is included in order to get a compact form for the multiplicative properties of the γ-factor. Equality (3.5) is an equality in $\mathbb{C}(q^{-s})$ between meromorphic continuations.

The existence of the functional equation which defines $\gamma(\pi \times \tau, \psi, s)$, when both π and τ are irreducible, follows from [AGRS10, MW10] (see [GGP12, p. 57]). The case of $l = n$ was proved in [GPR87]. For π of finite type one uses the theory of derivatives of Bernstein and Zelevinsky [BZ76, BZ77], as done by Soudry [Sou93, § 8] for the integrals of $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$.

In [Kap10b] we proved that $\gamma(\pi \times \tau, \psi, s)$ is multiplicative in both variables. Firstly, if τ is induced from P_{n_1,n_2} and the representation $\tau_1 \otimes \tau_2$ of A_{n_1,n_2},

$$\gamma(\pi \times \tau, \psi, s) = \gamma(\pi \times \tau_1, \psi, s)\gamma(\pi \times \tau_2, \psi, s).$$

Secondly, for π induced from P_k and the representation $\sigma \otimes \pi'$ of $L_k \cong \mathrm{GL}_k \times G_{l-k}$,

$$\gamma(\pi \times \tau, \psi, s) = \omega_\sigma(-1)^n\omega_\tau(-1)^k[\omega_\tau(2\gamma)^{-1}]\gamma(\sigma \times \tau, \psi, s)\gamma(\pi' \times \tau, \psi, s)\gamma(\sigma^* \times \tau, \psi, s).$$

Here $\gamma(\sigma \times \tau, \psi, s)$, $\gamma(\sigma^* \times \tau, \psi, s)$ are the γ-factors of $\mathrm{GL}_k \times \mathrm{GL}_n$ of [JPS83] and the factor $\omega_\tau(2\gamma)^{-1}$ appears only when $k = l$, in which case also $\gamma(\pi' \times \tau, \psi, s) \equiv 1$.

The L-factor was defined in [JPS83] as a gcd of the integrals. As explained in the introduction, we adapt their approach to our scenario, using the idea of Piatetski-Shapiro and Rallis [PR87, PR86] (see also [HKS96, Ike92]), which is to define a gcd for integrals with good sections. Let

$$\xi(\tau, \mathrm{good}, s) = \xi(\tau, \mathrm{hol}, s) \cup M^*(\tau^*, 1-s)\xi(\tau^*, \mathrm{hol}, 1-s)$$

be the set of good sections, i.e. either holomorphic sections or the images of such under the normalized intertwining operator. According to (3.1), $M^*(\tau, s)\xi(\tau, \mathrm{good}, s) = \xi(\tau^*, \mathrm{good}, 1-s)$; that is, the intertwining operator $M^*(\tau, s)$ is a bijection of good sections.

The integrals $\Psi(W, f_s, s)$ where $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$, $f_s \in \xi(\tau, \mathrm{good}, s)$ span a fractional ideal $\mathcal{I}_{\pi\times\tau}(s)$ of $\mathbb{C}[q^{-s}, q^s]$ which contains the constant 1 (see Proposition 4.1). Hence, it admits a unique generator in the form $P(q^{-s})^{-1}$, with $P \in \mathbb{C}[X]$ such that $P(0) = 1$. This generator is what we call the gcd of the integrals $\Psi(W, f_s, s)$. Denote $\gcd(\pi \times \tau, s) = P(q^{-s})^{-1}$. Here the character ψ is absent. We now show that the gcd is indeed independent of ψ.

Recall that we have a fixed non-trivial additive character ψ of the field. Any other such character ψ' takes the form $\psi'(x) = \psi(cx)$ for some $c \neq 0$. Changing ψ effectively changes $\mathcal{W}(\pi, \psi_\gamma^{-1})$, $\mathcal{W}(\tau, \psi)$ and the character ψ_γ of $R_{l,n}$ (for $l < n$).

CLAIM 3.1. *Let $\mathcal{I}'_{\pi\times\tau}(s)$ be the fractional ideal spanned by the integrals $\Psi(W, f_s, s)$ with the character ψ in the construction replaced with ψ'. Then $\mathcal{I}'_{\pi\times\tau}(s) = \mathcal{I}_{\pi\times\tau}(s)$.*

Proof. Suppose that $\psi'(x) = \psi(cx)$. Assume that $l \leqslant n$. Let $a = \mathrm{diag}(c^{l-1}, \ldots, c^2, c) \in A_{l-1}$, $t = \mathrm{diag}(a, I_2, a^*) \in T_{G_l}$, $b = \mathrm{diag}(c^{-1}, c^{-2}, \ldots, c^{-(n-l)}) \in A_{n-l}$, $d = \mathrm{diag}(a, 1, b) \in A_n$ and $d' = \mathrm{diag}(b^*, a, 1) \in A_n$. If $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$, the function $g \mapsto W(tg)$ belongs to $\mathcal{W}(\pi, {}^t\psi_\gamma^{-1})$ where ${}^t\psi_\gamma(u) = \psi_\gamma(tut^{-1})$. Similarly for τ realized in $\mathcal{W}(\tau, \psi)$ and $f_s \in V(\tau, s)$, the mapping $y \mapsto f_s(h, dy)$ lies in $\mathcal{W}(\tau, {}^d\psi)$. Then $\mathcal{I}'_{\pi\times\tau}(s)$ is spanned by integrals of the form

$$\int_{U_{G_l}\backslash G_l} \int_{R_{l,n}} W(tg) f_s(w_{l,n} rg, d)({}^{d'}\psi_\gamma)(r)\, dr\, dg \quad (W \in \mathcal{W}(\pi, \psi_\gamma^{-1}), f_s \in \xi(\tau, \mathrm{good}, s)).$$

We may move d to the first argument of f_s and conjugate it by $w_{l,n}$; it normalizes $R_{l,n}$ and the integral becomes

$$C(d) \int_{U_{G_l}\backslash G_l} \int_{R_{l,n}} W(tg) f_s(w_{l,n} r({}^{w_{l,n}}d)g, 1)\psi_\gamma(r)\, dr\, dg,$$

where $C(d) \in \mathbb{C}[q^{-s}, q^s]^*$. Changing $g \mapsto t^{-1}gt$ in the integral yields, for some $\delta(t) \in \mathbb{C}^*$,

$$\delta(t)C(d) \int_{U_{G_l}\backslash G_l} \int_{R_{l,n}} t \cdot W(g)({}^{w_{l,n}}d) \cdot f_s(w_{l,n} rg, 1)\psi_\gamma(r)\, dr\, dg,$$

which is just $\delta(t)C(d)\Psi(t \cdot W, {}^{w_{l,n}}d \cdot f_s, s) \in \mathcal{I}_{\pi\times\tau}(s)$.

Regarding the case of $l > n$, set $a = \mathrm{diag}(c^{l-n-1}, \ldots, c^2, c) \in A_{l-n-1}$, $b = \mathrm{diag}(c^{l-1}, c^{l-2}, \ldots, c^{l-n}) \in A_n$ and $t = \mathrm{diag}(b, a, I_2, a^*, b^*) \in T_{G_l}$. Then for some $C'(t) \in \mathbb{C}[q^{-s}, q^s]^*$,

$$\int_{U_{H_n}\backslash H_n} \int_{R^{l,n}} W(trw^{l,n}h) f_s(h, b)\, dr\, dh = C'(t)\Psi({}^{w^{l,n}}t \cdot W, b \cdot f_s, s).$$

The result follows from this. $\square$

By the results regarding the existence of a functional equation mentioned above, the quotients

$$\frac{\Psi(W, f_s, s)}{\gcd(\pi \times \tau, s)}, \quad \frac{\Psi(W, M^*(\tau, s) f_s, 1-s)}{\gcd(\pi \times \tau^*, 1-s)},$$

which by definition belong to $\mathbb{C}[q^{-s}, q^s]$, are also proportional. There exists a proportionality factor $\epsilon(\pi \times \tau, \psi, s)$ satisfying, for all $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ and $f_s \in \xi(\tau, \mathrm{good}, s)$,

$$\frac{\Psi(W, M^*(\tau, s) f_s, 1-s)}{\gcd(\pi \times \tau^*, 1-s)} = c(l, \tau, \gamma, s)^{-1} \epsilon(\pi \times \tau, \psi, s) \frac{\Psi(W, f_s, s)}{\gcd(\pi \times \tau, s)}. \tag{3.6}$$

Combining (3.5) and (3.6) we derive the relation

$$\gamma(\pi \times \tau, \psi, s) = \epsilon(\pi \times \tau, \psi, s) \frac{\gcd(\pi \times \tau^*, 1-s)}{\gcd(\pi \times \tau, s)}. \tag{3.7}$$

Equality (3.7) resembles the relation between the γ, L and ϵ-factors of Shahidi [Sha90], where the gcd is replaced with the L-function. By Shahidi's definitions, when π and τ are standard modules, the γ-factor and L-function are multiplicative in their inducing data, and hence so is the ϵ-factor. Here $\gamma(\pi \times \tau, \psi, s)$ is multiplicative, but in order to deduce this for $\epsilon(\pi \times \tau, \psi, s)$ we still need to establish proper multiplicative properties for the gcd. Currently this task seems difficult. We prove another fundamental property of the ϵ-factor, namely that it is invertible.

CLAIM 3.2. *We have $\epsilon(\pi \times \tau, \psi, s) \in \mathbb{C}[q^{-s}, q^s]^*$.*

Proof of Claim 3.2. Take $W_i, f_s^{(i)}$, $i = 1, \ldots, k$, such that $\gcd(\pi \times \tau, s) = \sum_{i=1}^k \Psi(W_i, f_s^{(i)}, s)$. Plugging $W_i, f_s^{(i)}$ into (3.6) and summing, we get $\epsilon(\pi \times \tau, \psi, s) \in \mathbb{C}[q^{-s}, q^s]$. According to the definition of $\epsilon(\pi \times \tau^*, \psi, 1-s)$, (3.1), (3.6) and because $c(l, \tau, \gamma, s) c(l, \tau^*, \gamma, 1-s) = 1$, we have $\epsilon(\pi \times \tau, \psi, s) \epsilon(\pi \times \tau^*, \psi, 1-s) = 1$, whence $\epsilon(\pi \times \tau, \psi, s)$ is a unit. $\square$

Remark 3.3. For ξ, ϕ a pair of representations of GL_k, GL_r (respectively), the relation between the γ-factor and the L and ϵ-factors of [JPS83] is given by

$$\gamma(\phi \times \xi, \psi, s) = \epsilon(\phi \times \xi, \psi, s) \frac{L(\phi^* \times \xi^*, 1-s)}{L(\phi \times \xi, s)}. \tag{3.8}$$

We mention a useful, simple observation which follows from the multiplicative properties of the γ-factor stated above.

CLAIM 3.3. *Let $\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_a}}^{\mathrm{GL}_n}(\tau_1 \otimes \cdots \otimes \tau_a)$ be irreducible, $a \geqslant 1$. Suppose that for some $P \in \mathbb{C}[q^{-s}, q^s]$, $\gcd(\pi \times \tau, s) = \prod_{i=1}^a \gcd(\pi \times \tau_i, s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) P$. Then $\gcd(\pi \times \tau^*, 1-s) \simeq \prod_{i=1}^a \gcd(\pi \times \tau_i^*, 1-s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) P$. A similar assertion holds when π is induced from $\sigma \otimes \pi'$, e.g. if $\gcd(\pi \times \tau, s) = L(\sigma \times \tau, s) \gcd(\pi' \times \tau, s) L(\sigma^* \times \tau, s) M_\tau(s) P$, $\gcd(\pi \times \tau^*, 1-s) \simeq L(\sigma \times \tau^*, 1-s) \gcd(\pi' \times \tau^*, 1-s) L(\sigma^* \times \tau^*, 1-s) M_\tau(s) P$.*

Proof of Claim 3.3. In general there exist $M^{-1}, Q \in \mathbb{C}[q^{-s}, q^s]$ with $\gcd(\pi \times \tau^*, 1-s) = \prod_{i=1}^a \gcd(\pi \times \tau_i^*, 1-s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) M Q$. Utilizing the multiplicativity of $\gamma(\pi \times \tau, \psi, s)$ in τ,

$$\prod_{i=1}^a \frac{\gcd(\pi \times \tau_i^*, 1-s)}{\gcd(\pi \times \tau_i, s)} \simeq \frac{\gcd(\pi \times \tau^*, 1-s)}{\gcd(\pi \times \tau, s)} = \prod_{i=1}^a \frac{\gcd(\pi \times \tau_i^*, 1-s)}{\gcd(\pi \times \tau_i, s)} \frac{MQ}{P}.$$

Hence $MQ \simeq P$. For π induced from $\sigma \otimes \pi'$ one uses (3.8). $\square$

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4. Properties of the integrals

4.1 Special f_s and W

The following two results describe a section and a Whittaker function that will be used to study the integrals. The first lemma was proved in a slightly different form in [Kap12, Claim 4.1].

Lemma 4.1. *There is a constant $k_0 \geqslant 0$ depending on the embedding of G_l in H_n such that the following holds. Assume $l \leqslant n$. Let $W_\tau \in \mathcal{W}(\tau, \psi)$ and $k > k_0$ be such that W_τ is right-invariant by $\mathcal{N}_{\mathrm{GL}_n,k-k_0}$. Then there exists $f_s \in \xi(\tau, \mathrm{std}, s)$ and a measurable subset $O_k \subset \mathcal{N}_{G_l,k-k_0}$ with $\mathrm{vol}(O_k) > 0$ such that for any $b \in \mathrm{GL}_n$ and $v \in \overline{V_{l-1}} \rtimes G_1 < G_l$, $\int_{R_{l,n}} f_s(w_{l,n} r v, b)\psi_\gamma(r)\, dr$ equals $|\gamma|^{l(\frac{1}{2}n+s-\frac{1}{2})} W_\tau(bt_\gamma)$ if $v \in O_k$ and zero otherwise. Here $t_\gamma = \mathrm{diag}(\gamma I_l, I_{n-l})$ if $|\gamma| \neq 1$ and $t_\gamma = I_n$ otherwise.*

Actually, the function is $f_s = \mathrm{ch}_{(t_\gamma^{-1} w_{l,n})\mathcal{N}_{H_n,k}, cW_\tau, s}$, with $c > 0$ a volume constant. To prove the lemma first write $v = v_0 x$ with $v_0 \in \overline{V_{l-1}}$, $x \in G_1$ and observe that $f_s(w_{l,n} r v, b)$ vanishes unless the image of x in H_1 belongs to $\mathcal{N}_{H_1,k}$. Then we may assume that $v = v_0$ and the non-constant coordinates of v_0 and r are essentially seen to belong to $\mathcal{P}^k$.

The second lemma is a straightforward adaptation of a similar claim of Soudry [Sou93] (inside the proof of Proposition 6.1). The proof is skipped.

Lemma 4.2. *Let $0 \leqslant j < l$ and $W_0 \in \mathcal{W}(\pi, \psi_\gamma^{-1})$. For any k large enough (depending on W_0) there exists $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ such that for any $v = \left(\begin{smallmatrix} I_j & \\ u & b \end{smallmatrix}\right) \in \mathrm{GL}_{l-1} < P_{l-1}$ with $b \in \overline{B_{\mathrm{GL}_{l-j-1}}}$ and $a \in \mathrm{GL}_j < P_{l-1}$, $W(av) = W_0(a)$ if $v \in \mathcal{N}_{G_l,k}$ and zero otherwise. We can take W which, in addition, vanishes unless the last row of a lies in $\eta_j + M_{1\times j}(\mathcal{P}^k)$ ($\eta_j = (0, \ldots, 0, 1)$).*

Now we can prove the following result.

Proposition 4.1. *There exist W and $f_s \in \xi(\tau, \mathrm{hol}, s)$ such that $\Psi(W, f_s, s) = 1$, for all s.*

Proof of Proposition 4.1. The proof follows the arguments of Soudry [Sou93, § 6]. If $l > n$, apply Lemma 4.2 along with $f_s = \mathrm{ch}_{\mathcal{N}_{H_n,k}, W', s} \in \xi(\tau, \mathrm{std}, s)$, where $W' \in \mathcal{W}(\tau, \psi)$ is such that $W'(1) \neq 0$. For $l \leqslant n$ also use Lemma 4.2, and $f_s \in \xi(\tau, \mathrm{hol}, s)$ is selected by Lemma 4.1. □

When describing the integral $\Psi(W, f_s, s)$ for a good section f_s, it is often convenient to assume that f_s is holomorphic or even standard.

Proposition 4.2. *For any $f_s \in \xi(\tau, \mathrm{good}, s)$ there exist $P_i \in \mathbb{C}[q^{-s}, q^s]$, $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$ such that $\Psi(W, f_s, s) = \ell_{\tau^*}(1-s) \sum_{i=1}^k P_i \Psi(W, f_s^{(i)}, s)$. If $f_s \in \xi(\tau, \mathrm{hol}, s)$, the factor $\ell_{\tau^*}(1-s)$ may be dropped.*

Proof of Proposition 4.2. This follows from the definitions and rationality of $M^*(\tau^*, 1-s)$. □

4.2 The inner integration over $R_{l,n}$

In the case of $l < n$, the integral $\Psi(W, f_s, s)$ contains an inner integration over the unipotent subgroup $R_{l,n}$. The properties of this integration resemble those of the Whittaker functional, proved by Casselman and Shalika [CS80] (see also [Sha78, Sha81]) and we follow their line of arguments. Define a functional on $V(\tau, s)$ by

$$f_s \mapsto \Omega(f_s) = \int_{R_{l,n}} f_s(w_{l,n} r, 1)\psi_\gamma(r)\, dr.$$

There exists a constant $s_1 > 0$ depending only on τ such that if $\Re(s) > s_1$, this integral is absolutely convergent for all f_s (see [Sou93, § 4.5]). According to the (global) construction of the integral in [Kap12] or by a direct verification,

$$\Omega(f_s) \in \mathrm{Hom}_{N_{n-l}}(V(\tau, s), \psi_\gamma^{-1}), \tag{4.1}$$

where $N_{n-l} = Z_{n-l} \ltimes U_{n-l}$ (here Z_{n-l} is considered as a subgroup of the Levi part of Q_{n-l}) and ψ_γ denotes the character of N_{n-l} defined for $z \in Z_{n-l}$ and $u \in U_{n-l}$ by $\psi_\gamma(zu) = \psi(z)\psi(e_{n-l}ue_\gamma)$ (with e_{n-l}, e_γ given in § 2.2; see also Remark 2.2). Specifically,

$$\psi_\gamma(x) = \psi\Bigg(\sum_{i=1}^{n-l-1} x_{i,i+1} + x_{n-l,n} + \gamma x_{n-l,n+2}\Bigg).$$

The results in the next few paragraphs imply that as a function of s, in its domain of absolute convergence, $\Omega(f_s)$ is a polynomial in $\mathbb{C}[q^{-s}, q^s]$, for $f_s \in \xi(\tau, \mathrm{hol}, s)$. Hence it has an analytic continuation by which it is defined for all s.

Define for $N < N_{n-l}$ a compact open subgroup and $f_s \in V(\tau, s)$ the function $f_s^{N,\psi_\gamma} \in V(\tau, s)$ by

$$f_s^{N,\psi_\gamma} = \mathrm{vol}(N)^{-1} \int_N \psi_\gamma(n) n \cdot f_s \, dn.$$

The following claim shows that Ω (defined for $\Re(s) > s_1$) is invariant for such a twist of f_s.

CLAIM 4.1. *For any $f_s \in V(\tau, s)$, compact open $N < N_{n-l}$ and $g \in G_l$, $\Omega(g \cdot f_s^{N,\psi_\gamma}) = \Omega(g \cdot f_s)$.*

Proof of Claim 4.1. A direct consequence of (4.1) and the fact that, according to the embedding $G_l < H_n$ described in § 2.2, G_l normalizes N_{n-l} and stabilizes ψ_γ. □

Fix $s \in \mathbb{C}$ arbitrarily. We use the filtration of $V(\tau, s)$ according to the geometrical lemma [BZ77, 2.12]. We follow the exposition of Muić [Mui08, § 3] (see also [BZ77, Cas95]). Consider the decomposition $H_n = \coprod_{w \in \mathcal{A}} C(w)$ ($\coprod$ is a disjoint union), where $\mathcal{A}$ is a set of representatives for $Q_n \backslash H_n / Q_{n-l}$ and $C(w) = Q_n w Q_{n-l}$. One can take

$$\mathcal{A} = \left\{ w_r = \begin{pmatrix} I_r &&&&&& \\ &&&&& I_{n-l-r} & \\ && I_l &&&& \\ &&& (-1)^{n-l-r} &&& \\ &&&& I_l && \\ & I_{n-l-r} &&&&& \\ &&&&&& I_r \end{pmatrix} : r = 0, \ldots, n-l \right\}. \tag{4.2}$$

According to the special ordering defined on the Bruhat cells, $w_0 > \cdots > w_{n-l}$. Let $C^{\geqslant w_r} = \coprod_{w \geqslant w_r} C(w)$. In the following we consider the elements of $V(\tau, s)$ as functions of one variable, i.e. functions defined on H_n taking values in U, the space of τ. The space $V(\tau, s)$ as a representation of Q_{n-l} is filtered by the subspaces $F_{w_r}(s) = \{f_s \in V(\tau, s) : \mathrm{supp}(f_s) \subset C^{\geqslant w_r}\}$ (e.g. $F_{w_{n-l}}(s) = V(\tau, s)$). Here $\mathrm{supp}(f_s)$ denotes the support of f_s. Also let $Q_n^{w_r} = {}^{w_r}Q_n \cap Q_{n-l}$.

Fix r and consider also the decomposition $Q_{n-l} = \coprod_{\eta \in \mathcal{A}(r)} Q_n^{w_r} \eta G_l N_{n-l}$, where $\mathcal{A}(r)$ is a (finite) set of representatives for $Q_n^{w_r} \backslash Q_{n-l} / G_l N_{n-l}$. We describe the set $\mathcal{A}(r)$. For $k \geqslant 0$, let S_k be the group of permutations of $\{1, \ldots, k\}$ ($S_0 = \{1\}$). Denote the natural association of S_k with the permutation matrices in GL_k by $\sigma \mapsto b(\sigma)$ ($b(\sigma)_{i,j} = \delta_{i,\sigma(j)}$). The space $Q_l \backslash H_l / G_l$ is trivial containing a single element which we represent using the identity, when G_l is quasi-split. Otherwise it contains two more elements [GPR87, § 3], which may be taken to be

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$\mathrm{diag}(1,-1,1)\cdot J_3\cdot \jmath(-\gamma,\mp\beta)\in H_1<H_l$ (the matrix $\jmath(\cdot,\cdot)$ was defined in § 2.2). Then

$$\mathcal{A}(r)=\{\mathrm{diag}(b(\sigma),\xi,b(\sigma)^*):\sigma\in(S_r\times S_{n-l-r})\backslash S_{n-l},\xi\in Q_l\backslash H_l/G_l\}.$$

Here S_r (respectively S_{n-l-r}) is embedded in S_{n-l} as the subgroup of permutations of $\{1,\ldots,r\}$ (respectively of $\{r+1,\ldots,n-l\}$). We order the elements of $\mathcal{A}(r)$ according to the special ordering defined on the Bruhat cells.

We order the set of pairs $\{(w_r,\eta):0\leqslant r\leqslant n-l,\eta\in\mathcal{A}(r)\}$ lexicographically, i.e. $(w',\eta')>(w,\eta)$ if $w'>w$ or both $w'=w$, $\eta'>\eta$. Let $C(w_r,\eta)=Q_nw_r\eta G_lN_{n-l}$ and $C^{\geqslant(w_r,\eta)}=\coprod_{(w',\eta')\geqslant(w_r,\eta)}C(w',\eta')$. The space $F_{w_r}(s)$ as a representation of G_lN_{n-l} is filtered by the subspaces $F_{w_r,\eta}(s)=\{f_s\in V(\tau,s):\mathrm{supp}(f_s)\subset C^{\geqslant(w_r,\eta)}\}$, where η varies over $\mathcal{A}(r)$.

Claim 4.2. *For any $r>0$, $\eta\in\mathcal{A}(r)$ and $f_s\in F_{w_r,\eta}(s)$ there exists a compact open subgroup $N<N_{n-l}$ such that $f_s^{N,\psi_\gamma}\in F_{w',\eta'}(s)$ where $(w',\eta')>(w_r,\eta)$. Furthermore, N depends only on the support of f_s (and on ψ_γ).*

Proof of Claim 4.2. Fix $r>0$ and η. It is enough to find N such that for all $x\in G_lN_{n-l}$,

$$f_s^{N,\psi_\gamma}(w_r\eta x)=\mathrm{vol}(N)^{-1}\int_N f_s(w_r\eta xn)\psi_\gamma(n)\,dn=0.$$

Regard the function $f'_s=\lambda(\eta^{-1}w_r^{-1})(f_s|_{C(w_r)})$ as a function on G_lN_{n-l}. There is a compact set $C\subset G_lN_{n-l}$ such that $\mathrm{supp}(f'_s)=({}^{\eta}(Q_n^{w_r})\cap G_lN_{n-l})C$. Denote by C_{G_l}, $C_{N_{n-l}}$ the projections of C on G_l, N_{n-l} (respectively). These are compact sets.

It is possible to show that there exists a compact subgroup $O<{}^{w_r\eta}U_n\cap N_{n-l}$ (which depends on r and η) on which $\psi_\gamma|_O\not\equiv 1$. For instance if $r<n-l$, $\eta=\mathrm{diag}(b(\sigma),\xi,b(\sigma)^*)$ and there is some $1\leqslant j<n-l$ such that $\sigma(j)\leqslant r$ and $\sigma(j+1)>r$, we let O be the image in N_{n-l} of the subgroup of Z_{n-l} consisting of matrices with 1 on the diagonal, an arbitrary element of $\mathcal{P}^{-k}$ in the $(j,j+1)$th coordinate and zero elsewhere. Here $k\geqslant 0$ depends on ψ_γ.

We can take a compact open subgroup $N<N_{n-l}$ such that $C_{N_{n-l}}\subset N$ and for all $g\in C_{G_l}$, $O\subset {}^{g^{-1}}N$. This follows from the fact that G_l normalizes N_{n-l}. We see that N depends only on the support of f_s in $C(w_r,\eta)$ (and on ψ_γ).

Assume that for some $x\in G_lN_{n-l}$, $f_s^{N,\psi_\gamma}(w_r\eta x)\neq 0$. Then $x\in({}^{\eta}(Q_n^{w_r})\cap G_lN_{n-l})C_{G_l}N$, and hence it is enough to prove that, for all $g\in C_{G_l}$, $f_s^{N,\psi_\gamma}(w_r\eta g)=0$. Since G_l stabilizes ψ_γ, up to a volume constant (depending on g) $f_s^{N,\psi_\gamma}(w_r\eta g)$ equals

$$\int_{{}^{g^{-1}}N} f_s(w_r\eta ng)\psi_\gamma(n)\,dn.$$

This vanishes because $O<{}^{g^{-1}}N$ and we can factor the integral through O and obtain an inner integration of the non-trivial character $\psi_\gamma|_O$ which vanishes. □

Again we treat s as a parameter and study the behavior of the functional on holomorphic sections. Let F_{w_r} be the set of $f_s\in\xi(\tau,\mathrm{hol},s)$ such that for all $s\in\mathbb{C}$, $f_s\in F_{w_r}(s)$. Since, for all s, $F_{w_r}(s)$ is a Q_{n-l}-space, so is F_{w_r}. Also note that F_{w_r} is a $\mathbb{C}[q^{-s},q^s]$-module, i.e. if $f_s\in F_{w_r}$, $P\cdot f_s\in F_{w_r}$ for $P\in\mathbb{C}[q^{-s},q^s]$. Similarly for $\eta\in\mathcal{A}(r)$ we define $F_{w_r,\eta}$ as the set of $f_s\in\xi(\tau,\mathrm{hol},s)$ such that for all $s\in\mathbb{C}$, $f_s\in F_{w_r,\eta}(s)$. Then $F_{w_r,\eta}$ is a G_lN_{n-l}-space and a $\mathbb{C}[q^{-s},q^s]$-module.

The above claim has the following useful corollary, showing how to use twists by some N in order to put holomorphic sections in F_{w_0}. The key point is that N will be independent of s.

COROLLARY 4.1. *For any $f_s \in \xi(\tau, \mathrm{hol}, s)$ there exists a compact open subgroup $N < N_{n-l}$, independent of s, such that $f_s^{N,\psi_\gamma} \in F_{w_0}$.*

Proof of Corollary 4.1. We start with $f_s \in \xi(\tau, \mathrm{std}, s)$. The support of standard sections is independent of s, hence $f_s \in F_{w_r,\eta}$ for some r and $\eta \in \mathcal{A}(r)$. Argue, by induction on r and η, that there exists a subgroup N independent of s with $f_s^{N,\psi_\gamma} \in F_{w_0}$. For $r=0$ this is trivial. Let $r>0$.

For every s, $f_s \in F_{w_r,\eta}(s)$. By Claim 4.2 there is a subgroup N', depending only on the support of f_s which is independent of s, such that $f_s^{N',\psi_\gamma} \in F_{w',\eta'}(s)$ with $(w',\eta') > (w_r,\eta)$. Thus $f_s^{N',\psi_\gamma} \in F_{w',\eta'}$. The next step is to apply the induction hypothesis, but note that $f_s^{N',\psi_\gamma} \in \xi(\tau, \mathrm{hol}, s)$ might not be standard.

Write f_s^{N',ψ_γ} as in (2.2), i.e. $f_s^{N',\psi_\gamma} = \sum_{i=1}^m P_i \cdot f_s^{(i)}$ with $P_i \neq 0$, $f_s^{(i)} = \mathrm{ch}_{k_iN,v_i,s} \in \xi(\tau, \mathrm{std}, s)$ and such that if $k_{i_1} = \cdots = k_{i_c}$, $v_{i_1}, \ldots, v_{i_c}$ are linearly independent. Suppose that for some i, $f_s^{(i)} \notin F_{w',\eta'}$. Since the support of $f_s^{(i)}$ is independent of s, then $f_s^{(i)} \notin F_{w',\eta'}(s)$ for all s. In fact, there is some $x \in k_iN$ which does not belong to $C^{\geqslant(w',\eta')}$ ($x \in \mathrm{supp}(f_s^{(i)})$). Let $\{i_1,\ldots,i_c\}$ be a maximal set of indices such that $k_{i_1} = \cdots = k_{i_c} = k_i$. Choose s_0 such that $P_{i_j}(q^{-s_0}, q^{s_0}) = \alpha_j \neq 0$ for some $1 \leqslant j \leqslant c$. Now on the one hand, since $x \notin C^{\geqslant(w',\eta')}$, $f_{s_0}^{N',\psi_\gamma}(x) = 0$. On the other hand $f_{s_0}^{N',\psi_\gamma}(x) = \alpha_1 v_{i_1} + \cdots + \alpha_c v_{i_c}$, contradicting the fact that $v_{i_1},\ldots,v_{i_c}$ are linearly independent. Hence, for each i, $f_s^{(i)} \in F_{w',\eta'}$. Now the induction hypothesis shows that $(f_s^{(i)})^{N_i,\psi_\gamma} \in F_{w_0}$, for a subgroup N_i independent of s.

In general, if $O_1 < O_2 < N_{n-l}$ are compact open, then $(f_s^{O_1,\psi_\gamma})^{O_2,\psi_\gamma} = f_s^{O_2,\psi_\gamma}$. Then if we take N containing N' and all of the subgroups N_i, we have

$$f_s^{N,\psi_\gamma} = (f_s^{N',\psi_\gamma})^{N,\psi_\gamma} = \left(\sum_{i=1}^m P_i \cdot f_s^{(i)}\right)^{N,\psi_\gamma} = \sum_{i=1}^m P_i \cdot ((f_s^{(i)})^{N_i,\psi_\gamma})^{N,\psi_\gamma} \in F_{w_0}.$$

This establishes the claim for standard sections.

Now if $f_s \in \xi(\tau, \mathrm{hol}, s)$, then write, as above, $f_s = \sum_{i=1}^m P_i \cdot f_s^{(i)}$. For each i we have N_i independent of s, $(f_s^{(i)})^{N_i,\psi_\gamma} \in F_{w_0}$, whence, for N containing all of the subgroups N_i, $f_s^{N,\psi_\gamma} \in F_{w_0}$. □

Looking at (4.2) one sees that $C(w_0) = C(w_{l,n})$, so in fact we may replace w_0 with $w_{l,n}$. Let $f_s \in \xi(\tau, \mathrm{std}, s)$. The next proposition shows that $\Omega(f_s)$, initially defined for $\Re(s) > s_1$ to ensure the absolute convergence of the integral, equals an element in $\mathbb{C}[q^{-s}, q^s]$. Therefore $\Omega(f_s)$ has an analytic continuation by which it can be defined for all s. These results extend to $f_s \in \xi(\tau, \mathrm{hol}, s)$. The proposition is also the main tool in writing the Iwasawa decomposition in § 4.3.

PROPOSITION 4.3. *Let $f_s \in \xi(\tau, \mathrm{std}, s)$. There exist $P_i \in \mathbb{C}[q^{-s}, q^s]$ and $W_i \in \mathcal{W}(\tau, \psi)$, $i = 1, \ldots, m$, such that, for all $a \in A_{l-1} < T_{G_l}$ and s,*

$$\Omega(a \cdot f_s) = |\det a|^{l-\frac{1}{2}n+s-\frac{1}{2}} \sum_{i=1}^m P_i W_i(\mathrm{diag}(a, I_{n-l+1})). \tag{4.3}$$

In addition, for the split case there exists a constant $k > 0$ such that the following holds. For $t = ax \in T_{G_l}$ with $x \in G_1 < T_{G_l}$ write $x = \mathrm{diag}(b, b^{-1})$. Then there exist $P_i \in \mathbb{C}[q^{-s}, q^s]$ and

$W_i \in \mathcal{W}(\tau, \psi)$, which depend on whether $\lfloor x\rfloor = b$ or $\lfloor x\rfloor = b^{-1}$, so that, for all t satisfying $[x] > q^k$ and s,

$$\Omega(t \cdot f_s) = (|\det a|[x]^{-1})^{l-\frac{1}{2}n+s-\frac{1}{2}} \sum_{i=1}^{m} P_i W_i(\mathrm{diag}(a, \lfloor x\rfloor, I_{n-l})). \tag{4.4}$$

In both (4.3) and (4.4) the left-hand side is defined by the integral for $\Re(s) > s_1$ and the equality for all s is in the sense of analytic continuation.

Proof of Proposition 4.3. Select N for f_s by Corollary 4.1. Note that GL_{l-1} (as a subgroup of $P_{l-1} < G_l$) normalizes $R_{l,n}$. Additionally, G_l fixes ψ_γ. Using this and Claim 4.1, yields, for $\Re(s) > s_1$,

$$\Omega(a \cdot f_s) = \Omega(a \cdot f_s^{N,\psi_\gamma}) = |\det a|^{l-\frac{1}{2}n+s-\frac{1}{2}} \int_{R_{l,n}} f_s^{N,\psi_\gamma}(w_{l,n}r, \mathrm{diag}(a, I_{n-l+1}))\psi_\gamma(r)\, dr. \tag{4.5}$$

Since $f_s^{N,\psi_\gamma} \in \xi(\tau, \mathrm{hol}, s) \cap F_{w_0}$, we can write, as in (2.2),

$$f_s^{N,\psi_\gamma} = \sum_{i=1}^{m} P_i f_s^{(i)} \quad (P_i \in \mathbb{C}[q^{-s}, q^s],\, f_s^{(i)} = \mathrm{ch}_{k_i N, v_i, s} \in \xi(\tau, \mathrm{std}, s)). \tag{4.6}$$

As in the proof of Corollary 4.1 we deduce that, for each i, $f_s^{(i)} \in F_{w_0}$. Hence each $\lambda(w_{l,n}^{-1})f_s^{(i)}$, as a function on Q_{n-l}, is compactly supported modulo $Q_n^{w_{l,n}}$ (similar to f_s^{N,ψ_γ}) and this support is independent of s (in contrast with f_s^{N,ψ_γ}). That is, there is a compact set $B_i \subset Q_{n-l}$ such that, for all s, the support of $\lambda(w_{l,n}^{-1})f_s^{(i)}$ equals $Q_n^{w_{l,n}}B_i$. One can show that $R_{l,n} \cap Q_n^{w_{l,n}}B_i$ is contained in a compact subset of $R_{l,n}$. Roughly, this is because $R_{l,n} \cap Q_n^{w_{l,n}} = \{1\}$ (but a more direct calculation is needed). Then let $N_i' < R_{l,n}$ be a compact open subgroup such that $\lambda(w_{l,n}^{-1})f_s^{(i)}$ is right-invariant by N_i', $\psi_\gamma|_{N_i'} \equiv 1$ and $R_{l,n} \cap Q_n^{w_{l,n}}B_i \subset \coprod_{j=1}^{m_i} r_{i,j}N_i'$ for some elements $r_{i,j} \in R_{l,n}$ and $m_i \geqslant 1$. It follows that there exist constants $c_{i,j} \in \mathbb{C}$ such that, for all s and a,

$$\int_{R_{l,n}} f_s^{(i)}(w_{l,n}r, \mathrm{diag}(a, I_{n-l+1}))\psi_\gamma(r)\, dr = \sum_{j=1}^{m_i} c_{i,j} f_s^{(i)}(w_{l,n}r_{i,j}, \mathrm{diag}(a, I_{n-l+1})).$$

In particular the integral $\Omega(a \cdot f_s^{N,\psi_\gamma})$ is absolutely convergent for all s. Next, there are $P_{i,j} \in \mathbb{C}[q^{-s}, q^s]$ and $W_{i,j} \in \mathcal{W}(\tau, \psi)$ such that

$$c_{i,j} f_s^{(i)}(w_{l,n}r_{i,j}, \mathrm{diag}(a, I_{n-l+1})) = P_{i,j}W_{i,j}(\mathrm{diag}(a, I_{n-l+1}))$$

for all s and a. Therefore, we conclude that

$$\int_{R_{l,n}} f_s^{N,\psi_\gamma}(w_{l,n}r, \mathrm{diag}(a, I_{n-l+1}))\psi_\gamma(r)\, dr = \sum_{i=1}^{m} P_i \sum_{j=1}^{m_i} P_{i,j}W_{i,j}(\mathrm{diag}(a, I_{n-l+1})).$$

This establishes (4.3).

Let $k_0 > 0$ be such that f_s is right-invariant by $\mathcal{N}_{H_n,k_0}$. Let $k \geqslant k_0$ be as in Lemma 2.1. For $[x] > q^k$ we write $x = m_x u_x h n_x$ as specified by the lemma. The choice of h depends on whether $|b| < |b|^{-1}$ or vice versa, but is otherwise independent of x. We now assume that $|b| < |b|^{-1}$ and fix h. Since $\psi_\gamma(r) = \psi(r_{n-l,n})$ and ${}^{w_{l,n}^{-1}}m_x \in M_n$,

$$\Omega(t \cdot f_s) = (|\det a|[x]^{-1})^{l-\frac{1}{2}n+s-\frac{1}{2}} \int_{R_{l,n}} f_s^{N,\psi_\gamma}(w_{l,n}ru_xh, \mathrm{diag}(a, \lfloor x\rfloor, I_{n-l}))\psi(\lfloor x\rfloor^{-1}r_{n-l,n})\, dr.$$

Conjugating r by u_x and noting that ${}^{w_{l,n}^{-1}}u_x \in U_n$, the dr-integration equals

$$\int_{R_{l,n}} f_s^{N,\psi_\gamma}(w_{l,n}rh, \mathrm{diag}(a, \lfloor x \rfloor, I_{n-l}))\psi^\star(r)\,dr.$$

Here $\psi^\star(r) = \psi(\gamma c r_{n-l,n+1})$ where c is the constant of Lemma 2.1. By our fixing of h, $\psi^\star(r)$ no longer depends on x (since c is fixed). We can continue as above and write f_s^{N,ψ_γ} as in (4.6); for each i, $h \cdot (\lambda(w_{l,n}^{-1}) f_s^{(i)})$ is still compactly supported modulo $Q_n^{w_{l,n}}$ (because $h \in Q_{n-l}$). Thus, for each $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$,

$$\int_{R_{l,n}} f_s^{(i)}(w_{l,n}rh, \mathrm{diag}(a, \lfloor x \rfloor, I_{n-l}))\psi^\star(r)\,dr = \sum_{j=1}^{m_i} P'_{i,j} W'_{i,j}(\mathrm{diag}(a, \lfloor x \rfloor, I_{n-l})).$$

Here $P'_{i,j} \in \mathbb{C}[q^{-s}, q^s]$, $W'_{i,j} \in \mathcal{W}(\tau, \psi)$. Now equality (4.4) follows.

Note that the proof for $|b| > |b|^{-1}$ is identical but the actual polynomials $P'_{i,j}$ and Whittaker functions $W'_{i,j}$ may vary, because they depend on h and $\psi^\star$. $\square$

Remark 4.1. Observe that if $l = n$, in which case the dr-integration is trivial, we can simply put $\Omega(h \cdot f_s) = f_s(w_{l,n}h, 1)$ $(h \in H_n)$ and Proposition 4.3 remains valid.

4.3 Iwasawa decomposition for $\Psi(W, f_s, s)$

We show how to write $\Psi(W, f_s, s)$ as a finite sum of integrals over a torus. We do this for $f_s \in \xi(\tau, \mathrm{std}, s)$; using Proposition 4.2, a similar form holds for $f_s \in \xi(\tau, \mathrm{good}, s)$. Write an element $x \in G_1 < T_{G_l}$ as $x = \mathrm{diag}(\alpha, \alpha^{-1})$ and define for $k > 0$, $G_1^{0,k} = \{x \in G_1 : [x] > q^k, \lfloor x \rfloor = \alpha\}$ and $G_1^{\infty,k} = \{x \in G_1 : [x] > q^k, \lfloor x \rfloor = \alpha^{-1}\}$. In addition for a set Λ let ch_Λ be the characteristic function of Λ.

Proposition 4.4. *For each integral $\Psi(W, f_s, s)$, $f_s \in \xi(\tau, \mathrm{std}, s)$, there exist integrals $I_s^{(1)}, \ldots, I_s^{(m)}$ such that, for all s, $\Psi(W, f_s, s) = \sum_{i=1}^m I_s^{(i)}$. If $l \leqslant n$, each $I_s^{(i)}$ is of the form*

$$\begin{cases} P \displaystyle\int_{A_{l-1}} \int_{G_1} \mathrm{ch}_\Lambda(x) W^\diamond(ax) W'(\mathrm{diag}(a, \lfloor x \rfloor, I_{n-l})) & \\ \qquad (|\det a| \cdot [x]^{-1})^{l-\frac{1}{2}n+s-\frac{1}{2}} \delta_{B_{G_l}}^{-1}(a)\,dx\,da & \textit{split } G_l, \\ & \\ P \displaystyle\int_{A_{l-1}} W^\diamond(a) W'(\mathrm{diag}(a, I_{n-l+1})) |\det a|^{l-\frac{1}{2}n+s-\frac{1}{2}} \delta_{B_{G_l}}^{-1}(a)\,da & \textit{quasi-split } G_l. \end{cases} \tag{4.7}$$

Here $P \in \mathbb{C}[q^{-s}, q^s]$, $W^\diamond \in \mathcal{W}(\pi, \psi_\gamma^{-1})$, $W' \in \mathcal{W}(\tau, \psi)$. In the split case, Λ is either $G_1^{0,k}$ or $G_1^{\infty,k}$ for a constant $k > 0$, or a compact open subgroup of G_1. When $l > n$, $I_s^{(i)}$ takes the form

$$P \int_{A_n} W^\diamond(\mathrm{diag}(a, I_{2(l-n)}, a^*)) W'(a) |\det a|^{\frac{3}{2}n-l+s+\frac{1}{2}} \delta_{B_{H_n}}^{-1}(a)\,da. \tag{4.8}$$

Remark 4.2. Note that in the above integrals, except for P and the exponents of $|\det a|$ and $[x]$, none of the terms depend on s.

Proof of Proposition 4.4. We prove the result for $\Re(s) \gg 0$. Specifically, s belongs to a right half-plane depending only on the representations. Then the equality holds for all s in the sense of meromorphic continuations, since $\Psi(W, f_s, s)$ and each of the integrals $I_s^{(i)}$ extend to functions in $\mathbb{C}(q^{-s})$. In fact, using the asymptotic expansion of Whittaker functions (see [JPS83, § 2.5],

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[CS80, § 6], and [LM09]) one shows that $I_s^{(i)}$ is equal to a sum of products of Tate-type integrals and its meromorphic continuation is proved exactly as in [JPS83, § 2.7].

First assume that $l \leqslant n$. Applying the Iwasawa decomposition of G_l to the integral, it reduces to a sum of integrals of the form

$$\int_{A_{l-1}} \int_{G_1} W^{\diamond}(ax)\Omega(ax \cdot f'_s)\delta_{B_{G_l}}^{-1}(a)\, dx\, da, \tag{4.9}$$

where $W^{\diamond} \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ and $f'_s \in \xi(\tau, \mathrm{hol}, s)$. This writing depends on the subgroups of K_{G_l} for which W and f_s are right-invariant, and, since these are independent of s, this decomposition is independent of s. This means that there exist a finite number of integrals, each of the form (4.9), such that for all s in the right half-plane their sum equals $\Psi(W, f_s, s)$. Henceforth, whenever we write such a decomposition, it will be independent of s in this sense.

If G_l is quasi-split, then integral (4.9) equals a sum of integrals over A_{l-1}, and we apply Proposition 4.3 to obtain integrals of the form (4.7). If G_l is split, the integration over G_1 is divided into integrations over $G_1^{0,k}$, $G_1^{\infty,k}$ and $\{x \in G_1 : [x] \leqslant q^k\}$, with $k > 0$ taken according to Proposition 4.3. In the first two domains we can again apply the proposition to obtain a sum of integrals of the form (4.7). The last domain is compact and treated as in the quasi-split case.

The result for $l > n$ follows again from the Iwasawa decomposition and from the fact that the support in r of the function $(a, r) \mapsto W(ar)$, where $a \in \mathrm{GL}_n < L_n$ and $r \in R^{l,n}$, is contained in a compact set independent of a (see [Sou93, Lemma 4.1]). □

This proposition has the following corollary, which is a direct consequence of the properties of Whittaker functions for supercuspidal representations (see [CS80, § 6]).

Corollary 4.2. *Let π be a supercuspidal representation and τ be an irreducible supercuspidal representation. Assume that if $l = n = 1$, G_1 is quasi-split. Then $\Psi(W, f_s, s)$ is holomorphic for $f_s \in \xi(\tau, \mathrm{hol}, s)$. In particular,* $\gcd(\pi \times \tau, s) \in \ell_{\tau^*}(1 - s)\mathbb{C}[q^{-s}, q^s]$.

If π and τ are irreducible unitary supercuspidal representations (in particular, tempered) for which Conjecture 1.1 holds, Theorem 1.1 will imply that $\gcd(\pi \times \tau, s) = L(\pi \times \tau, s)$, and then Corollary 4.2 shows that $L(\pi \times \tau, s)^{-1}$ divides $L(\tau, S^2, 2s)^{-1}$, since $\ell_{\tau^*}(1 - s) = L(\tau, S^2, 2s)$ (see § 3.2, Theorem 3.2).

4.4 Realization of τ induced from $\tau_1 \otimes \tau_2$

When $\tau = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau_2)$, it is convenient for the manipulations of $\Psi(W, f_s, s)$ to use an explicit integral formula for the Whittaker functional on τ. To ensure convergence of this (Jacquet) integral, the representations τ_i are twisted using an auxiliary complex parameter ζ as in [JPS83, Kap10b, Sha78, Sou93, Sou00]. Set $\varepsilon_1 = \tau_1|\det|^{\zeta}$, $\varepsilon_2 = \tau_2|\det|^{-\zeta}$ and suppose that ε_i is realized in $\mathcal{W}(\varepsilon_i, \psi)$. Let $\varepsilon = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\varepsilon_1 \otimes \varepsilon_2)$. Assume that ζ is such that ε is irreducible; this holds for all but a discrete subset of $\mathbb{C}$ (because τ was taken to be irreducible). Let $Q_{n_1,n_2} < H_n$ be the standard parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$. As explained in § 2.5 we have spaces of standard, holomorphic and rational sections for the induced representation $\Pi_{s_1,s_2} = \mathrm{Ind}_{Q_{n_1,n_2}}^{H_n}((\varepsilon_1 \otimes \varepsilon_2)\alpha^{(s_1,s_2)})$, where s_1 and s_2 are complex parameters (i.e., $\underline{s} = (s_1, s_2)$). Consider also $\Pi'_{s_1,s_2} = \mathrm{Ind}_{Q_{n_1}}^{H_n}((\varepsilon_1 \otimes \mathrm{Ind}_{Q_{n_2}}^{H_{n_2}}(\varepsilon_2\alpha^{s_2}))\alpha^{s_1})$ (for $b_1 \in \mathrm{GL}_{n_1}$, $h_2 \in H_{n_2}$, $\alpha^{s_1}(\mathrm{diag}(b_1, h_2, b_1^*)) = \alpha^{s_1}(b_1)$). These representations are isomorphic via

$$(h, b_1, h_2, b_2) \mapsto f(h_2h, b_1, b_2), \quad (h, b_1, b_2) \mapsto \varphi(h, b_1, I_{2n_2+1}, b_2). \tag{4.10}$$

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Here f belongs to the space of Π_{s_1,s_2}, φ belongs to the space of Π'_{s_1,s_2}, $h \in H_n$, $h_2 \in H_{n_2}$, and $b_i \in \mathrm{GL}_{n_i}$. Furthermore (4.10) defines isomorphisms between $\Pi = \mathrm{Ind}_{Q_{n_1,n_2}\cap K_{H_n}}^{K_{H_n}}(\varepsilon_1 \otimes \varepsilon_2)$ and $\Pi' = \mathrm{Ind}_{Q_{n_1}\cap K_{H_n}}^{K_{H_n}}(\varepsilon_1 \otimes \mathrm{Ind}_{Q_{n_2}\cap K_{H_{n_2}}}^{K_{H_{n_2}}}(\varepsilon_2))$. Then, for φ in the space of Π', we can first regard it as a function in the space of Π, extend it to a function in the space of Π_{s_1,s_2} using the Iwasawa decomposition and finally consider it as a function in the space of Π'_{s_1,s_2}. Therefore we may realize the space $\xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{std}, (s_1, s_2)) = \xi_{Q_{n_1,n_2}}^{H_n}(\varepsilon_1 \otimes \varepsilon_2, \mathrm{std}, (s_1, s_2))$ by extending functions of the space of Π' to functions of the space of Π'_{s_1,s_2}. Then we can also realize $\xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{hol}, (s_1, s_2))$ as the space of elements $\sum_{i=1}^m P_i \varphi_{s_1,s_2}^{(i)}$ where $P_i \in \mathbb{C}[q^{\mp s_1}, q^{\mp s_2}]$, $\varphi_{s_1,s_2}^{(i)} \in \xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{std}, (s_1, s_2))$, and similarly realize $\xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{rat}, (s_1, s_2))$.

Now consider the case of $s_1 = s_2 = s$. Any function in $V_{Q_n}^{H_n}(\mathcal{W}(\varepsilon, \psi), s)$ is the image of a function in $V_{Q_n}^{H_n}(\varepsilon, s)$ under the application of a Whittaker functional to ε. The representations $\mathrm{Ind}_{Q_n}^{H_n}(\varepsilon\alpha^s)$ and $\Pi'_{s,s}$ are also isomorphic, according to

$$(h, b_1, h_2, b_2) \mapsto f(h_2 h, I_n, b_1, b_2), \quad (h, b, b_1, b_2) \mapsto |\det b|^{-\frac{1}{2}n-s+\frac{1}{2}}\varphi(bh, b_1, I_{2n_2+1}, b_2).$$

Here $f \in V_{Q_n}^{H_n}(\varepsilon, s)$, φ belongs to the space of $\Pi'_{s,s}$ and $b \in \mathrm{GL}_n$. These isomorphisms also define isomorphisms between $\mathrm{Ind}_{Q_n\cap K_{H_n}}^{K_{H_n}}(\varepsilon)$ and Π'.

Any $\varphi_s \in \xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{hol}, (s, s))$ defines a function $f'_s \in \xi_{Q_n}^{H_n}(\mathcal{W}(\varepsilon, \psi), \mathrm{hol}, s)$ by

$$f'_s(h, b) = |\det b|^{-\frac{1}{2}n-s+\frac{1}{2}} \int_{Z_{n_2,n_1}} \varphi_s(\omega_{n_1,n_2} z b h, I_{n_1}, I_{2n_2+1}, I_{n_2})\psi^{-1}(z)\, dz. \tag{4.11}$$

Here ω_{n_1,n_2} is the image of $\left(\begin{smallmatrix} & I_{n_1} \\ I_{n_2} & \end{smallmatrix}\right)$ in Q_n. The Jacquet integral (4.11) always has a sense as a principal value, but there exists a $\zeta_0 > 0$, which depends only on τ_1 and τ_2, such that for all ζ with $\Re(\zeta) > \zeta_0$ it is absolutely convergent for all s and φ_s. The integral $\Psi(W, \varphi_s, s)$ is just $\Psi(W, f'_s, s)$ with formula (4.11); e.g. if $l \leqslant n$, then

$$\Psi(W, \varphi_s, s) = \int_{U_{G_l}\backslash G_l} W(g) \int_{R_{l,n}} \int_{Z_{n_2,n_1}} \varphi_s(\omega_{n_1,n_2} z w_{l,n} r g, 1, 1, 1)\psi^{-1}(z)\psi_\gamma(r)\, dz\, dr\, dg. \tag{4.12}$$

We say that this integral is absolutely convergent if it is convergent when we replace W, φ_s with $|W|, |\varphi_s|$ and drop the characters. The following claim relates $\Psi(W, f'_s, s)$ to $\Psi(W, \varphi_s, s)$.

CLAIM 4.3. *Suppose that, for W and $\varphi_s \in \xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{hol}, (s, s))$, $\Psi(W, \varphi_s, s)$ is absolutely convergent at s. Then $\Psi(W, \varphi_s, s) = \Psi(W, f'_s, s)$ at s, where f'_s is defined by (4.11).*

Proof. Assume, for instance, that $l \leqslant n$, the other case being similar. We use the idea of [JPS83, p. 424]. Let s be such that $\Psi(W, \varphi_s, s)$ is absolutely convergent. Then $\Psi(W, f'_s, s)$ is also absolutely convergent. The smoothness of W and φ_s implies that the dz-integration in $\Psi(W, \varphi_s, s)$ is absolutely convergent for all $g \in \mathrm{supp}(W)$ and $r \in R_{l,n}$. Then according to Fubini's theorem this dz-integration can be replaced with $f'_s(w_{l,n} r g, 1)$ (defined by principle value). This shows that the integrals $\Psi(W, \varphi_s, s)$ and $\Psi(W, f'_s, s)$ are equal at s. $\square$

Suppose that τ is realized in $\mathcal{W}(\tau, \psi)$ and let $f_s \in \xi_{Q_n}^{H_n}(\tau, \mathrm{hol}, s)$. Then there is a section $\varphi_s \in \xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{hol}, (s, s))$ with the following properties. Let f'_s be defined by (4.11) and write $f'_s = \sum_{i=1}^m P_i\, \mathrm{ch}_{k_i N, W_\zeta^{(i)}, s}$ with $P_i \in \mathbb{C}[q^{-s}, q^s]$, $W_\zeta^{(i)} \in \mathcal{W}(\varepsilon, \psi)$ as in (2.2). Now if we let ζ vary, the data (m, k_i, N, P_i) is independent of ζ, $W_\zeta^{(i)}$ is right-invariant by $({}^{k_i^{-1}}N) \cap Q_n$ and for

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each $b \in \mathrm{GL}_n$ there is $Q \in \mathbb{C}[q^{-\zeta}, q^{\zeta}]$ such that, for all ζ, $W_{\zeta}^{(i)}(b) = Q$. Moreover, if we put $\zeta = 0$ in this expression for f'_s we get that $f'_s = f_s$.

4.5 The gcd of ramified twists

Let τ be an irreducible representation of GL_n and let μ be a (unitary) character of F^*, extended to GL_n by $b \mapsto \mu(\det(b))$. We show that if μ^2 is sufficiently ramified, a twist of τ by μ removes all poles except perhaps those of the intertwining operator.

Proposition 4.5. *For μ such that μ^2 is sufficiently ramified,* $\gcd(\pi \times (\tau\mu), s) \in \ell_{(\tau\mu)^*}(1-s)\mathbb{C}[q^{-s}, q^s]$.

This resembles a result of [JPS83, Proposition 2.13] and is proved in the same manner, with the aid of Propositions 4.2 and 4.4.

Example 4.1. Let τ' be a unitary irreducible supercuspidal representation of GL_n and take a character μ for which μ^2 is sufficiently ramified (with respect to τ', π), so that no unramified twist of $\tau = \tau'\mu$ (or τ^*) would be self-dual. Then $L(\tau, S^2, 2s) = 1$ by [Sha92, Theorem 6.2]. Using (3.2) and Theorem 3.2, $\ell_{\tau^*}(1-s) = 1$ and $\gcd(\pi \times \tau, s) = 1$.

There is an analogous result for a ramified character π of G_1, proved similarly.

Proposition 4.6. *If π is a sufficiently ramified character or G_1 is quasi-split,* $\gcd(\pi \times \tau, s) \in \ell_{\tau^*}(1-s)\mathbb{C}[q^{-s}, q^s]$.

5. Embedding poles in induced representations

5.1 Embedding poles in $\pi \times \tau$

Let τ be an irreducible representation of GL_n. We consider cases where the poles of the Rankin–Selberg $\mathrm{GL}_k \times \mathrm{GL}_n$ integrals are contained in $\gcd(\pi \times \tau, s)$. This provides a weak lower bound on $\gcd(\pi \times \tau, s)$, which will be strengthened in the process of proving Theorem 1.4.

Lemma 5.1. *Let $\pi = \mathrm{Ind}_{P_k}^{G_l}(\sigma \otimes \pi')$ with $1 \leqslant k < l$, where σ is a representation of GL_k and π' is a representation of G_{l-k}. Then $L(\sigma \times \tau, s) \in \gcd(\pi \times \tau, s)\mathbb{C}[q^{-s}, q^s]$.*

Proof of Lemma 5.1. Let $W_\sigma \in \mathcal{W}(\sigma, \psi^{-1})$ and $W_\tau \in \mathcal{W}(\tau, \psi)$ be arbitrary. Suppose that W_σ (respectively W_τ) is right-invariant by $\mathcal{N}_{\mathrm{GL}_k, k_0}$ (respectively $\mathcal{N}_{\mathrm{GL}_n, k_0}$). Select φ in the space of π with support in $\overline{P_k}\mathcal{N}_{G_l, k_1}$, $k_1 \gg k_0$, which is right-invariant by $\mathcal{N}_{G_l, k_1}$ and such that $\varphi(a, I_k, I_{2(l-k)}) = \delta_{P_k}^{-\frac{1}{2}}(a)W_\sigma(a)$ for $a \in \mathrm{GL}_k$. Then φ defines a Whittaker function $W_\varphi \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ by the principal value of the Jacquet integral

$$W_\varphi(g) = \int_{V_k} \varphi(vg, 1, 1)\psi_\gamma(v)\, dv. \tag{5.1}$$

Let W_1 be as in Lemma 4.2, with j to be specified below, defined for W_φ, with $k_2 \gg k_1$ (i.e. W_φ is W_0 in the lemma). The additional condition in the lemma for W_1 concerning the last row of $a \in \mathrm{GL}_j$ is not imposed yet. If $n < l$, set $W = (w^{l,n})^{-1} \cdot W_1$ and otherwise $W = W_1$. Finally let $f_s = \mathrm{ch}_{\mathcal{N}_{H_n, k_3}, W_\tau, s} \in \xi(\tau, \mathrm{std}, s)$, $k_3 \gg k_2$. The proof varies according to the relative sizes of k, l and n.

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(i) $n < k$. In the selection of W above, Lemma 4.2 is applied with $j = n$. We may write the dh-integral in $\Psi(W, f_s, s)$ over $\overline{Q_n}$. The integral becomes

$$\int_{Z_n\backslash\mathrm{GL}_n}\int_{\overline{U_n}}\bigg(\int_{R^{l,n}} W(rw^{l,n}au)\,dr\bigg)f_s(u,a)|\det a|^{-\frac{1}{2}n+s-\frac{1}{2}}\,du\,da.$$

By our choice of f_s, $f_s(u, a) = 0$ unless $u \in \mathcal{N}_{H_n,k_3}$, in which case $f_s(u, a) = f_s(1, a)$. Taking k_3 large enough, W is right-invariant for such elements u. Therefore the du-integration can be ignored. According to the selection of W and W_1 the dr-integration is ignored. Now substitute (5.1) for W_φ. Since $n < k$, a stabilizes ψ_γ and normalizes V_k. The integral equals

$$\int_{Z_n\backslash\mathrm{GL}_n}\bigg(\int_{V_k}\varphi(v,\mathrm{diag}(a, I_{k-n}), 1)\psi_\gamma(v)\,dv\bigg)W_\tau(a)|\det a|^{s-\frac{1}{2}(k-n)}\,da.$$

By our choice of φ, the dv-integration is discarded (up to a constant depending on k_1) and we obtain the $\mathrm{GL}_k\times\mathrm{GL}_n$ integral of [JPS83],

$$\int_{Z_n\backslash\mathrm{GL}_n} W_\sigma(\mathrm{diag}(a, I_{k-n}))W_\tau(a)|\det a|^{s-\frac{1}{2}(k-n)}\,da.$$

(In the notation of [JPS83, § 2.4] this is $\Psi(s, W_\sigma, W_\tau; 0)$; an integral of type (3).) Because W_σ and W_τ are arbitrary, this shows that $L(\sigma\times\tau, s) \in \gcd(\pi\times\tau, s)\mathbb{C}[q^{-s}, q^s]$.

(ii) $k < n < l$. In choosing W_1 we take $j = k$. Proceeding as above we reach

$$\int_{Z_k\backslash\mathrm{GL}_k}\bigg(\int_{V_k}\varphi(va, 1, 1)\psi_\gamma(v)\,dv\bigg)W_\tau(\mathrm{diag}(a, I_{n-k}))|\det a|^{-\frac{1}{2}n+k-l+s+\frac{1}{2}}\,da. \tag{5.2}$$

The difference between this case and the previous is that here $a \in \mathrm{GL}_k$ does not stabilize ψ_γ. Write $a = x_a y_a b_a$ with $x_a \in F^*$ in the center of GL_k, $y_a \in Y_k$ (Y_k is the mirabolic subgroup) and $b_a \in K_{\mathrm{GL}_k}$. Since W_τ is majorized by a gauge and $k < n$, $W_\tau(\mathrm{diag}(a, I_{n-k}))$ vanishes unless $|x_a| < c$ where c is some positive constant depending on W_τ. Then if we choose k_1 large enough, depending on the constant c, $\psi_\gamma({}^{a^{-1}}v)$ will be identically 1 whenever $|x_a| < c$, and thus $\int_{V_k}\varphi(v, a, 1)\psi_\gamma({}^{a^{-1}}v)\,dv = W_\sigma(a)$ and the integral is

$$\int_{Z_k\backslash\mathrm{GL}_k} W_\sigma(a)W_\tau(\mathrm{diag}(a, I_{n-k}))|\det a|^{s-\frac{1}{2}(n-k)}\,da.$$

(iii) $n = k$. Contrary to case (i), a does not stabilize ψ_γ, while the method of case (ii) falls short because the properties of W_τ do not allow to bound x_a. Following Cogdell and Piatetski-Shapiro [CP, § 3], it is enough to obtain two types of integral:

$$\int_{Z_k\backslash\mathrm{GL}_k} W_\sigma(a)W_\tau(a)\Phi(\eta_k a)|\det a|^s\,da, \tag{5.3}$$

$$\int_{Z_k\backslash Y_k} W_\sigma(y)W_\tau(y)|\det y|^{s-1}d_r y. \tag{5.4}$$

Here $\Phi \in \mathcal{S}(F^k)$ ($\mathcal{S}(F^k)$; the space of Schwartz functions on F^k) satisfies $\Phi(0) \neq 0$ and any such Φ is sufficient, $\eta_k = (0, \ldots, 0, 1)$. In fact, denote the $\mathrm{GL}_k\times\mathrm{GL}_k$ integral by $\Psi(W_\sigma, W_\tau, \Phi, s)$ (this is (5.3) with an arbitrary Φ). Assume that $L(\sigma\times\tau, s)$ has a pole at s_0. Then the residue of $\Psi(W_\sigma, W_\tau, \Phi, s_0)$ defines a non-trivial trilinear form on $\mathcal{W}(\sigma, \psi^{-1})\times\mathcal{W}(\tau, \psi)\times\mathcal{S}(F^k)$. If this form vanishes identically on the subspace $\mathcal{S}_0 = \{\Phi \in \mathcal{S}(F^k) : \Phi(0) = 0\}$, it represents a non-trivial trilinear form on $\mathcal{W}(\sigma, \psi^{-1})\times\mathcal{W}(\tau, \psi)\times\mathcal{S}_0\backslash\mathcal{S}(F^k)$. Then the pole is obtained by (5.3) with

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any Φ such that $\Phi(0)\neq 0$. Otherwise, s_0 is a pole of some $\Psi(W_\sigma, W_\tau, \Phi, s)$ with $\Phi\in\mathcal{S}_0$, in which case, using the Iwasawa decomposition, the integral is seen to be equal to a sum of integrals, each of the form (5.4).

Start as in case (i). In general our selection of φ implies that

$$\int_{V_k}\varphi(va,1,1)\psi_\gamma(v)\,dv=\delta_{P_k}^{\frac{1}{2}}(a)W_\sigma(a)\int_{V_k\cap\mathcal{N}_{G_l,k_1}}\psi_\gamma({}^{a^{-1}}v)\,dv.$$

Writing $a=x_a y_a b_a$ with the above notation we see that there is a constant $c>0$ depending on ψ_γ and on k_1, such that for $|x_a|\geqslant c$ the integral on the right-hand side vanishes and for $|x_a|<c$ we get $\delta_{P_k}^{\frac{1}{2}}(a)W_\sigma(a)$ multiplied by a measure constant depending on k_1. Let $B_c=\{a\in\mathrm{GL}_k:|x_a|<c\}$. The integral becomes

$$\int_{Z_k\backslash\mathrm{GL}_k}W_\sigma(a)W_\tau(a)\,\mathrm{ch}_{B_c}(a)|\det a|^s\,da.$$

We can replace $\mathrm{ch}_{B_c}(a)$ with $\Phi(\eta_k a)$, for some $\Phi\in\mathcal{S}(F^k)$ such that $\Phi(0)\neq 0$. Hence we have (5.3).

Now we turn to integrals of the form (5.4), where we can assume that $k>1$. Here W_1 is chosen with $j=k$ and such that it vanishes on $a\in\mathrm{GL}_k$ unless $\eta_k a$ lies in $\eta_k+M_{1\times k}(\mathcal{P}^{k_2})$. The constant k_2 is large enough so that W_φ is right-invariant on $\mathcal{N}_{G_l,k_2}$. As in case (i) the integral becomes

$$\int_{K_{\mathrm{GL}_k}}\int_{Z_k\backslash Y_k}\int_{\mathrm{GL}_1}W_1(\mathrm{diag}(yxb,I_{2(l-k)},(yxb)^*))W_\tau(yxb)|\det y|^{\frac{1}{2}k-l+s-\frac{1}{2}}|x|^{A+s}\,dx\,d_ry\,db,$$

where the integral was factored using $\mathrm{GL}_k=P_{k-1,1}K_{\mathrm{GL}_k}$, $x=\mathrm{diag}(I_{k-1},x)$ and A is some constant. Denote $b=\left(\begin{smallmatrix}b_1&b_2\\b_3&b_4\end{smallmatrix}\right)\in K_{\mathrm{GL}_k}=\mathrm{GL}_k(\mathcal{O})$, where $b_1\in M_{k-1\times k-1}$. The condition imposed by W_1 on $\eta_k yxb=\eta_k xb$ is that $xb_3\in M_{1\times k-1}(\mathcal{P}^{k_2})$ and $xb_4\in 1+\mathcal{P}^{k_2}$. When this is fulfilled, $|x|=1$, $W_1(yxb)=W_\varphi(yxb)$ and $xb\in(Y_k\cap K_{\mathrm{GL}_k})\mathcal{N}_{\mathrm{GL}_k,k_2}$. Since W_φ and W_τ are right-invariant on $\mathcal{N}_{\mathrm{GL}_k,k_2}$ and the measure d_ry is invariant for translations on the right, we may ignore the $dx\,db$-integration. Plugging in (5.1) and noticing that Y_k stabilizes $\psi_\gamma(v)$ leads to (5.4).

(iv) $l\leqslant n$. Choose W_1 with $j=k$. Now f_s is selected according to Lemma 4.1 for $t_\gamma^{-1}\cdot W_\tau$, with $k_3\gg k_2$ large so that W is right-invariant by $\mathcal{N}_{G_l,k_3-c_0}$ (t_γ as in Lemma 4.1 and c_0 is the constant k_0 of Lemma 4.1). We use the formula

$$\int_{U_{G_l}\backslash G_l}F(g)\,dg$$
$$=\int_{\overline{B_{\mathrm{GL}_{l-1-k}}}}\int_{G_1}\int_{\overline{V_{l-1}}}\int_{Z_k\backslash\mathrm{GL}_k}\int_{\overline{Z_{k,l-1-k}}}F(ambvx)\delta_{P_k}^{-1}(a)\delta(b)\,dm\,da\,dv\,dx\,db,$$

where $\mathrm{GL}_{l-1-k}<G_{l-k}$, $Z_{k,l-1-k}<L_{l-1}$ and δ is some modulus character. The properties of f_s imply that the $dr\,dv\,dx$-integration can be ignored (the dr-integration is over $R_{l,n}$) and the choice of W implies that the $dm\,db$-integration may be disregarded. The integral reduces to an integral over $Z_k\backslash\mathrm{GL}_k$, and after plugging in the formula for W_φ we reach (5.2) and continue as in case (ii) using the fact that W_τ is majorized by a gauge. □

Lemma 5.2. *Assume that $l>n$. Let $\pi=\mathrm{Ind}_{P_l}^{G_l}(\sigma)$ where σ is a representation of GL_l. Then $L(\sigma\times\tau,s)\in\gcd(\pi\times\tau,s)\mathbb{C}[q^{-s},q^s]$. A similar relation holds for $\pi=\mathrm{Ind}_{\kappa(\overline{P_l})}^{G_l}(\sigma)$ ($\kappa=\mathrm{diag}(I_{l-1},J_2,I_{l-1})$, see § 2.1).*

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Proof of Lemma 5.2. The arguments of Lemma 5.1(i) apply here, as well, with φ, W_1, $W=(w^{l,n})^{-1}\cdot W_1$ and f_s selected similarly. Now if $n<l-1$, then GL_n stabilizes $\psi_\gamma(v)=\psi(-\gamma v_{l-1,l+1})$ and the result follows. Otherwise if $n=l-1$, then continue as in case (ii) of the lemma, using a bound of W_σ by a gauge (instead of W_τ). □

COROLLARY 5.1. *Let π and τ be as in Theorem 1.3. If $k<l$ or $k=l>n$, $L(\sigma\times\tau,s)\in\gcd(\pi\times\tau,s)\mathbb{C}[q^{-s},q^s]$.*

5.2 Embedding poles of $\pi\times\tau$

Let π and τ be a pair of representations of G_l and GL_n (respectively). In contrast with § 5.1, here we show that $\mathcal{I}_{\pi\times\tau}(s)\subset\mathcal{I}_{\pi\times\varepsilon}(s)$ for a representation ε induced from τ and another auxiliary representation. This result will be used to prove Theorem 1.3. It suggests an inductive passage from $\pi\times\tau$ to $\pi\times\varepsilon$ which increases n while preserving the original poles, and hence any upper bound of $\gcd(\pi\times\varepsilon,s)$ implies a similar bound of $\gcd(\pi\times\tau,s)$.

LEMMA 5.3. *Let τ_1 be a representation of GL_m such that $l<m+n$ and $\varepsilon=\mathrm{Ind}_{P_{m,n}}^{\mathrm{GL}_{m+n}}(\tau_1\otimes\tau)$ is irreducible. Then for any $f_s\in\xi_{Q_n}^{H_n}(\tau,\mathrm{hol},s)$ there is $f'_s\in\xi_{Q_{m+n}}^{H_{m+n}}(\varepsilon,\mathrm{hol},s)$, where ε is realized in $\mathcal{W}(\varepsilon,\psi)$, such that for all W, $\Psi(W,f_s,s)=\Psi(g_0\cdot W,f'_s,s)$. Here $g_0=I_{2l}$ unless $l>n$ and G_l is quasi-split, in which case $g_0\in G_l$ depends only on γ.*

Proof of Lemma 5.3. We use the notation and results of § 4.4, for $\zeta=0$. We will define $\varphi_s\in\xi_{Q_{m,n}}^{H_{m+n}}(\tau_1\otimes\tau,\mathrm{hol},(s,s))$ such that $I_1=\Psi(g_0\cdot W,\varphi_s,s)$ is absolutely convergent for $\Re(s)\gg 0$ and equals $\Psi(W,f_s,s)$. Then f'_s is defined by (4.11) and according to Claim 4.3, for all $\Re(s)\gg0$, $\Psi(g_0\cdot W,f'_s,s)=I_1=\Psi(W,f_s,s)$. Hence $\Psi(g_0\cdot W,f'_s,s)=\Psi(W,f_s,s)$, as functions in $\mathbb{C}(q^{-s})$. The proof depends on the relative sizes of l and n.

(i) $l\leqslant n$. We introduce the following integral, deduced from I_1 by a formal manipulation:

$$I_2=\int_{U_m}\int_{U_{G_l}\backslash G_l}W(g)\int_{R_{l,n}}\varphi_s(w'u,1,w_{l,n}r'({}^{b_{n,m}}g),1)\psi_\gamma(r')\psi_\gamma(u)\,dr'\,dg\,du,$$

where $w'\in H_n$ is a Weyl element satisfying ${}^{(w')^{-1}}U_m=\overline{U_m}$ ($\overline{U_m}$ is the unipotent subgroup opposed to U_m) and $b_{n,m}=\mathrm{diag}(I_n,(-1)^m,I_n)$. The subgroup $R_{l,m+n}$ (in I_1) is decomposed as $R_{l,n}\ltimes R_m$ where $R_m=R_{l,m+n}\cap U_m$, $dr=dr'dr_m$ and, in I_2, $du=dz\,dr_m$.

CLAIM 5.1. *For s such that I_2 is absolutely convergent we have $I_1=I_2$.*

This claim was proved in [Kap10b] by a simple application of Fubini's theorem (see [Sou00, Lemma 3.2]). We follow the argument of [Sou00, Lemma 3.4] to find a specific φ_s for which $I_2=\Psi(W,f_s,s)$, almost what we need. Set $N=\mathcal{N}_{H_{m+n},k}$, $k\gg0$. Let $\varphi_s\in\xi_{Q_{m,n}}^{H_{m+n}}(\tau_1\otimes\tau,\mathrm{hol},(s,s))$ be such that, as a function on H_{m+n}, $\mathrm{supp}(\varphi_s)=Q_mw'N$, φ_s is right-invariant by N and $\varphi_s(xw',1,1,y)=(b_{n,m}\cdot\lambda(b_{n,m})f_s)(x,y)$ $(x\in H_n,\ y\in\mathrm{GL}_n)$. Note that ${}^{w'}N=N$. Then $w'u\in\mathrm{supp}(\varphi_s)$ if and only if ${}^{(w')^{-1}}u\in N$. Formally, for this φ_s,

$$I_2=c\int_{U_{G_l}\backslash G_l}W(g)\int_{R_{l,n}}\varphi_s(w',1,w_{l,n}r'({}^{b_{n,m}}g),1)\psi_\gamma(r')\,dr'\,dg=c\Psi(W,f_s,s).$$

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Here $c=\mathrm{vol}(U_m\cap N)$. The same reasoning shows that

$$\int_{U_m}\int_{U_{G_l}\backslash G_l}\int_{R_{l,n}}|W|(g)|\varphi_s|(w'u,1,w_{l,n}r'({}^{b_{n,m}}g),1)\,dr'dg\,du$$
$$=c\int_{U_{G_l}\backslash G_l}\int_{R_{l,n}}|W|(g)|f_s|(w_{l,n}r'g,1)\,dr'\,dg. \tag{5.5}$$

We still need to show that $\Psi(W,f'_s,s)=c\Psi(W,f_s,s)$. To this end we use the idea of [JPS83, p. 424]. Since $\Psi(W,f_s,s)$ is absolutely convergent for $\Re(s)\gg 0$ (i.e. the right-hand side of (5.5) is finite), the left-hand side of (5.5) is finite. Hence we can apply Claim 5.1 to conclude firstly that $I_1=I_2=c\Psi(W,f_s,s)$ and secondly, by Fubini's theorem, for $\Re(s)\gg 0$,

$$\int_{U_{G_l}\backslash G_l}\int_{R_{l,m+n}}\int_{Z_{n,m}}|W|(g)|\varphi_s|(\omega_{m,n}zw_{l,m+n}rg,1,1,1)\,dz\,dr\,dg<\infty.$$

This proves that I_1 is absolutely convergent.

(ii) $l>n$. The method is similar to the above but the integral manipulations and selection of φ_s are more involved. The proof is omitted. □

Lemma 5.3 shows that the poles of $\gcd(\pi\times\tau,s)$ originating from holomorphic sections appear in $\gcd(\pi\times\varepsilon,s)$. However, $\gcd(\pi\times\tau,s)$ may also contain poles due to non-holomorphic sections. Under a certain assumption, these poles will also be included in $\gcd(\pi\times\varepsilon,s)$.

Corollary 5.2. *Let τ_1 and $\varepsilon=\mathrm{Ind}_{P_{m,n}}^{\mathrm{GL}_{m+n}}(\tau_1\otimes\tau)$ be as in Lemma 5.3. If the operator $M^*(\tau_1,s)M^*(\tau_1\otimes\tau^*,(s,1-s))$ is holomorphic,* $\gcd(\pi\times\tau,s)\in\gcd(\pi\times\varepsilon,s)\mathbb{C}[q^{-s},q^s]$.

Proof of Corollary 5.2. We still need to show that for any $f_{1-s}\in\xi(\tau^*,\mathrm{hol},1-s)$ there exists $f'_s\in\xi(\varepsilon,\mathrm{good},s)$ such that $\Psi(W,M^*(\tau^*,1-s)f_{1-s},s)=\Psi(g_0\cdot W,f'_s,s)$. Let $\varphi_{s,1-s}\in\xi_{Q_{m,n}}^{H_{m+n}}(\tau_1\otimes\tau^*,\mathrm{hol},(s,1-s))$ be defined as in Lemma 5.3, using f_{1-s}. Denote $\varphi'_s=M^*(\tau^*,1-s)\varphi_{s,1-s}\in\xi_{Q_{m,n}}^{H_{m+n}}(\tau_1\otimes\tau,\mathrm{rat},(s,s))$. Our assumption on the operators along with (3.1) show that we can find $\Phi_{1-s}\in\xi_{Q_{n,m}}^{H_{m+n}}(\tau^*\otimes\tau_1^*,\mathrm{hol},(1-s,1-s))$ such that $M^*(\tau^*\otimes\tau_1,(1-s,s))M^*(\tau_1^*,1-s)\Phi_{1-s}=\varphi_{s,1-s}$. Using (3.3) with ε^*, $\varphi'_s=M^*(\varepsilon^*,1-s)\Phi_{1-s}$. Therefore φ'_s defines $f'_s\in\xi(\varepsilon,\mathrm{good},s)$ by (4.11) and, as in the lemma, $\Psi(W,M^*(\tau^*,1-s)f_{1-s},s)=\Psi(g_0\cdot W,f'_s,s)$. □

We will resort to Corollary 5.2 for given representations π and τ, with the luxury of selecting τ_1. The following demonstrates how to select τ_1 so that the corollary would be applicable.

Corollary 5.3. *Let τ be an irreducible representation of GL_n. For any $m>\max(n,l-n)$ and unitary irreducible supercuspidal representation τ_1 of GL_m twisted by a sufficiently ramified character, we have $M_{\tau_1}(s)=\ell_{\tau^*\otimes\tau_1}(1-s)=\ell_{\tau_1\otimes\tau^*}(s)=1$ and $\gcd(\pi\times\tau,s)\in\gcd(\pi\times\varepsilon,s)\mathbb{C}[q^{-s},q^s]$, for any representation π of G_l.*

Proof of Corollary 5.3. Note that the properties of τ and τ_1 imply that ε is irreducible (see [Zel80, § 9]). Once we show that $M_{\tau_1}(s)=\ell_{\tau^*\otimes\tau_1}(1-s)=\ell_{\tau_1\otimes\tau^*}(s)=1$, the result follows from Corollary 5.2. The fact that $M_{\tau_1}(s)=1$ is obtained as explained in Example 4.1. Next we show that $\ell_{\tau^*\otimes\tau_1}(1-s)=\ell_{\tau_1\otimes\tau^*}(s)=1$. Choose $\phi^*=\mathrm{Ind}_{P_{n_1,\ldots,n_k}}^{\mathrm{GL}_n}(\phi_1^*\otimes\cdots\otimes\phi_k^*)$ with irreducible supercuspidal representations ϕ_i^* such that τ^* is a sub-representation of ϕ^*. Since whenever $m>n$ (and τ_1 is irreducible supercuspidal) [JPS83, Proposition 8.1(i)] implies that $L(\phi_i^*\times\tau_1^*,1-2s)=1$, Theorem 3.3 shows that each $M(\phi_i^*\otimes\tau_1,(1-s,s))$, $1\leqslant i\leqslant k$, is holomorphic. According to [Sha81, Theorem 2.1.1], so is $M(\tau^*\otimes\tau_1,(1-s,s))$. Now the fact

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that $L(\tau \times \tau_1, 2s) = 1$ implies that $M^*(\tau^* \otimes \tau_1, (1-s, s))$ is holomorphic. The same reasoning applies to $M^*(\tau_1 \otimes \tau^*, (s, 1-s))$. □

6. Computation of the gcd

6.1 Tempered representations

Assume that π and τ are tempered representations. The following proposition is the key ingredient in the proof of Theorem 1.1.

Proposition 6.1. *The integral $\Psi(W, f_s, s)$ with $f_s \in \xi(\tau, \mathrm{std}, s)$ has no poles for $\Re(s) > 0$.*

We derive the theorem first.

Proof of Theorem 1.1. We prove that $L(\pi \times \tau, s)^{-1}$ divides $\gcd(\pi \times \tau, s)^{-1}$, $\gcd(\pi \times \tau, s) \in L(\pi \times \tau, s) M_\tau(s)\mathbb{C}[q^{-s}, q^s]$ and under a certain assumption on the intertwining operators, $\gcd(\pi \times \tau, s) = L(\pi \times \tau, s)$.

Recall that by Conjecture 1.1, $\gamma(\pi \times \tau, \psi, s)$ given by (3.5) equals (up to an invertible factor in $\mathbb{C}[q^{-s}, q^s]$) the corresponding γ-factor of Shahidi. By Shahidi's definition of the γ-factor [Sha90] and (3.7),

$$\frac{\gcd(\pi \times \tau^*, 1-s)}{\gcd(\pi \times \tau, s)} \simeq \gamma(\pi \times \tau, \psi, s) \simeq \frac{L(\pi \times \tau^*, 1-s)}{L(\pi \times \tau, s)}. \tag{6.1}$$

Here the L-functions on the right-hand side are the ones defined by Shahidi. By [CS98, § 4], the L-function $L(\pi \times \tau, s)$ is holomorphic for $\Re(s) > 0$ and $L(\pi \times \tau^*, 1-s)$ is holomorphic for $\Re(s) < 1$. Therefore the quotient on the right-hand side is reduced and it follows immediately that $L(\pi \times \tau, s)^{-1}$ divides $\gcd(\pi \times \tau, s)^{-1}$.

The integrals $\Psi(W, f_s, s)$ with $f_s \in \xi(\tau, \mathrm{hol}, s)$ span a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$ which contains 1, according to Proposition 4.1. Thus there is a polynomial $P_0 \in \mathbb{C}[X]$ with $P_0(0) = 1$ and of minimal degree such that $P_0(q^{-s})\Psi(W, f_s, s) \in \mathbb{C}[q^{-s}, q^s]$ for all W and $f_s \in \xi(\tau, \mathrm{hol}, s)$. Put $\gcd_0(\pi \times \tau, s) = P_0(q^{-s})^{-1}$. Then $\gcd_0(\pi \times \tau, s)^{-1}$ divides $\gcd(\pi \times \tau, s)^{-1}$ and, by virtue of Proposition 4.2,

$$\gcd(\pi \times \tau, s) = \gcd_0(\pi \times \tau, s)\ell_{\tau^*}(1-s)P(q^{-s}, q^s),$$

where $P \in \mathbb{C}[q^{-s}, q^s]$ divides $\ell_{\tau^*}(1-s)^{-1}$. Similarly write $\gcd(\pi \times \tau^*, 1-s)$ using $\widetilde{P} \in \mathbb{C}[q^{-s}, q^s]$ which divides $\ell_\tau(s)^{-1}$. Consider the quotient

$$\frac{\gcd(\pi \times \tau^*, 1-s)}{\gcd(\pi \times \tau, s)} = \frac{\gcd_0(\pi \times \tau^*, 1-s)\ell_\tau(s)\widetilde{P}(q^{-s}, q^s)}{\gcd_0(\pi \times \tau, s)\ell_{\tau^*}(1-s)P(q^{-s}, q^s)}.$$

Proposition 6.1 implies that $\gcd_0(\pi \times \tau^*, 1-s)^{-1}$ and $\gcd_0(\pi \times \tau, s)^{-1}$ are relatively prime. However, $\gcd_0(\pi \times \tau, s)$ and $\ell_\tau(s)\widetilde{P}(q^{-s}, q^s)$ may have common factors and factors of $\ell_{\tau^*}(1-s)$ $P(q^{-s}, q^s)$ may also appear in the numerator. Canceling common factors and using (6.1) we see that $\gcd_0(\pi \times \tau, s) \in \ell_\tau(s)L(\pi \times \tau, s)\mathbb{C}[q^{-s}, q^s]$. Therefore $\gcd(\pi \times \tau, s) \in L(\pi \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^s]$.

Now assume that the intertwining operators $L(\tau, S^2, 2s-1)^{-1}M(\tau, s)$ and $L(\tau^*, S^2, 1-2s)^{-1}M(\tau^*, 1-s)$ are holomorphic. According to (3.2), with a minor abuse of notation,

$$M^*(\tau^*, 1-s) = \gamma(\tau^*, S^2, \psi, 1-2s)M(\tau^*, 1-s) \simeq \frac{L(\tau, S^2, 2s)}{L(\tau^*, S^2, 1-2s)}M(\tau^*, 1-s).$$

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Since we assume that $L(\tau^*, S^2, 1-2s)^{-1}M(\tau^*, 1-s)$ is holomorphic, the poles of $M^*(\tau^*, 1-s)$ are contained in the poles of $L(\tau, S^2, 2s)$, which by Theorem 3.1 lie in $\Re(s) \leqslant 0$. Thus by Propositions 4.2 and 6.1 the poles of $\Psi(W, f_s, s)$ for $f_s \in \xi(\tau, \text{good}, s)$ are in $\Re(s) \leqslant 0$ and the same is true for $\gcd(\pi \times \tau, s)$. Similarly we see that the poles of $\gcd(\pi \times \tau^*, 1-s)$ are in $\Re(s) \geqslant 1$. Thus the quotient $\gcd(\pi \times \tau^*, 1-s)\gcd(\pi \times \tau, s)^{-1}$ is reduced. Then (6.1) implies that $\gcd(\pi \times \tau, s) = L(\pi \times \tau, s)$. □

Proof of Proposition 6.1. Consider the case of $l \leqslant n$ and split G_l. By Proposition 4.4 it is enough to prove that the integral

$$\int_{A_{l-1}} \int_{G_1} \mathrm{ch}_\Lambda(x)|W|(ax)\delta_{B_{G_l}}^{-\frac{1}{2}}(a)(\delta_{B_{\mathrm{GL}_n}}^{-\frac{1}{2}} \cdot |\det|^{\Re(s)} \cdot |W'|)(\mathrm{diag}(a, \lfloor x \rfloor, I_{n-l}))\, dx\, da \tag{6.2}$$

is convergent for $\Re(s) > 0$ and this convergence is uniform for $\Re(s)$ in a compact set. We may already take a real s. Since $|\lfloor x \rfloor| \leqslant 1$ and W' vanishes away from zero ($W'(t) = 0$ for $t \in A_n$ unless all simple roots evaluated at t are close enough to zero; see [CS80, § 6.1]), the coordinates of a are all bounded from above. Hence we may simply replace $\mathrm{ch}_\Lambda(x)$ in the last integral by a non-negative Schwartz function $\Phi \in \mathcal{S}(F^l)$, which is a function of a and $\lfloor x \rfloor$. By the Cauchy–Schwarz inequality, integral (6.2) is bounded by the product of square roots of the following two integrals:

$$\int_{A_{l-1}} \int_{G_1} |W|^2(ax)|\det a|^s[x]^{-s}\delta_{B_{G_l}}^{-1}(a)\, dx\, da, \tag{6.3}$$

$$\int_{A_{l-1}} \int_{G_1} \Phi^2(a, \lfloor x \rfloor)(\delta_{B_{\mathrm{GL}_n}}^{-1} \cdot |\det|^s \cdot |W'|^2)(\mathrm{diag}(a, \lfloor x \rfloor, I_{n-l}))\, dx\, da. \tag{6.4}$$

Replace the dx-integration in (6.4) with an integration over $a_l \in F^*$. Then (6.4) is a sum of two integrals: the first with $|a_l| \leqslant 1$, the second with $|a_l| > 1$, both bounded by an integral of the form

$$\int_{A_l} |W'|^2(a)\Phi^2(a)|\det a|^s \delta_{B_{\mathrm{GL}_n}}^{-1}(a)\, da. \tag{6.5}$$

Integrals (6.3) and (6.5) converge for $s > 0$, uniformly for s in a compact set, since π and τ are tempered. This follows by using the asymptotic expansion of Whittaker functions (see [CS80, § 6], and also [LM09]) and the fact that the exponents of a tempered representation are non-negative [Wal03, Proposition III.2.2].

For $l \leqslant n$ and quasi-split G_l one just repeats the above arguments, ignoring the integration over G_1. The case of $l > n$ is similar (the properties of W imply that the coordinates of $a \in A_n$ are bounded from above). □

6.2 Lower bound for an irreducible π and a tempered τ

Let π be an irreducible (generic) representation of G_l. According to [Mui01] we may assume that π is a standard module. Namely, $\pi = \mathrm{Ind}_{P_k}^{G_l}(\sigma \otimes \pi')$ where $\sigma = \mathrm{Ind}_{P_{k_1,\dots,k_m}}^{\mathrm{GL}_k}(\sigma_1 \otimes \cdots \otimes \sigma_m)$ (if $m = 1$, $\sigma = \sigma_1$), $\sigma_i = |\det|^{e_i}\sigma_i'$, σ_i' is a square-integrable representation of GL_{k_i}, $e_i \in \mathbb{R}$, $0 > e_1 \geqslant \cdots \geqslant e_m$ and π' is a tempered representation of G_{l-k} (see also [Mui04]). Note that if $k = l$, it is also possible that $\pi = \mathrm{Ind}_{{}^\kappa P_l}^{G_l}(\sigma)$.

Proof of Theorem 1.4. Let τ be tempered such that the prescribed intertwining operators are holomorphic. Using Corollary 5.1 (valid for our range of k, l and n), for some $P_1, P_2 \in \mathbb{C}[q^{-s}, q^s]$,

$$L(\sigma \times \tau, s) = \gcd(\pi \times \tau, s)P_1, \quad L(\sigma \times \tau^*, 1-s) = \gcd(\pi \times \tau^*, 1-s)P_2. \tag{6.6}$$

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According to the multiplicativity of $\gamma(\pi \times \tau, \psi, s)$, (3.7), (3.8) and (6.6),

$$\frac{P_2 L(\sigma^* \times \tau^*, 1-s) \gcd(\pi' \times \tau^*, 1-s)}{P_1 L(\sigma^* \times \tau, s) \gcd(\pi' \times \tau, s)} \simeq 1. \tag{6.7}$$

As proved in § 6.1 the poles of $\gcd(\pi' \times \tau, s)$ lie in $\Re(s) \leqslant 0$, because π' and τ are tempered (this did not rely on Conjecture 1.1). According to [JPS83, Theorem 3.1],

$$L(\sigma^* \times \tau, s) \in \prod_{i=1}^{m} L(\sigma_i^* \times \tau, s)\mathbb{C}[q^{-s}, q^s].$$

Hence any pole of $L(\sigma^* \times \tau, s)$ is contained in some $L(\sigma_i^* \times \tau, s) = L((\sigma_i')^* \times \tau, s - e_i)$, and since σ_i' is square-integrable we get from [JPS83, § 8] that the poles of $L((\sigma_i')^* \times \tau, s - e_i)$ lie in $\Re(s) \leqslant e_i < 0$. We conclude that the poles of the denominator of (6.7) are in $\Re(s) \leqslant 0$.

In a similar manner we prove that the poles of the numerator of (6.7) are in $\Re(s) \geqslant 1$. Specifically, the poles of $\gcd(\pi' \times \tau^*, 1-s)$ are in $\Re(1-s) \leqslant 0$ because π' and τ^* are tempered, and any pole of $L(\sigma^* \times \tau^*, 1-s)$ appears in some $L((\sigma_i')^* \times \tau^*, 1-s-e_i)$.

Thus any pole appearing in the denominator must be canceled by P_1, i.e. $P_1 L(\sigma^* \times \tau, s) \gcd(\pi' \times \tau, s) \in \mathbb{C}[q^{-s}, q^s]$. Together with (6.6) this gives the result. □

7. Integrals and Laurent series

7.1 Functions and Laurent series

The integral $\Psi(W, f_s, s)$ is by meromorphic continuation a rational function in q^{-s}. In order to bound the possible poles of the integral in their case, Jacquet *et al.* [JPS83] regarded it as a Laurent series. In this section we provide a few background definitions and results, explicating the connection between integrals and series.

Let $\Sigma(X) = \mathbb{C}[[X, X^{-1}]]$ be the complex vector space of formal Laurent series, e.g. of $\sum_{m \in \mathbb{Z}} a_m X^m$, $a_m \in \mathbb{C}$. It is an $R(X) = \mathbb{C}[X, X^{-1}]$-module with torsion. A $\Sigma \in \Sigma(X)$ is said to be absolutely convergent at $s_0 \in \mathbb{C}$ if the complex series obtained from Σ by replacing X with q^{-s_0} is absolutely convergent. Then the value of Σ at s_0 is the value of the complex series.

Let $f = f(s)$ be a complex-valued function defined on some domain $D_f \subset \mathbb{C}$, where a domain will always refer to a subset containing a non-empty open set. We say that f has a representation $\Sigma \in \Sigma(X)$ in D_f if, for all $s_0 \in D_f$, Σ is absolutely convergent at s_0 and equals $f(s_0)$. Note that such a representation is unique, i.e. if Σ' also represents f in D_f, then $\Sigma = \Sigma'$ in $\Sigma(X)$. When clear from the context, we omit the domain and say that f is representable by Σ.

7.2 Representations of integrals as series

Let Γ be an l-space, i.e. a Hausdorff, locally compact zero-dimensional topological space (see [BZ76, 1.1]), with a Borelian measure dx. For a ring R, denote by $C^\infty(\Gamma, R)$ the set of locally constant functions $\phi : \Gamma \to R$. This is a ring with the pointwise product. We usually take R to be the polynomial ring $\mathbb{C}[q^{-s}, q^s]$, or $\mathbb{C}(q^{-s})$.

Example 7.1. The function α^s belongs to $C^\infty(\mathrm{GL}_n, \mathbb{C}[q^{-s}, q^s])$.

Example 7.2. Let $f_s \in \xi(\tau, \mathrm{hol}, s) = \xi_{Q_n}^{H_n}(\tau, \mathrm{hol}, s)$ where τ is realized in $\mathcal{W}(\tau, \psi)$. For a fixed s, $f_s \in V(\tau, s)$, and then, for a fixed h, $b \mapsto f_s(h, b)$ belongs to $\mathcal{W}(\tau, \psi)$ (see § 3.1).

Write $f_s = \sum_{i=1}^{m} P_i(q^{-s}, q^s) f_s^{(i)}$ where $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$. For $h \in H_n$ write $h = buk$ with $b \in \mathrm{GL}_n \cong M_n$, $u \in U_n$ and $k \in K_{H_n}$ according to the Iwasawa decomposition. Then $f_s^{(i)}(h,1) = |\det b|^{s-\frac{1}{2}} \delta_{Q_n}^{\frac{1}{2}}(b) f_s^{(i)}(k,b)$ and, since $f_s^{(i)}(k,b)$ is independent of s, $f_s^{(i)}(h,1) \in \mathbb{C}[q^{-s}, q^s]$. Hence the function $h \mapsto f_s(h,1) = \sum_{i=1}^{m} P_i(q^{-s}, q^s) f_s^{(i)}(h,1)$ is a locally constant function on H_n with values in $\mathbb{C}[q^{-s}, q^s]$. Therefore we may regard f_s as an element of $C^\infty(H_n, \mathbb{C}[q^{-s}, q^s])$. Similarly, $f_s \in \xi(\tau, \mathrm{rat}, s)$ may be viewed as an element of $C^\infty(H_n, \mathbb{C}(q^{-s}))$ (the function $h \mapsto f_s(h,1)$ takes values in $\mathbb{C}(q^{-s})$).

For a non-empty finite subset of integers M, let $\mathcal{V}_M = \mathrm{Span}_{\mathbb{C}}\{q^{-sj} : j \in M\} \subset \mathbb{C}[q^{-s}, q^s]$. Denote the empty set by $\emptyset$. Let $\phi \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$. The M-support of ϕ is the set of $x \in \Gamma$ such that $0 \neq \phi(x) \in \mathcal{V}_M$ and for all $\emptyset \neq M' \subsetneq M$, $\phi(x) \notin \mathcal{V}_{M'}$. We denote this set (which may be empty) by $\mathrm{supp}_M(\phi)$. Since ϕ is locally constant, $\mathrm{supp}_M(\phi)$ is an open subset. Note that $x \in \mathrm{supp}_M(\phi)$ means that M is exactly the subset of integers m such that q^{-sm} appears in the polynomial $\phi(x)$ with a nonzero coefficient.

Claim 7.1. *If $\mathrm{supp}(\phi) \neq \emptyset$ ($\mathrm{supp}(\phi)$ being the support of ϕ), there is a unique (necessarily countable) collection of non-empty finite subsets of integers $\{M_i\}_{i\in I}$, such that $\mathrm{supp}_{M_i}(\phi) \neq \emptyset$ for all $i \in I$ and $\mathrm{supp}(\phi) = \coprod_{i\in I} \mathrm{supp}_{M_i}(\phi)$.*

Proof of Claim 7.1. For any $x \in \mathrm{supp}(\phi)$, $\phi(x)$ is a nonzero polynomial in q^{-s}, q^s whence there is a unique finite set $\emptyset \neq M \subset \mathbb{Z}$ containing precisely the integers m such that q^{-sm} appears in $\phi(x)$ with a nonzero coefficient. Hence $\phi(x) \in \mathcal{V}_M$ and, for all $\emptyset \neq M' \subsetneq M$, $\phi(x) \notin \mathcal{V}_{M'}$. Thus $x \in \mathrm{supp}_M(\phi)$. It follows that $\mathrm{supp}(\phi) = \bigcup_{i\in I} \mathrm{supp}_{M_i}(\phi)$ for some collection of sets $\{M_i\}_{i\in I}$.

If $x \in \mathrm{supp}_M(\phi) \bigcap \mathrm{supp}_N(\phi)$ for another non-empty finite subset N, then $\phi(x) \in \mathcal{V}_M \bigcap \mathcal{V}_N$. Since $\phi(x) \neq 0$ we obtain $M \cap N \neq \emptyset$. If $M \bigcap N = M$, we get that $\emptyset \neq M \subsetneq N$ and $\phi(x) \in \mathcal{V}_M$, which contradicts the fact that $x \in \mathrm{supp}_N(\phi)$. But now $\emptyset \neq M \cap N \subsetneq M$ satisfies $\phi(x) \in \mathcal{V}_{M\cap N}$ contradicting that $x \in \mathrm{supp}_M(\phi)$. This shows that $\mathrm{supp}(\phi) = \coprod_{i\in I} \mathrm{supp}_{M_i}(\phi)$ and the collection $\{M_i\}_{i\in I}$ is unique. $\square$

We say that Γ can be divided into the simple supports of ϕ if $\mathrm{supp}(\phi) = \emptyset$ or if the collection $\{M_i\}_{i\in I}$ of Claim 7.1 satisfies $M_i = \{m_i\}$ for some $m_i \in \mathbb{Z}$ for all $i \in I$.

Example 7.3. The l-space GL_n can be divided into the simple supports of α^s.

Example 7.4. Let $P = \sum_{j\in\mathbb{Z}} a_j q^{-sj} \in \mathbb{C}[q^{-s}, q^s]$ (i.e. $a_j = 0$ for almost all j). The function $\phi(x) = P$ defined on Γ trivially belongs to $C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$. Assuming that $P \neq 0$, we have that $\mathrm{supp}(\phi) = \Gamma = \mathrm{supp}_M(\phi)$ where $M = \{j \in \mathbb{Z} : a_j \neq 0\}$. This example can be extended by replacing the coefficients a_j with functions in $C^\infty(\Gamma, \mathbb{C})$.

Example 7.5. Let $f_s \in \xi(\tau, \mathrm{std}, s)$ be regarded as a function in $C^\infty(H_n, \mathbb{C}[q^{-s}, q^s])$, as explained in Example 7.2, i.e. the actual function is $h \mapsto f_s(h,1)$. For $h \in H_n$, $f_s(h,1) = |\det b|^{s-\frac{1}{2}} \delta_{Q_n}^{\frac{1}{2}}(b) f_s(k,b)$ where $h = buk$ is written according to Example 7.2. Since $f_s(k,b)$ is independent of s,

$$\mathrm{supp}_{\{m\}}(f_s) = \{h \in H_n : h = buk, |\det b| = q^{-m}, f_s(k,b) \neq 0\}.$$

This shows that H_n can be divided into the simple supports of f_s.

Denote by $P_m : \mathbb{C}[q^{-s}, q^s] \to \mathbb{C}$ the function given by $P_m(\sum_{j\in\mathbb{Z}} a_j q^{-sj}) = a_m$. The function $x \mapsto P_m(\phi(x))$ is locally constant (hence measurable).

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DEFINITION 7.1. Let $\phi \in C^{\infty}(\Gamma, \mathbb{C}[q^{-s}, q^{s}])$. Assume that for all $m \in \mathbb{Z}$, $\int_{\Gamma} |P_m(\phi(x))|\, dx < \infty$. Then we define a Laurent series in $\Sigma(X)$,

$$\oint_{\Gamma} \phi(x)\, dx = \sum_{m\in\mathbb{Z}} X^m \int_{\Gamma} P_m(\phi(x))\, dx.$$

We define a series as above to be strongly convergent at s if

$$\sum_{m\in\mathbb{Z}} q^{-\Re(s)m} \int_{\Gamma} |P_m(\phi(x))|\, dx < \infty,$$

a condition stronger than being absolutely convergent at s (where the absolute value surrounds the dx-integral).

Example 7.6. Let $a \in C^{\infty}(\Gamma, \mathbb{C}) \subset C^{\infty}(\Gamma, \mathbb{C}[q^{-s}, q^{s}])$. Then $\mathrm{supp}(a) = \mathrm{supp}_{\{0\}}(a)$, and provided that $\int_{\Gamma} |a(x)|\, dx < \infty$ then $\oint_{\Gamma} a(x)\, dx = \int_{\Gamma} a(x)\, dx$ is a constant term as an element of $\Sigma(X)$.

Next we define an integral for $\phi \in C^{\infty}(\Gamma, \mathbb{C}[q^{-s}, q^{s}])$. Any fixed $s_0 \in \mathbb{C}$ induces a homomorphism $\mathbb{C}[q^{-s}, q^{s}] \to \mathbb{C}$ by evaluation. Denote by $[\phi(x)](s_0)$ the value of $\phi(x)$ under this homomorphism. For example, if $x \in \mathrm{supp}_M(\phi)$, then $\phi(x) = \sum_{j\in M} a_j(x) q^{-sj}$ where $0 \neq a_j(x) \in \mathbb{C}$ and $[\phi(x)](s_0) = \sum_{j\in M} a_j(x) q^{-s_0 j} \in \mathbb{C}$.

For any fixed $s \in \mathbb{C}$, the integral $\Phi(s) = \int_{\Gamma} [\phi(x)](s)\, dx$ is absolutely convergent if $\int_{\Gamma} |[\phi(x)](s)|\, dx < \infty$. If there is a domain $D \subset \mathbb{C}$ such that, for all $s \in D$, $\Phi(s)$ is absolutely convergent, then $s \mapsto \Phi(s)$ is a complex-valued function on D. The following results show how to use the series defined above to represent the complex function Φ in the sense of § 7.1.

We introduce the following notation that will be used repeatedly below. For $\phi \in C^{\infty}(\Gamma, \mathbb{C}[q^{-s}, q^{s}])$, denote $\Phi_{\phi}(s) = \int_{\Gamma} [\phi(x)](s)\, dx$ and $\Sigma_{\phi} = \oint_{\Gamma} \phi(x)\, dx$ (assuming these are defined). Let $D \subset \mathbb{C}$ be a domain. We write $\Sigma_{\phi} \sim_D \Phi_{\phi}$ if, for all $s \in D$, $\Phi_{\phi}(s)$ is absolutely convergent, Σ_{ϕ} is strongly convergent at s and Σ_{ϕ} represents Φ_{ϕ} in D.

LEMMA 7.1. *Let $\phi \in C^{\infty}(\Gamma, \mathbb{C}[q^{-s}, q^{s}])$ be such that Γ can be divided into its simple supports. Assume that $\Phi_{\phi}(s)$ is absolutely convergent in a domain $D \subset \mathbb{C}$. Then Σ_{ϕ} is defined and $\Sigma_{\phi} \sim_D \Phi_{\phi}$.*

Proof of Lemma 7.1. If $x \in \mathrm{supp}_{\{m\}}(\phi)$, write $\phi(x) = q^{-sm} a_m(x)$ with $0 \neq a_m(x) \in \mathbb{C}$. In D,

$$\begin{aligned}\Phi_{\phi}(s) &= \sum_{m\in\mathbb{Z}} \int_{\mathrm{supp}_{\{m\}}(\phi)} [\phi(x)](s)\, dx = \sum_{m\in\mathbb{Z}} \int_{\mathrm{supp}_{\{m\}}(\phi)} q^{-sm} a_m(x)\, dx \\ &= \sum_{m\in\mathbb{Z}} q^{-sm} \int_{\mathrm{supp}_{\{m\}}(\phi)} P_m(\phi(x))\, dx = \sum_{m\in\mathbb{Z}} q^{-sm} \int_{\Gamma} P_m(\phi(x))\, dx.\end{aligned}$$

Note that whenever $\mathrm{supp}_{\{m\}}(\phi) \neq \emptyset$, $a_m(x)$ is defined. Each of the integrals $\int_{\Gamma} P_m(\phi(x))\, dx$ is absolutely convergent, because $\Phi(s)$ is such for $s \in D$. Hence Σ_{ϕ} is defined,

$$\Sigma_{\phi} = \sum_{m\in\mathbb{Z}} X^m \int_{\Gamma} P_m(\phi(x))\, dx.$$

It is strongly convergent since

$$\sum_{m\in\mathbb{Z}} q^{-\Re(s)m}\int_{\Gamma}|P_m(\phi(x))|\,dx = \sum_{m\in\mathbb{Z}} q^{-\Re(s)m}\int_{\mathrm{supp}_{\{m\}}(\phi)}|P_m(\phi(x))|\,dx$$
$$= \sum_{m\in\mathbb{Z}}\int_{\mathrm{supp}_{\{m\}}(\phi)}|[\phi(x)](s)|\,dx = \int_{\Gamma}|[\phi(x)](s)|\,dx < \infty.$$

Evidently, replacing X with q^{-s} in Σ_ϕ we obtain $\Phi_\phi(s)$, showing that Σ_ϕ represents Φ_ϕ in D. □

More generally, we have the following lemma.

Lemma 7.2. *Let $\phi\in C^\infty(\Gamma,\mathbb{C}[q^{-s},q^s])$ be such that Σ_ϕ is defined. Assume that $\Phi_\phi(s)$ is absolutely convergent and Σ_ϕ is strongly convergent in a domain $D\subset\mathbb{C}$. Then $\Sigma_\phi\sim_D\Phi_\phi$.*

Proof of Lemma 7.2. According to the assumptions, it is left to show that Σ_ϕ represents Φ_ϕ in D. For $x\in\mathrm{supp}_M(\phi)$ write $\phi(x)=\sum_{j\in M}a_j(x)q^{-sj}$ where $0\neq a_j(x)\in\mathbb{C}$. Let $\mathcal{Z}$ be the (countable) set of finite non-empty subsets of $\mathbb{Z}$. Observe that for a fixed $m\in\mathbb{Z}$,

$$\int_{\Gamma}|P_m(\phi(x))|\,dx = \sum_{\{M\in\mathcal{Z}:m\in M\}}\int_{\mathrm{supp}_M(\phi)}|P_m(\phi(x))|\,dx = \sum_{\{M\in\mathcal{Z}:m\in M\}}\int_{\mathrm{supp}_M(\phi)}|a_m(x)|\,dx. \tag{7.1}$$

(The summation is over all subsets $M\in\mathcal{Z}$ containing m.) Note that $\mathrm{supp}_M(\phi)$ may be empty, in which case the dx-integration over $\mathrm{supp}_M(\phi)$ vanishes. Also, for any two distinct $M,N\in\mathcal{Z}$, as we proved in Claim 7.1 the sets $\mathrm{supp}_M(\phi)$, $\mathrm{supp}_N(\phi)$ are disjoint.

Let $s\in D$. Since Σ_ϕ is strongly convergent in D, using (7.1) yields

$$\sum_{m\in\mathbb{Z}} q^{-\Re(s)m}\sum_{\{M\in\mathcal{Z}:m\in M\}}\int_{\mathrm{supp}_M(\phi)}|a_m(x)|\,dx<\infty. \tag{7.2}$$

We also have

$$\Phi_\phi(s) = \sum_{M\in\mathcal{Z}}\int_{\mathrm{supp}_M(\phi)}[\phi(x)](s)\,dx = \sum_{M\in\mathcal{Z}}\int_{\mathrm{supp}_M(\phi)}\sum_{j\in M}a_j(x)q^{-sj}\,dx. \tag{7.3}$$

For fixed M and $j\in M$, $\int_{\mathrm{supp}_M(\phi)}|a_j(x)|\,dx<\infty$ because it is majorized by $q^{\Re(s)j}$ times (7.2). Hence we may change the order of summation and integration in (7.3) and obtain

$$\sum_{M\in\mathcal{Z}}\sum_{j\in M}q^{-sj}\int_{\mathrm{supp}_M(\phi)}a_j(x)\,dx.$$

Again using (7.2) we change the order of summation,

$$\Phi_\phi(s) = \sum_{M\in\mathcal{Z}}\sum_{j\in M}q^{-sj}\int_{\mathrm{supp}_M(\phi)}a_j(x)\,dx = \sum_{m\in\mathbb{Z}}q^{-sm}\sum_{\{M\in\mathcal{Z}:m\in M\}}\int_{\mathrm{supp}_M(\phi)}a_m(x)\,dx$$
$$= \sum_{m\in\mathbb{Z}}q^{-sm}\sum_{\{M\in\mathcal{Z}:m\in M\}}\int_{\mathrm{supp}_M(\phi)}P_m(\phi(x))\,dx = \sum_{m\in\mathbb{Z}}q^{-sm}\int_{\Gamma}P_m(\phi(x))\,dx.$$

This shows that Σ_ϕ represents Φ_ϕ in D. □

Next we consider the integral of a sum of holomorphic sections.

Lemma 7.3. *Let $\phi_1,\phi_2\in C^\infty(\Gamma,\mathbb{C}[q^{-s},q^s])$ and put $\phi=\phi_1+\phi_2$. If Σ_{ϕ_i} is defined, $i=1,2$, then Σ_ϕ is defined. Moreover if $\Sigma_{\phi_i}\sim_D\Phi_{\phi_i}$ for both i, then $\Sigma_\phi\sim_D\Phi_\phi$ and $\Sigma_\phi=\Sigma_{\phi_1}+\Sigma_{\phi_2}$.*

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Proof of Lemma 7.3. The series Σ_ϕ is defined because $\Sigma_{\phi_1}, \Sigma_{\phi_2}$ are defined and $P_m(\phi(x)) = P_m(\phi_1(x)) + P_m(\phi_2(x))$. Now consider $s \in D$. Since $\Phi_{\phi_1}(s)$, $\Phi_{\phi_2}(s)$ are absolutely convergent, so is $\Phi_\phi(s)$ and $\Phi_\phi(s) = \Phi_{\phi_1}(s) + \Phi_{\phi_2}(s)$. Also Σ_ϕ is strongly convergent, because Σ_{ϕ_1}, Σ_{ϕ_2} are. By Lemma 7.2, $\Sigma_\phi \sim_D \Phi_\phi$. Since the series $\Sigma_{\phi_1} + \Sigma_{\phi_2}$ also represents Φ_ϕ in D, then $\Sigma_\phi = \Sigma_{\phi_1} + \Sigma_{\phi_2}$. □

As in Example 7.4, let $P \in \mathbb{C}[q^{-s}, q^s]$ be considered as an element of $C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$; then $P \cdot \phi \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$. This defines a structure of a $\mathbb{C}[q^{-s}, q^s]$-module on $C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$. The next lemma shows that the $\oint$ operation commutes with multiplication by a polynomial.

LEMMA 7.4. *Let $\phi \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$ be such that Σ_ϕ is defined and assume that $\Sigma_\phi \sim_D \Phi_\phi$. Then, for any $P \in \mathbb{C}[q^{-s}, q^s]$, $\Sigma_{P\phi}$ is defined, $\Sigma_{P\phi} \sim_D \Phi_{P\phi}$ and $P(X, X^{-1})\Sigma_\phi = \Sigma_{P\phi}$, i.e.*

$$P(X, X^{-1}) \oint_\Gamma \phi(x)\, dx = \oint_\Gamma (P\phi)(x)\, dx.$$

Proof of Lemma 7.4. Write $P = \sum_{j \in M} a_j q^{-sj}$, $\phi_j = a_j q^{-sj} \cdot \phi$. It follows from the definitions that $\Sigma_{\phi_j} = \oint_\Gamma \phi_j(x)\, dx$ is defined and in D, Σ_{ϕ_j} is strongly convergent and $\Phi_{\phi_j}(s) = \int_\Gamma [\phi_j(x)](s)\, dx$ is absolutely convergent. By Lemma 7.2, $\Sigma_{\phi_j} \sim_D \Phi_{\phi_j}$. Since $P\phi = \sum_{j \in M} \phi_j$, Lemma 7.3 shows that $\Sigma_{P\phi} = \oint_\Gamma (P\phi)(x)\, dx$ is defined and $\Sigma_{P\phi} \sim_D \Phi_{P\phi}$. For $s \in D$, $\Phi_{P\phi}(s) = P(q^{-s}, q^s)\Phi_\phi(s)$, whence $P(X, X^{-1})\Sigma_\phi$ also represents $\Phi_{P\phi}$ in D, showing that $\Sigma_{P\phi} = P(X, X^{-1})\Sigma_\phi$. □

Let $\Gamma \times \Gamma'$ be a product of l-spaces (this is also an l-space) and let $\phi \in C^\infty(\Gamma \times \Gamma', \mathbb{C}[q^{-s}, q^s])$. For any $x \in \Gamma$ define a function $\phi(x, \cdot) \in C^\infty(\Gamma', \mathbb{C}[q^{-s}, q^s])$ by $x' \mapsto \phi(x, x')$. We say that ϕ is smooth in Γ if the function $x \mapsto \phi(x, \cdot)$ belongs to $C^\infty(\Gamma, C^\infty(\Gamma', \mathbb{C}[q^{-s}, q^s]))$. Put $\Sigma_\phi = \oint_{\Gamma \times \Gamma'} \phi(x, x') d(x, x')$, $\Phi_\phi(s) = \int_{\Gamma \times \Gamma'} [\phi(x, x')](s) d(x, x')$. We prove a weak analogue of Fubini's theorem.

LEMMA 7.5. *Let $\phi \in C^\infty(\Gamma \times \Gamma', \mathbb{C}[q^{-s}, q^s])$ be smooth in Γ, for which Σ_ϕ is defined. Assume $\Sigma_\phi \sim_D \Phi_\phi$. Then, for all $x \in \Gamma$, the series $\Sigma_{\phi(x,\cdot)} = \oint_{\Gamma'} \phi(x, x')\, dx'$ is defined and $\Sigma_{\phi(x,\cdot)} \sim_D \Phi_{\phi(x,\cdot)}$, where $\Phi_{\phi(x,\cdot)}(s) = \int_{\Gamma'} [\phi(x, x')](s)\, dx'$. Further suppose that, for all $x \in \Gamma$, $\Sigma_{\phi(x,\cdot)} \in R(X)$. Then the function $\phi_{\Gamma'}(x) = \Sigma_{\phi(x,\cdot)}(q^{-s}, q^s)$ belongs to $C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$, $\Sigma_{\phi_{\Gamma'}} = \oint_\Gamma \phi_{\Gamma'}(x)\, dx$ is defined, $\Sigma_{\phi_{\Gamma'}} \sim_D \Phi_\phi$, and $\Sigma_\phi = \Sigma_{\phi_{\Gamma'}}$.*

Proof of Lemma 7.5. Let $x \in \Gamma$. Since ϕ is smooth in Γ, there is a compact open neighborhood $N_x \subset \Gamma$ of x, such that $\phi(x, \cdot) = \phi(n_x, \cdot)$ for all $n_x \in N_x$. Put $c_x = \mathrm{vol}(N_x)^{-1}$. Because Σ_ϕ is defined, for all m we have

$$\int_{\Gamma'} |P_m(\phi(x, x'))|\, dx' \leqslant c_x \int_{\Gamma \times \Gamma'} |P_m(\phi(x, x'))|\, d(x, x') < \infty.$$

Therefore $\Sigma_{\phi(x,\cdot)}$ is defined. The coefficient of X^m in $\Sigma_{\phi(x,\cdot)}$ is $\int_{\Gamma'} P_m(\phi(x, x'))\, dx'$. In addition, since Σ_ϕ is strongly convergent at $s \in D$,

$$\sum_{m \in \mathbb{Z}} q^{-\Re(s)m} \int_{\Gamma'} |P_m(\phi(x, x'))|\, dx \leqslant c_x \sum_{m \in \mathbb{Z}} q^{-\Re(s)m} \int_{\Gamma \times \Gamma'} |P_m(\phi(x, x'))|\, d(x, x') < \infty.$$

Hence $\Sigma_{\phi(x,\cdot)}$ is strongly convergent. Also $\Phi_{\phi(x,\cdot)}(s)$ is absolutely convergent in D. According to Lemma 7.2, we have $\Sigma_{\phi(x,\cdot)} \sim_D \Phi_{\phi(x,\cdot)}$.

Now suppose that $\Sigma_{\phi(x,\cdot)} \in R(X)$ for all x. Then $\phi_{\Gamma'} \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$ because ϕ is smooth in Γ. By Tonelli's theorem and using the fact that Σ_ϕ is defined,

$$\int_\Gamma |P_m(\phi_{\Gamma'}(x))|\, dx = \int_\Gamma \left| \int_{\Gamma'} P_m(\phi(x,x'))\, dx' \right| dx \leqslant \int_{\Gamma\times\Gamma'} |P_m(\phi(x,x'))| d(x,x') < \infty.$$

It follows that $\Sigma_{\phi_{\Gamma'}}$ is defined. Since Σ_ϕ is strongly convergent for $s \in D$, so is $\Sigma_{\phi_{\Gamma'}}$. In fact,

$$\sum_{m\in\mathbb{Z}} q^{-\Re(s)m} \int_\Gamma |P_m(\phi_{\Gamma'}(x))|\, dx \leqslant \sum_{m\in\mathbb{Z}} q^{-\Re(s)m} \int_{\Gamma\times\Gamma'} |P_m(\phi(x,x'))|\, d(x,x') < \infty.$$

Now $\Phi_{\phi_{\Gamma'}}(s) = \int_\Gamma [\phi_{\Gamma'}(x)](s)\, dx$ is absolutely convergent in D because

$$\int_\Gamma |[\phi_{\Gamma'}(x)](s)|\, dx \leqslant \int_\Gamma \int_{\Gamma'} |[\phi(x,x')](s)|\, dx'\, dx = \int_{\Gamma\times\Gamma'} |[\phi(x,x')](s)|\, d(x,x') < \infty.$$

Appealing to Lemma 7.2, we have that $\Sigma_{\phi_{\Gamma'}} \sim_D \Phi_{\phi_{\Gamma'}}$ and since, for $s \in D$, $\Phi_{\phi_{\Gamma'}}(s) = \int_\Gamma \int_{\Gamma'} [\phi(x,x')](s)\, dx' dx$, by Fubini's theorem we have that $\Phi_{\phi_{\Gamma'}}(s) = \Phi_\phi(s)$ for all $s \in D$. Hence $\Sigma_{\phi_{\Gamma'}} \sim_D \Phi_\phi$ and $\Sigma_\phi = \Sigma_{\phi_{\Gamma'}}$. □

We may extend the definitions and results of § 7.1 and this section to functions $f(s_1, \ldots, s_k)$ in k variables. Then we consider the space $C^\infty(\Gamma, \mathbb{C}[q^{\mp s_1}, \ldots, q^{\mp s_k}])$ and, for instance, $\mathcal{V}_M = \mathrm{Span}_{\mathbb{C}}\{q^{-s_1 j_1 - \ldots - s_k j_k} : (j_1, \ldots, j_k) \in M\}$ where $\emptyset \neq M \subset \mathbb{Z}^k$ is finite.

7.3 Laurent representation for $\Psi(W, f_s, s)$

We will use the series described in § 7.2 to represent the integral $\Psi(W, f_s, s)$. Recall that there is a domain $D \subset \mathbb{C}$ of absolute convergence for $\Psi(W, f_s, s)$, $f_s \in \xi(\tau, \mathrm{hol}, s)$, which depends only on the representations π and τ (see § 3.1). Since for $f_s \in \xi(\tau, \mathrm{good}, s)$ we have $\ell_{\tau^*}(1-s)^{-1} f_s \in \xi(\tau, \mathrm{hol}, s)$, we can assume that $\Psi(W, f_s, s)$ is absolutely convergent in D for all good sections, and D still depends only on the representations.

LEMMA 7.6. *Let $f_s \in \xi(\tau, \mathrm{hol}, s)$. Consider $\Psi(W, f_s, s)$ as a function of s, defined in D. It is representable in $\Sigma(X)$ by a strongly convergent series $\Sigma(W, f_s, s)$ with finitely many negative coefficients.*

Proof of Lemma 7.6. We prove the case of $l \leqslant n$, the other case being similar. Since $U_{G_l}\overline{B_{G_l}}$ is a dense open subset of G_l such that its complement is a subset of zero measure, we can replace the dg-integration in $\Psi(W, f_s, s)$ with an integration over $\overline{B_{G_l}}$. Then the integral takes the form

$$\Psi(W, f_s, s) = \int_{\overline{B_{G_l}}} \int_{R_{l,n}} W(g) f_s(w_{l,n} r g, 1) \psi_\gamma(r)\, dr\, dg.$$

Here dg is actually a right-invariant Haar measure on $\overline{B_{G_l}}$. Let $\Gamma = \overline{B_{G_l}} \times R_{l,n}$. For $f_s \in \xi(\tau, \mathrm{hol}, s)$, set $\phi_{f_s}(g, r) = W(g) f_s(w_{l,n} r g, 1) \psi_\gamma(r) \in \mathbb{C}[q^{-s}, q^s]$ $(g \in \overline{B_{G_l}},\ r \in R_{l,n})$ and note that $\phi_{f_s} \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$. Then $\Phi_{\phi_{f_s}}(s) = \Psi(W, f_s, s)$ is absolutely convergent in D.

Assume first that $f_s \in \xi(\tau, \mathrm{std}, s)$. For $(g, r) \in \Gamma$ write $w_{l,n} r g = buk$ with $b \in \mathrm{GL}_n \cong M_n$, $u \in U_n$ and $k \in K_{H_n}$ according to the Iwasawa decomposition. Then

$$\phi_{f_s}(g, r) = W(g) |\det b|^{s-\frac{1}{2}} \delta_{Q_n}^{\frac{1}{2}}(b) f_s(k, b) \psi_\gamma(r)$$

and, since the only factor depending on s is $|\det b|^s$ (because $f_s(k, b)$ is independent of s),

$$\mathrm{supp}_{\{m\}}(\phi_{f_s}) = \{(g, r) \in \Gamma : w_{l,n} r g = buk,\ |\det b| = q^{-m},\ W(g) f_s(k, b) \neq 0\}.$$

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Thus Γ can be divided into the simple supports of ϕ_{f_s} (see also Example 7.5). Applying Lemma 7.1, we get $\Sigma_{\phi_{f_s}} \sim_D \Phi_{\phi_{f_s}}$. Regarding $f_s \in \xi(\tau, \mathrm{hol}, s)$, write $f_s = \sum_{i=1}^m P_i f_s^{(i)}$ for $P_i \in \mathbb{C}[q^{-s}, q^s]$, $f_s^{(i)} \in \xi(\tau, \mathrm{std}, s)$. Then $\Sigma_{\phi_{f_s^{(i)}}} \sim_D \Phi_{\phi_{f_s^{(i)}}}$ for all i and, by Lemma 7.4, $\Sigma_{P_i\phi_{f_s^{(i)}}} \sim_D \Phi_{P_i\phi_{f_s^{(i)}}}$. Since $\sum_{i=1}^m P_i\phi_{f_s^{(i)}} = \phi_{f_s}$, Lemma 7.3 implies that $\Sigma_{\phi_{f_s}} \sim_D \Phi_{\phi_{f_s}}$, and we set $\Sigma(W, f_s, s) = \Sigma_{\phi_{f_s}}$.

Regarding the negative coefficients, decompose $\Psi(W, f_s, s)$ as in Proposition 4.4. Clearly each integral $I_s^{(j)}$ in this decomposition is representable as a series, by formally replacing q^{-s} with X. We argue that each series has a finite number of negative coefficients, i.e., nonzero coefficients of X^{-k} with $k > 0$. Looking at (4.7), each coordinate of a is bounded from above because $W^\diamond \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ vanishes away from zero and by definition $[x]^{-1} = ||\lfloor x\rfloor|| \leqslant 1$. Hence $|\det a| \cdot [x]^{-1}$ is bounded from above. Note that the properties of $W' \in \mathcal{W}(\tau, \psi)$ could similarly be used to bound the coordinates of a. When $l > n$ we consider (4.8) instead of (4.7) for this argument. □

For any $f_s \in \xi(\tau, \mathrm{good}, s)$ there exists a $P \in \mathbb{C}[q^{-s}, q^s]$ which divides $\ell_{\tau^*}(1-s)^{-1}$, such that $Pf_s \in \xi(\tau, \mathrm{hol}, s)$. According to the above, $\Psi(W, Pf_s, s)$ defined in D is representable in $\Sigma(X)$.

8. Upper bound in the second variable

8.1 Outline

In this section we prove Theorem 1.2. We revisit the arguments of [Kap10b], where we proved the multiplicative property of the γ-factor in the second variable. The proof relied on manipulations of integrals, involving the application of three functional equations, for $\pi \times \tau_2$, $\tau_1 \times \tau_2$ and $\pi \times \tau_1$, to $\Psi(W, f_s, s)$. Here we reinterpret the passages as passages between Laurent series, and apply the functional equations to the series $\Sigma(W, f_s, s)$ which represents $\Psi(W, f_s, s)$. We utilize the notation and results of § 7 (e.g. $\Sigma(X) = \mathbb{C}[[X, X^{-1}]]$, $R(X) = \mathbb{C}[X, X^{-1}]$).

The overall structure of the proof already appeared in [JPS83]. Roughly, the proportionality factor between $\Psi(W, f_s, s)$ and $\Psi(W, M^*(\tau, s)f_s, 1-s)$ is calculated using (3.3). Each of the intertwining operators on the right-hand side of (3.3) appears as a result of applying a functional equation to an inner integral. For example, $M^*(\tau_2, s)$ appears once we apply the functional equation of $\pi \times \tau_2$.

As explained in the introduction our technique extends to the Rankin–Selberg integrals of $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. In particular, one can apply the results of § 7 to the integral manipulations in Soudry's proof of the multiplicativity for the γ-factors ([Sou93, § 11] and [Sou00]).

8.2 Interpretation of functional equations

The functional equations, at first defined between meromorphic continuations, can sometimes be interpreted as equations in $\Sigma(X)$, by substituting X for q^{-s}, as observed in [JPS83, (§ 4.3)]. Consider a typical functional equation:

$$\begin{aligned} &\gcd(\pi \times \tau, s)^{-1}\Psi(W, f_s, s) \\ &\quad = c(l, \tau, \gamma, s)\epsilon(\pi \times \tau, \psi, s)^{-1} \gcd(\pi \times \tau^*, 1-s)^{-1}\Psi(W, M^*(\tau, s)f_s, 1-s). \end{aligned} \tag{8.1}$$

This equality is, *a priori*, between meromorphic continuations, but, by the definition of the gcd and because of Claim 3.2, both sides are actually polynomials. Then (8.1) may be reinterpreted as an equality in $R(X)$ by replacing q^{-s} with X. If $\gcd(\pi \times \tau, s)^{-1}\Psi(W, f_s, s) = B(q^{-s}, q^s)$ and the right-hand side equals $\widetilde{B}(q^{-s}, q^s)$ for some $B, \widetilde{B} \in \mathbb{C}[q^{-s}, q^s]$, equality (8.1) implies that $B(X, X^{-1}) = \widetilde{B}(X, X^{-1})$ in $R(X)$.

Let $G(q^{-s})^{-1} = \gcd(\pi \times \tau, s)^{-1}$ and denote by $G(X)^{-1} \in R(X)$ (in fact, $G(X)^{-1} \in \mathbb{C}[X]$) the polynomial obtained by replacing q^{-s} with X. Similarly put $\widetilde{G}(q^{-s}, q^s)^{-1} = c(l, \tau, \gamma, s)\epsilon(\pi \times \tau, \psi, s)^{-1} \gcd(\pi \times \tau^*, 1 - s)^{-1}$ and form $\widetilde{G}(X, X^{-1})^{-1} \in R(X)$.

If $f_s \in \xi(\tau, \mathrm{hol}, s)$, by Lemma 7.6 there exists a series $\Sigma(W, f_s, s)$ which represents $\Psi(W, f_s, s)$ in some domain of absolute convergence. Hence in $\Sigma(X)$ it holds that $G(X)^{-1}\Sigma(W, f_s, s) = B(X, X^{-1})$. If also $M^*(\tau, s)f_s \in \xi(\tau^*, \mathrm{hol}, 1 - s)$, equality (8.1) may be interpreted in $\Sigma(X)$ as

$$G(X)^{-1}\Sigma(W, f_s, s) = \widetilde{G}(X, X^{-1})^{-1}\Sigma(W, M^*(\tau, s)f_s, 1 - s).$$

However, it may be the case that $M^*(\tau, s)f_s$ has poles (due to the intertwining operator). Let $0 \neq P \in \mathbb{C}[q^{-s}, q^s]$ be such that $P(q^{-s}, q^s)M^*(\tau, s)f_s$ is holomorphic. Since

$$P(q^{-s}, q^s)\Psi(W, M^*(\tau, s)f_s, 1 - s) = \Psi(W, P(q^{-s}, q^s)M^*(\tau, s)f_s, 1 - s),$$

we can multiply both sides of (8.1), which are polynomials, by P and reach the equivalent equation

$$\begin{aligned} &P(q^{-s}, q^s) \gcd(\pi \times \tau, s)^{-1}\Psi(W, f_s, s) \\ &\quad = c(l, \tau, \gamma, s)\epsilon(\pi \times \tau, \psi, s)^{-1} \gcd(\pi \times \tau^*, 1 - s)^{-1}\Psi(W, P(q^{-s}, q^s)M^*(\tau, s)f_s, 1 - s). \end{aligned}$$

Now this may be interpreted in $\Sigma(X)$ as

$$P(X, X^{-1})G(X)^{-1}\Sigma(W, f_s, s) = \widetilde{G}(X, X^{-1})^{-1}\Sigma(W, P(q^{-s}, q^s)M^*(\tau, s)f_s, 1 - s).$$

We also mention that if an integral $\Psi(W, f_s, s)$ for $f_s \in \xi(\tau, \mathrm{hol}, s)$ is already a polynomial, then $\Sigma(W, f_s, s) \in R(X)$ and the analytic continuation of $\Psi(W, f_s, s)$ equals $(\Sigma(W, f_s, s))(q^{-s}, q^s)$.

8.3 Proof of Theorem 1.2

The proof is based on the following lemma.

Lemma 8.1. *Suppose that $\tau = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau_2)$ is irreducible. Let $P_1, P_2, P_3 \in \mathbb{C}[X]$, normalized by $P_1(0) = P_2(0) = P_3(0) = 1$, and of minimal degree such that*

$$P_1(q^{-s}) \gcd(\pi \times \tau_2^*, 1 - s)^{-1}M^*(\tau_2, s),$$
$$P_2(q^{-s})M^*(\tau_1 \otimes \tau_2^*, (s, 1 - s)),$$
$$P_3(q^{-s}) \gcd(\pi \times \tau_1^*, 1 - s)^{-1}M^*(\tau_1, s)$$

are holomorphic. Set $P_{\pi\times\tau} = P_1P_2P_3 \in \mathbb{C}[X]$. Then, for $f_s \in \xi(\tau, \mathrm{hol}, s)$,

$$\Psi(W, f_s, s) \in \gcd(\pi \times \tau_1, s) \gcd(\pi \times \tau_2, s)P_{\pi\times\tau}(q^{-s})^{-1}\mathbb{C}[q^{-s}, q^s], \tag{8.2}$$
$$\Psi(W, M^*(\tau, s)f_s, 1 - s) \in \gcd(\pi \times \tau_1^*, 1 - s) \gcd(\pi \times \tau_2^*, 1 - s)P_{\pi\times\tau}(q^{-s})^{-1}\mathbb{C}[q^{-s}, q^s]. \tag{8.3}$$

Before proving the lemma we use it to derive the theorem. We argue by induction on k. Assume first that $k = 2$, $\tau = \mathrm{Ind}_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau_2)$. By applying Lemma 8.1 to τ^*, for $f_{1-s} \in \xi(\tau^*, \mathrm{hol}, 1 - s)$,

$$\Psi(W, M^*(\tau^*, 1 - s)f_{1-s}, s) \in \gcd(\pi \times \tau_1, s) \gcd(\pi \times \tau_2, s)P_{\pi\times\tau^*}(q^{s-1})^{-1}\mathbb{C}[q^{-s}, q^s]. \tag{8.4}$$

We have $P_{\pi\times\tau}(q^{-s})^{-1} \in \ell_{\tau_1}(s)\ell_{\tau_1\otimes\tau_2^*}(s)\ell_{\tau_2}(s)\mathbb{C}[q^{-s}, q^s]$, according to the definitions of $P_{\pi\times\tau}$, P_1, $\ell_{\tau_2}(s)$ (P_2, etc.). Similarly, $P_{\pi\times\tau^*}(q^{s-1})^{-1} \in \ell_{\tau_1^*}(1 - s)\ell_{\tau_2^*\otimes\tau_1}(1 - s)\ell_{\tau_2^*}(1 - s)\mathbb{C}[q^{-s}, q^s]$. We also apply the lemma to τ and $f_s \in \xi(\tau, \mathrm{hol}, s)$. Combining (8.2) and (8.4) we see that, for any $f_s \in \xi(\tau, \mathrm{good}, s)$,

$$\Psi(W, f_s, s) \in \gcd(\pi \times \tau_1, s) \gcd(\pi \times \tau_2, s) M_{\tau_1}(s) \ell_{\tau_1 \otimes \tau_2^*}(s) \ell_{\tau_2^* \otimes \tau_1}(1-s) M_{\tau_2}(s) \mathbb{C}[q^{-s}, q^s].$$

Now for the general case let $k > 2$. Put $G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s) = \prod_{i=1}^k \gcd(\pi \times \tau_i, s)$. Let $\eta = \mathrm{Ind}_{P_{n_2,\dots,n_k}}^{\mathrm{GL}_{n-n_1}}(\tau_2 \otimes \cdots \otimes \tau_k)$. Applying the induction hypothesis to η^* yields

$$\gcd(\pi \times \eta^*, 1-s) \in G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_2^*)}(1-s) M_{\tau_k^* \otimes \cdots \otimes \tau_2^*}(1-s) \mathbb{C}[q^{-s}, q^s].$$

Apply Lemma 8.1 to $\tau = \mathrm{Ind}_{P_{n_1,n-n_1}}^{\mathrm{GL}_n}(\tau_1 \otimes \eta)$. By the minimality of P_1 and since $\ell_\eta(s) \in M_{\tau_k^* \otimes \cdots \otimes \tau_2^*}(1-s)\mathbb{C}[q^{-s}, q^s]$ (because of the multiplicativity of $M^*(\eta, s)$, see (3.4)),

$$P_1(q^{-s})^{-1} \gcd(\pi \times \eta^*, 1-s) \in G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_2^*)}(1-s) M_{\tau_k^* \otimes \cdots \otimes \tau_2^*}(1-s) \mathbb{C}[q^{-s}, q^s].$$

Also $\ell_{\tau_1 \otimes \eta^*}(s) \in \prod_{j=2}^k \ell_{\tau_1 \otimes \tau_j^*}(s) \mathbb{C}[q^{-s}, q^s]$ and $\ell_{\tau_1}(s) \in M_{\tau_1^*}(1-s)\mathbb{C}[q^{-s}, q^s]$. Hence

$$\begin{aligned}
&P_2(q^{-s})^{-1} P_1(q^{-s})^{-1} \gcd(\pi \times \eta^*, 1-s) \\
&\quad \in G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_2^*)}(1-s) M_{\tau_k^* \otimes \cdots \otimes \tau_2^*}(1-s) \prod_{j=2}^k \ell_{\tau_1 \otimes \tau_j^*}(s) \mathbb{C}[q^{-s}, q^s], \\
&P_{\pi \times \tau}(q^{-s})^{-1} \gcd(\pi \times \tau_1^*, 1-s) \gcd(\pi \times \eta^*, 1-s) \\
&\quad \in G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_1^*)}(1-s) M_{\tau_k^* \otimes \cdots \otimes \tau_1^*}(1-s) \mathbb{C}[q^{-s}, q^s].
\end{aligned} \tag{8.5}$$

Then it follows from (8.3) (η replaces τ_2) and (8.5) that, for $f_s \in \xi(\tau, \mathrm{hol}, s)$,

$$\Psi(W, M^*(\tau, s) f_s, 1-s) \in G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_1^*)}(1-s) M_{\tau_k^* \otimes \cdots \otimes \tau_1^*}(1-s) \mathbb{C}[q^{-s}, q^s]. \tag{8.6}$$

Using the multiplicativity of $\gamma(\pi \times \tau, \psi, s)$ and (3.7),

$$\frac{G_{\pi \times (\tau_k^* \otimes \cdots \otimes \tau_1^*)}(1-s)}{G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s)} \simeq \frac{\gcd(\pi \times \tau^*, 1-s)}{\gcd(\pi \times \tau, s)} \simeq \frac{\gcd(\pi \times \tau_1^*, 1-s) \gcd(\pi \times \eta^*, 1-s)}{\gcd(\pi \times \tau_1, s) \gcd(\pi \times \eta, s)}.$$

Then similarly to the proof of Claim 3.3, from (8.5) we deduce

$$P_{\pi \times \tau}(q^{-s})^{-1} \gcd(\pi \times \tau_1, s) \gcd(\pi \times \eta, s) \in G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_k}(s) \mathbb{C}[q^{-s}, q^s]. \tag{8.7}$$

Combining (8.2) with (8.7) we assert that, for $f_s \in \xi(\tau, \mathrm{hol}, s)$,

$$\Psi(W, f_s, s) \in G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_k}(s) \mathbb{C}[q^{-s}, q^s].$$

As in the case of $k = 2$, repeating the arguments above for τ^* and taking $f_{1-s} \in \xi(\tau^*, \mathrm{hol}, 1-s)$, (8.6) becomes

$$\Psi(W, M^*(\tau^*, 1-s) f_{1-s}, s) \in G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_k}(s) \mathbb{C}[q^{-s}, q^s].$$

We conclude that $\gcd(\pi \times \tau, s) \in G_{\pi \times (\tau_1 \otimes \cdots \otimes \tau_k)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_k}(s) \mathbb{C}[q^{-s}, q^s]$.

The result we obtain is somewhat stronger. In the statement of the lemma we see, for instance, that if $\gcd(\pi \times \tau_2^*, 1-s)$ already contains the poles of $M^*(\tau_2, s)$, then $P_1 = 1$. Hence the factor $M_{\tau_2}(s)$ in the upper bound for $k = 2$ will be replaced with just $\ell_{\tau_2^*}(1-s)$. Applying induction more carefully, keeping track of the poles of intertwining operators which are canceled by the gcd factors, yields the following refinement of Theorem 1.2.

Corollary 8.1. *For all $1 \leqslant i \leqslant k$, let $P_i, \widetilde{P}_i \in \mathbb{C}[X]$ be normalized by $P_i(0) = \widetilde{P}_i(0) = 1$ and of minimal degree such that*

$$P_i(q^{-s}) \gcd(\pi \times \tau_i^*, 1-s)^{-1} M^*(\tau_i, s), \quad \widetilde{P}_i(q^{s-1}) \gcd(\pi \times \tau_i, s)^{-1} M^*(\tau_i^*, 1-s)$$

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are holomorphic. Let $Q_i \in \mathbb{C}[q^{-s}, q^s]$ be a least common multiple of $P_i(q^{-s})$ and $\widetilde{P}_i(q^{s-1})$. Then

$$\gcd(\pi \times \tau, s) \in \prod_{i=1}^{k} (\gcd(\pi \times \tau_i, s) Q_i(q^{-s}, q^s)^{-1}) \prod_{1 \leqslant i < j \leqslant k} \ell_{\tau_i \otimes \tau_j^*}(s) \ell_{\tau_j^* \otimes \tau_i}(1-s) \mathbb{C}[q^{-s}, q^s].$$

Proof of Lemma 8.1. We use the notation and results of § 4.4. Namely, replace τ with ε and prove the lemma for ε, keeping ζ fixed, $\Re(\zeta) \gg 0$ throughout the proof. Eventually we shall put $\zeta = 0$ in order to derive the result for τ, and thus any data we use will need to be analytic in ζ. This follows the pattern of [JPS83]. □

Remark 8.1. Alternatively, we could take ζ as a parameter and use Laurent series in two variables. There are minimal technical differences between these approaches.

The polynomials P_i are replaced with P_i^ζ as prescribed by the next claim.

Claim 8.1. *There exist $P_i^\zeta \in \mathbb{C}[X]$, $i = 1, 2, 3$, with coefficients that are polynomial in $q^{\mp\zeta}$, such that the operators $P_1^\zeta(q^{-s}) \gcd(\pi \times \varepsilon_2^*, 1-s)^{-1} M^*(\varepsilon_2, s)$, $P_2^\zeta(q^{-s}) M^*(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))$ and $P_3^\zeta(q^{-s}) \gcd(\pi \times \varepsilon_1^*, 1-s)^{-1} M^*(\varepsilon_1, s)$ are holomorphic, and $P_i^0 = P_i$ for $1 \leqslant i \leqslant 3$.*

Proof of Claim 8.1. Start with P_1^ζ. By definition, the zeros of $\gcd(\pi \times \varepsilon_2^*, 1-s)^{-1} = \gcd(\pi \times \tau_2^*, 1-(s-\zeta))^{-1}$ can be written in the form $(1 - aq^{-s+\zeta})$, $a \in \mathbb{C}^*$. The poles of $M^*(\varepsilon_2, s) = M^*(\tau_2, s-\zeta)$ are of the form $(1 - aq^{-s+\zeta})^{-1}$. The multiplicity of a factor $(1 - aq^{-s+\zeta})^{\pm 1}$ in either $\gcd(\pi \times \varepsilon_2^*, 1-s)^{-1}$ or $\ell_{\varepsilon_2}(s)$ does not change when we substitute 0 for ζ. Then P_1^ζ is chosen by taking suitable factors $(1 - aq^{-s+\zeta})$ and replacing q^{-s} with X. The argument for P_3^ζ is similar.

The poles of $M^*(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))$ appear either in $L(\varepsilon_1^* \times \varepsilon_2^*, 2-2s)$ (due to the normalization of the intertwining operator) or as poles of $M(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))$. An argument as in the proof of Corollary 5.3 shows that the latter poles appear in $L(\phi_i \times \theta_j^*, 2s-1)$, where ϕ_i, θ_j are irreducible supercuspidal representations independent of ζ (e.g. τ_2^* is a sub-representation of a representation induced from $\theta_1 \otimes \cdots \otimes \theta_m$). We let P_2^ζ be a product of factors $(1 - aX^2)$. □

Let $W \in \mathcal{W}(\pi, \psi_\gamma^{-1})$ and $f_s \in \xi_{Q_n}^{H_n}(\tau, \mathrm{hol}, s)$. As explained in § 4.4 (the last paragraph) we take $\varphi_s \in \xi_{Q_{n_1,n_2}}^{H_n}(\varepsilon_1 \otimes \varepsilon_2, \mathrm{hol}, (s, s))$ such that f'_s, defined by (4.11), satisfies $f'_s = f_s$ for $\zeta = 0$. The function $h \mapsto \varphi_s(h, 1, 1, 1)$ belongs to $C^\infty(H_n, \mathbb{C}[q^{-s}, q^s])$ (see Example 7.2). We replace $\Psi(W, f_s, s)$ with $\Psi(W, \varphi_s, s)$ (see e.g. (4.12)). There is a domain D in ζ and s, depending only on π, τ_1 and τ_2, such that $\Psi(W, \varphi_s, s)$ is absolutely convergent. This domain is of the form $\{\zeta, s \in \mathbb{C} : \Re(s) \gg \Re(\zeta) \gg 0\}$ (the parameters for all of the domains we consider are similar to those calculated in [Sou00]). Recall that we fix ζ; hence we refer to this domain as a domain in s. As in the proof of Lemma 7.6 one shows that $\Psi(W, \varphi_s, s)$ is represented by a series $\Sigma(W, \varphi_s, s) \in \Sigma(X)$ which is strongly convergent in D. For instance, when $l \leqslant n$, we take $\Gamma = \overline{B_{G_l}} \times R_{l,n} \times Z_{n_2,n_1}$ and define $\phi \in C^\infty(\Gamma, \mathbb{C}[q^{-s}, q^s])$ using the integrand (start with $\varphi_s \in \xi(\varepsilon_1 \otimes \varepsilon_2, \mathrm{std}, (s, s))$ then use Lemmas 7.4 and 7.3). During our series computations, it is permissable to suppress certain factors in $R(X)^*$ (see e.g. the proof of Claim 8.1). We indicate such factors whenever they occur.

First we apply the functional equation for $\pi \times \varepsilon_2$. For $h \in H_n$, $b_1 \in \mathrm{GL}_{n_1}$, denote by $\varphi_s(h, b_1, \cdot, \cdot)$ the function $(h_2, b_2) \mapsto \varphi_s(h, b_1, h_2, b_2)$ ($h_2 \in H_{n_2}$, $b_2 \in \mathrm{GL}_{n_2}$). Then $\varphi_s(h, b_1, \cdot, \cdot) \in \xi_{Q_{n_2}}^{H_{n_2}}(\varepsilon_2, \mathrm{hol}, s)$. The function $M^*(\varepsilon_2, s)\varphi_s \in \xi_{Q_{n_1,n_2}}^{H_n}(\varepsilon_1 \otimes \varepsilon_2^*, \mathrm{rat}, (s, 1-s))$ is obtained by

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applying the intertwining operator to $\varphi_s(h, b_1, \cdot, \cdot)$. Let

$$\varphi_{s,1-s} = P_1^{\zeta}(q^{-s}) \gcd(\pi \times \varepsilon_2^*, 1-s)^{-1} M^*(\varepsilon_2, s)\varphi_s \in \xi_{Q_{n_1,n_2}}^{H_n}(\varepsilon_1 \otimes \varepsilon_2^*, \mathrm{hol}, (s, 1-s)).$$

We use the notation $\Psi(W, \theta_{s,1-s}, (s, 1-s))$ to denote the Rankin–Selberg integral for $G_l \times \mathrm{GL}_n$ where $\theta_{s,1-s} \in V_{Q_{n_1,n_2}}^{H_n}(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))$ and integral (4.11) is applied to $\theta_{s,1-s}$. It is a triple integral. If $l \leqslant n$, this is (4.12) with $\theta_{s,1-s}$ replacing φ_s. The integral is absolutely convergent in a domain D^* of the form $\{\zeta, s \in \mathbb{C} : \Re(\zeta) \gg \Re(s) \gg 0\}$ (ζ is fixed, and D^* is a domain in s depending only on π, τ_1, τ_2). Since $\varphi_{s,1-s}$ is a holomorphic section, the proof of Lemma 7.6 shows that we have a series $\Sigma(W, \varphi_{s,1-s}, (s, 1-s))$ representing $\Psi(W, \varphi_{s,1-s}, (s, 1-s))$.

Let $G(\pi \times \varepsilon_i, X)^{-1} \in R(X)$ be the polynomial obtained from $\gcd(\pi \times \varepsilon_i, s)^{-1}$ by replacing q^{-s} with X, for $i = 1, 2$. We have the next claim.

CLAIM 8.2. *The following identity holds:*

$$P_1^{\zeta}(X) G(\pi \times \varepsilon_2, X)^{-1} \Sigma(W, \varphi_s, s) = \Sigma(W, \varphi_{s,1-s}, (s, 1-s)).$$

Proof of Claim 8.2. We prove this claim for the case of $l \leqslant n_2$. In the other cases the integral manipulations are different but the method of proof is similar. In particular when $n_2 < l \leqslant n$ the manipulations are a bit more involved, since we have an inner integral over $H_{n_2} \backslash G_{n_2+1}$. As we showed in [Kap10b], in D it is true that (see I_2 in Lemma 5.3 (i))

$$\Psi(W, \varphi_s, s) = \int_{U_{n_1}} \int_{\overline{B_{G_l}}} \int_{R_{l,n_2}} W(g)\varphi_s(w'u, 1, w_{l,n_2} r({}^{b_{n_2,n_1}} g), 1)\psi_\gamma(r)\psi_\gamma(u)\, dr\, dg\, du. \tag{8.8}$$

Here we replaced the dg-integration over $U_{G_l} \backslash G_l$ with an integration over $\overline{B_{G_l}}$, $w' \in H_n$ is a Weyl element, and $b_{n_2,n_1} = \mathrm{diag}(I_{n_2}, (-1)^{n_1}, I_{n_2}) \in \mathrm{GL}_{2n_2+1}$. If we substitute $|W|, |\varphi_s|$ for W, φ_s and drop the characters, the right-hand side of (8.8) is convergent in D. We transform this equality into an equality of series. Let $\Gamma = U_{n_1}$, $\Gamma' = \overline{B_{G_l}} \times R_{l,n_2}$. The function

$$\phi(u, (g, r)) = W(g)\varphi_s(w'u, 1, w_{l,n_2} r({}^{b_{n_2,n_1}} g), 1)\psi_\gamma(r)\psi_\gamma(u)$$

belongs to $C^\infty(\Gamma \times \Gamma', \mathbb{C}[q^{-s}, q^s])$. Since φ_s and ψ_γ are smooth, ϕ is smooth in Γ. As in the proof of Lemma 7.6 we see that $\Sigma_\phi = \oint_{\Gamma \times \Gamma'} \phi(u, (g, r)) d(u, (g, r))$ is defined and strongly convergent, and hence Lemma 7.2 implies that $\Sigma_\phi \sim_D \Phi_\phi$, where $\Phi_\phi = \int_{\Gamma \times \Gamma'} \phi(u, (g, r)) d(u, (g, r))$. Since Φ_ϕ equals the right-hand side of (8.8), $\Sigma(W, \varphi_s, s) = \Sigma_\phi$.

Apply Lemma 7.5 to Σ_ϕ. Then, for all $u \in \Gamma$, $\Sigma_{\phi(u,\cdot)}$ is defined and $\Sigma_{\phi(u,\cdot)} \sim_D \Phi_{\phi(u,\cdot)}$. Note that $\Phi_{\phi(u,\cdot)} = \psi_\gamma(u)\Psi(W, \varphi_s(w'u, 1, \cdot, \cdot), s)$ (the conjugation by b_{n_2,n_1} was omitted because it does not impact the argument, $\varphi_s(w'u, 1, \cdot, \cdot)$ should be replaced with the function $(h_2, b_2) \mapsto \varphi_s(w'u, 1, {}^{b_{n_2,n_1}} h_2, b_2))$. Set $Q = P_1^{\zeta}(q^{-s}) \gcd(\pi \times \varepsilon_2, s)^{-1} \in \mathbb{C}[q^{-s}]$. By Lemma 7.4, $\Sigma_{Q\phi(u,\cdot)} \sim_D Q(q^{-s})\psi_\gamma(u)\Psi(W, \varphi_s(w'u, 1, \cdot, \cdot), s)$. Since $\varphi_s(w'u, 1, \cdot, \cdot) \in \xi_{Q_{n_2}}^{H_{n_2}}(\varepsilon_2, \mathrm{hol}, s)$,

$$\Sigma_{Q\phi(u,\cdot)} = Q(X)\psi_\gamma(u)\Sigma(W, \varphi_s(w'u, 1, \cdot, \cdot), s) \in R(X).$$

Now apply Lemma 7.5 again, to $Q(X)\Sigma_\phi = \Sigma_{Q\phi}$ and obtain $(Q\phi)_{\Gamma'}(u) = \Sigma_{Q\phi(u,\cdot)}(q^{-s}, q^s)$, $\Sigma_{Q\phi} = \Sigma_{(Q\phi)_{\Gamma'}}$.

According to the functional equation for $G_l \times \mathrm{GL}_{n_2}$ and $\pi \times \varepsilon_2$, in $\mathbb{C}[q^{-s}, q^s]$ we have

$$\begin{aligned} &\gcd(\pi \times \varepsilon_2, s)^{-1}\Psi(W, \varphi_s(w'u, 1, \cdot, \cdot), s) \\ &\quad = c(l, \varepsilon_2, \gamma, s)\epsilon(\pi \times \varepsilon_2, \psi, s)^{-1} \gcd(\pi \times \varepsilon_2^*, 1-s)^{-1} \\ &\qquad \times \Psi(W, M^*(\varepsilon_2, s)\varphi_s(w'u, 1, \cdot, \cdot), 1-s). \end{aligned} \tag{8.9}$$

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The factors $c(l, \varepsilon_2, \gamma, s) = c(l, \tau_2, \gamma, s-\zeta)$ and $\epsilon(\pi \times \varepsilon_2, \psi, s) = \epsilon(\pi \times \tau_2, \psi, s-\zeta)$ belong to $\mathbb{C}[q^{\mp(s-\zeta)}]^*$ and may be ignored. Note that the integrals $\Psi(\cdot,\cdot,s)$ for $\pi \times \varepsilon_2$ are defined in some right half-plane $\{s : \Re(s-\zeta) \gg 0\}$ and D can be taken so that it intersects this right half-plane in a domain. Therefore the functional equation is applicable to the integrals $\Psi(W, \varphi_s(w'u, 1, \cdot, \cdot), s)$.

Let $\widetilde{Q} = P_1^{\zeta}(q^{-s}) \gcd(\pi \times \varepsilon_2^*, 1-s)^{-1} \in \mathbb{C}[q^{-s}, q^s]$. Since $\widetilde{Q}(q^{-s}, q^s)M^*(\varepsilon_2, s)$ is a holomorphic operator, as explained in § 8.2 we can multiply both sides of (8.9) by P_1^{ζ} and obtain an equality in $\Sigma(X)$,

$$Q(X)\Sigma(W, \varphi_s(w'u, 1, \cdot, \cdot), s) = \Sigma(W, \widetilde{Q}(q^{-s}, q^s)M^*(\varepsilon_2, s)\varphi_s(w'u, 1, \cdot, \cdot), 1-s). \tag{8.10}$$

The left-hand side of (8.10) equals $\psi_\gamma^{-1}(u)\Sigma_{Q\phi(u,\cdot)}$ and the right-hand side represents the integral $\Psi(W, \varphi_{s,1-s}(w'u, 1, \cdot, \cdot), 1-s)$, which is a polynomial.

Next, in D^* equality (8.8) is applicable with $\varphi_{s,1-s}$ replacing φ_s. Define ϕ^* similarly to ϕ but with $\varphi_{s,1-s}$ instead of φ_s. Then, as above, Lemmas 7.6 and 7.2 show that $\Sigma_{\phi^*} \sim_{D^*} \Phi_{\phi^*}$ and $\Sigma(W, \varphi_{s,1-s}, (s, 1-s)) = \Sigma_{\phi^*}$. The domain D^* is taken so that it intersects the left half-plane $\{s : \Re(1-s+\zeta) \gg 0\}$ where the integrals $\Psi(\cdot,\cdot,1-s)$ for $\pi \times \varepsilon_2^*$ on the right-hand side of (8.9) are defined (this means taking $\Re(\zeta)$ large enough). As above,

$$\Sigma_{\phi^*(u,\cdot)} \sim_{D^*} \Phi_{\phi^*(u,\cdot)} = \psi_\gamma(u)\Psi(W, \varphi_{s,1-s}(w'u, 1, \cdot, \cdot), 1-s),$$
$$\Sigma_{\phi^*(u,\cdot)} = \psi_\gamma(u)\Sigma(W, \widetilde{Q}(q^{-s}, q^s)M^*(\varepsilon_2, s)\varphi_s(w'u, 1, \cdot, \cdot), 1-s) \in R(X),$$

and Lemma 7.5 shows that $\Sigma_{\phi^*} = \Sigma_{(\phi^*)_{\Gamma'}}$. Now (8.10) implies that $\Sigma_{Q\phi(u,\cdot)} = \Sigma_{\phi^*(u,\cdot)}$, for all u. Hence $(Q\phi)_{\Gamma'} = (\phi^*)_{\Gamma'}$. Putting the pieces together, we have

$$Q(X)\Sigma(W, \varphi_s, s) = \Sigma_{Q\phi} = \Sigma_{(Q\phi)_{\Gamma'}} = \Sigma_{(\phi^*)_{\Gamma'}} = \Sigma_{\phi^*} = \Sigma(W, \varphi_{s,1-s}, (s, 1-s)). \qquad \square$$

We continue with the functional equation for $\varepsilon_1 \times \varepsilon_2$. For $h \in H_n$ denote by

$$h \cdot \varphi_{s,1-s}(\cdot, \cdot, I_{2n_2+1}, \cdot)$$

the function $(b_0, b_1, b_2) \mapsto \varphi_{s,1-s}(b_0 h, b_1, I_{2n_2+1}, b_2)$ $(b_0 \in \mathrm{GL}_n,\ b_i \in \mathrm{GL}_{n_i})$. Then

$$h \cdot \varphi_{s,1-s}(\cdot, \cdot, I_{2n_2+1}, \cdot) \in \xi_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\varepsilon_1|\det|^{n/2} \otimes \varepsilon_2^*|\det|^{n/2}, \mathrm{hol}, (s, 1-s)).$$

The function $M^*(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))\varphi_{s,1-s} \in \xi_{Q_{n_2,n_1}}^{H_n}(\varepsilon_2^* \otimes \varepsilon_1, \mathrm{rat}, (1-s, s))$ is defined by applying the intertwining operator to $h \cdot \varphi_{s,1-s}(\cdot, \cdot, I_{2n_2+1}, \cdot)$. Let

$$\varphi'_{1-s,s} = P_2^{\zeta}(q^{-s})M^*(\varepsilon_1 \otimes \varepsilon_2^*, (s, 1-s))\varphi_{s,1-s} \in \xi_{Q_{n_2,n_1}}^{H_n}(\varepsilon_2^* \otimes \varepsilon_1, \mathrm{hol}, (1-s, s)).$$

We show the following claim.

Claim 8.3. *We have $P_2^{\zeta}(X)\Sigma(W, \varphi_{s,1-s}, (s, 1-s)) = \Sigma(W, \varphi'_{1-s,s}, (1-s, s))$.*

Proof of Claim 8.3. The proof is similar to the proof of Claim 8.2 and described briefly. Assume for instance that $l \leqslant n$ (the proof when $l > n$ is almost identical). Replace the dg-integration in $\Psi(W, \varphi_{s,1-s}, (s, 1-s))$ with an integration over $\overline{B_{G_l}}$. Let $\Gamma = \overline{B_{G_l}} \times R_{l,n}$, $\Gamma' = Z_{n_2,n_1}$. Let D^* be as in Claim 8.2. Define $\phi \in C^\infty(\Gamma \times \Gamma', \mathbb{C}[q^{-s}, q^s])$ by

$$\phi((g, r), z) = W(g)\varphi_{s,1-s}(\omega_{n_1,n_2} z w_{l,n} r g, 1, 1, 1)\psi^{-1}(z)\psi_\gamma(r).$$

Then $\int_{\Gamma'} \phi((g, r), z)\, dz = W(g)\psi_\gamma(r)\Upsilon((w_{l,n} r g) \cdot \varphi_{s,1-s}(\cdot, \cdot, I_{2n_2+1}, \cdot))$, where Υ denotes a Whittaker functional (given by the Jacquet integral) on the space $V_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\varepsilon_1|\det|^{n/2} \otimes \varepsilon_2^*|\det|^{n/2}, (s, 1-s))$. According to the definition of the local coefficient [Sha81, § 3], if Υ^* denotes a similar Whittaker functional on $V_{P_{n_2,n_1}}^{\mathrm{GL}_n}(\varepsilon_2^*|\det|^{n/2}\otimes\varepsilon_1|\det|^{n/2},(1-s,s))$, then

$$\Upsilon((w_{l,n}rg)\cdot\varphi_{s,1-s}(\cdot,\cdot,I_{2n_2+1},\cdot))=\Upsilon^*(M^*(\varepsilon_1\otimes\varepsilon_2^*,(s,1-s))(w_{l,n}rg)\cdot\varphi_{s,1-s}(\cdot,\cdot,I_{2n_1+1},\cdot)). \tag{8.11}$$

This is, *a priori*, an equality (between meromorphic continuations) in $\mathbb{C}(q^{-s})$, but because

$$\varphi_{s,1-s}(\cdot,\cdot,I_{2n_2+1},\cdot)\in\xi_{P_{n_1,n_2}}^{\mathrm{GL}_n}(\varepsilon_1|\det|^{n/2}\otimes\varepsilon_2^*|\det|^{n/2},\mathrm{hol},(s,1-s))$$

it is actually in $\mathbb{C}[q^{-s},q^s]$. We define $\Gamma''=Z_{n_1,n_2}$ and $\phi^*\in C^\infty(\Gamma\times\Gamma'',\mathbb{C}[q^{-s},q^s])$ by

$$\phi^*((g,r),z)=W(g)\varphi'_{1-s,s}(\omega_{n_2,n_1}zw_{l,n}rg,1,1,1)\psi^{-1}(z)\psi_\gamma(r).$$

The integral $\Psi(W,\varphi'_{1-s,s},(1-s,s))$ is absolutely convergent in a domain D^{**}. Proceeding as in Claim 8.2, we use Lemmas 7.4–7.6 to conclude that

$$P_2^\zeta(X)\Sigma(W,\varphi_{s,1-s},(s,1-s))=\Sigma_{P_2^\zeta\phi}=\Sigma_{(P_2^\zeta\phi)_{\Gamma'}}=\Sigma_{(\phi^*)_{\Gamma''}}=\Sigma_{\phi^*}=\Sigma(W,\varphi'_{1-s,s},(1-s,s)).$$

The equality $\Sigma_{(P_2^\zeta\phi)_{\Gamma'}}=\Sigma_{(\phi^*)_{\Gamma''}}$ holds because, according to (8.11), $(P_2^\zeta\phi)_{\Gamma'}=(\phi^*)_{\Gamma''}$. □

Next we apply the functional equation for $\pi\times\varepsilon_1$. Write

$$\varphi^*_{1-s}=P_3^\zeta(q^{-s})\gcd(\pi\times\varepsilon_1^*,1-s)^{-1}M^*(\varepsilon_1,s)\varphi'_{1-s,s}\in\xi_{Q_{n_2,n_1}}^{H_n}(\varepsilon_2^*\otimes\varepsilon_1^*,\mathrm{hol},(1-s,1-s)).$$

CLAIM 8.4. *The following holds:*

$$P_3^\zeta(X)G(\pi\times\varepsilon_1,X)^{-1}\Sigma(W,\varphi'_{1-s,s},(1-s,s))=\Sigma(W,\varphi^*_{1-s},1-s).$$

This is proved analogously to Claim 8.2. Collecting Claims 8.2–8.4, we have

$$P_1^\zeta(X)P_2^\zeta(X)P_3^\zeta(X)G(\pi\times\varepsilon_1,X)^{-1}G(\pi\times\varepsilon_2,X)^{-1}\Sigma(W,\varphi_s,s)=\Sigma(W,\varphi^*_{1-s},1-s). \tag{8.12}$$

The final step is to put $\zeta=0$ in this equality. Looking at (3.3) we see that

$$\varphi^*_{1-s}=P_1^\zeta(q^{-s})P_2^\zeta(q^{-s})P_3^\zeta(q^{-s})\gcd(\pi\times\varepsilon_1^*,1-s)^{-1}\gcd(\pi\times\varepsilon_2^*,1-s)^{-1}M^*(\varepsilon,s)\varphi_s.$$

Let

$$f^*_{1-s}=P_{\pi\times\tau}(q^{-s})\gcd(\pi\times\tau_1^*,1-s)^{-1}\gcd(\pi\times\tau_2^*,1-s)^{-1}M^*(\tau,s)f_s.$$

The definition of $P_{\pi\times\tau}$ and (3.4) imply that $f^*_{1-s}\in\xi_{Q_n}^{H_n}(\tau^*,\mathrm{hol},1-s)$. Let $\gcd(\pi\times\tau_i,X)^{-1}\in R(X)$ be obtained from $\gcd(\pi\times\tau_i,s)^{-1}$ by replacing q^{-s} with X.

CLAIM 8.5. *Putting $\zeta=0$ in (8.12) gives*

$$P_{\pi\times\tau}(X)\gcd(\pi\times\tau_1,X)^{-1}\gcd(\pi\times\tau_2,X)^{-1}\Sigma(W,f_s,s)=\Sigma(W,f^*_{1-s},1-s). \tag{8.13}$$

Proof of Claim 8.5. The domain $D\subset\mathbb{C}\times\mathbb{C}$ in ζ and s of absolute convergence of $\Psi(W,\varphi_s,s)$ contains a domain $D_0=\{(\zeta,s):\zeta_0<\Re(\zeta)<\zeta_1,\Re(s)>s_0\}$, where ζ_i, s_0 are constants depending only on π,τ_1 and τ_2. Put $D_0'=\{\zeta:\zeta_0<\Re(\zeta)<\zeta_1\}$. The first step is to show that $\Sigma(W,\varphi_s,s)$ is of the form $\sum_{m=N}^\infty a_m(\zeta)X^m$, where N is independent of ζ and for, each m, $a_m:D_0'\to\mathbb{C}$ is an analytic function: a polynomial in $q^{\mp\zeta}$.

According to Claim 4.3, in D_0 we have $\Psi(W,\varphi_s,s)=\Psi(W,f'_s,s)$ where $f'_s\in\xi(\mathcal{W}(\varepsilon,\psi),\mathrm{hol},s)$ is defined by (4.11) using φ_s. Fix $\zeta\in D_0'$ and use Proposition 4.4 to write $\Psi(W,f'_s,s)$ as a sum of integrals, say if $l\leqslant n$, of the form (4.7). Here $W'\in\mathcal{W}(\tau,\psi)$ is replaced with $W'_\zeta\in\mathcal{W}(\varepsilon,\psi)$. The number of negative coefficients is bounded as in Lemma 7.6, using the fact that W vanishes away from zero. Now if we let ζ vary, this bound remains fixed because W is independent of ζ (it is also possible to use W'_ζ to obtain a bound independent of ζ). Then observe that there is a

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compact open subgroup $N < \mathrm{GL}_n$, independent of ζ, such that W'_ζ is right-invariant by N and, for a fixed $b \in \mathrm{GL}_n$, $W'_\zeta(b) \in \mathbb{C}[q^{-\zeta}, q^{\zeta}]$ (see the last paragraph of § 4.4). Since the coordinates of $a \in A_{l-1}$ are bounded from above, if G_l is split (respectively quasi-split), then for any $m \in \mathbb{Z}$, the set of a, x (respectively a) such that $|\det a| \cdot [x]^{-1} = q^m$ (respectively $|\det a| = q^m$) is compact in $A_{l-1} \times G_1$ (respectively A_{l-1}), whence the coefficient of X^m is a polynomial in $q^{\mp\zeta}$. The same applies when $l > n$, by considering (4.8).

Let $C \subset \mathbb{C}$ be a compact set containing 0 and $s_C > 0$ be a constant, depending only on π, τ_1, τ_2 and C, such that, in $D_1 = \{(\zeta, s) : \zeta \in C, \Re(s) > s_C\}$, $\Psi(W, f'_s, s)$ is absolutely convergent and $D_1 \cap D_0$ is a domain of $\mathbb{C} \times \mathbb{C}$. By Lemma 7.6, for any $\zeta \in C$, in the domain $\{s : \Re(s) > s_C\}$, the integral $\Psi(W, f'_s, s)$ has a representation $\Sigma(W, f'_s, s) = \sum_{m=N}^{\infty} b_m(\zeta) X^m$ with properties as above. Note that Lemma 7.6 is applicable because D_1 does not depend on f'_s or W. Therefore $b_m(\zeta) = a_m(\zeta)$ for all $\zeta \in \mathbb{C}, m \geqslant N$, and, for $\zeta = 0$, $\Sigma(W, f'_s, s)$ is the series representing $\Psi(W, f_s, s)$ (since for $\zeta = 0$, $f'_s = f_s$). It follows that putting $\zeta = 0$ in $\Sigma(W, \varphi_s, s)$ yields $\Sigma(W, f_s, s)$. The argument for $\Sigma(W, \varphi^*_{1-s}, 1-s)$ is similar: use Claim 8.1 and note that, for $\zeta = 0$, $M^*(\varepsilon, s) = M^*(\tau, s)$. □

According to Lemma 7.6, $\Sigma(W, f_s, s)$ (respectively $\Sigma(W, f^*_{1-s}, 1-s)$) is a Laurent series with finitely many negative (respectively positive) coefficients. Hence both sides of (8.13) are polynomials. Since $P_{\pi\times\tau}(q^{-s}) \gcd(\pi \times \tau_1, s)^{-1} \gcd(\pi \times \tau_2, s)^{-1} \Psi(W, f_s, s)$ is represented by the left-hand side of (8.13),

$$\Psi(W, f_s, s) \in \gcd(\pi \times \tau_1, s) \gcd(\pi \times \tau_2, s) P_{\pi\times\tau}(q^{-s})^{-1} \mathbb{C}[q^{-s}, q^s].$$

Additionally $\Psi(W, f^*_{1-s}, 1-s)$ is represented by the polynomial $\Sigma(W, f^*_{1-s}, 1-s)$, and, according to the definition of f^*_{1-s},

$$\Psi(W, M^*(\tau, s) f_s, 1-s) \in \gcd(\pi \times \tau_1^*, 1-s) \gcd(\pi \times \tau_2^*, 1-s) P_{\pi\times\tau}(q^{-s})^{-1} \mathbb{C}[q^{-s}, q^s]. \quad \square$$

8.4 Technical issues concerning good sections

We encountered two basic problems with the method of good sections, which further applications of this method may need to address.

First, the approach of Laurent series is not suited for rational sections. It is not clear how to associate a Laurent series to an element of $C^\infty(H_n, \mathbb{C}(q^{-s}))$. Indeed, in (8.1) the right-hand side was known to be in $\mathbb{C}[q^{-s}, q^s]$, but in order to pass to $\Sigma(W, M^*(\tau, s) f_s, 1-s)$ we had to ensure that $M^*(\tau, s) f_s$ did not have any poles. To this end we used the polynomials P_i of Lemma 8.1, leading to the additional poles of $M_{\tau_1 \otimes \cdots \otimes \tau_k}(s)$ in Theorem 1.2, some of which were eliminated in Corollary 8.1 by a more careful calculation.

The second problem concerns the lower bound. We seek a lower bound as in § 5.2, i.e. $\gcd(\pi \times \tau, s) \in \gcd(\pi \times \varepsilon, s) \mathbb{C}[q^{-s}, q^s]$, for a general irreducible representation ε induced from $\tau_1 \otimes \tau$. We need to show that we can embed the poles of $\Psi(W, M^*(\tau^*, 1-s) f_{1-s}, s)$, with $f_{1-s} \in \xi(\tau^*, \mathrm{hol}, 1-s)$, in $\gcd(\pi \times \varepsilon, s)$. However, the integral $\Psi(W, f'_s, s)$ for $f'_s \in \xi(\varepsilon, \mathrm{good}, s)$ might not contain the poles of $M^*(\tau^*, 1-s)$, due to zeros of $M^*(\varepsilon^*, 1-s)$. This difficulty was settled by assuming certain properties of τ_1, as in Corollary 5.3.

9. Upper bound in the first variable

9.1 Proof of Theorem 1.3: case $k < l$

As in § 8 we reinterpret the proof of [Kap10b], of the multiplicativity of $\gamma(\pi \times \tau, \psi, s)$ in the first variable, in terms of Laurent series.

LEMMA 9.1. *Assume that $k < l < n$. Let $P \in \mathbb{C}[X]$ with $P(0) = 1$ be of minimal degree such that the operators $P(q^{-s}) \gcd(\pi' \times \tau^*, 1-s)^{-1} M^*(\tau, s)$ and $P(q^{-s}) \gcd(\pi' \times \tau, s)^{-1} M^*(\tau^*, 1-s)$ are holomorphic (note that $P(q^{-s})^{-1} \in M_\tau(s)\mathbb{C}[q^{-s}, q^s]$). Then*

$$\gcd(\pi \times \tau, s) \in L(\sigma \times \tau, s) \gcd(\pi' \times \tau, s) L(\sigma^* \times \tau, s) P(q^{-s})^{-1} \mathbb{C}[q^{-s}, q^s].$$

In particular (1.1) holds for $a = 1$ and $k < l < n$.

The proof is similar to the proof of Lemma 8.1: it involves applying the functional equations of $\sigma \times \tau$, $\pi' \times \tau$ and $\sigma \times \tau^*$. We skip the proof, but show how the lemma implies the theorem for $k < l$ and general n and a. Put $G_{\pi' \times (\tau_1 \otimes \cdots \otimes \tau_a)}(s) = \prod_{i=1}^a \gcd(\pi' \times \tau_i, s)$. Suppose first that $l < n$. By Theorem 1.2 applied to $\pi' \times \tau$ and Claim 3.3, for some $P_0 \in \mathbb{C}[q^{-s}, q^s]$,

$$\begin{gathered}\gcd(\pi' \times \tau, s) = G_{\pi' \times (\tau_1 \otimes \cdots \otimes \tau_a)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) P_0, \\ \gcd(\pi' \times \tau^*, 1-s) \simeq G_{\pi' \times (\tau_1^* \otimes \cdots \otimes \tau_a^*)}(1-s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) P_0.\end{gathered} \tag{9.1}$$

Then the operators $P_0 \gcd(\pi' \times \tau^*, 1-s)^{-1} M^*(\tau, s)$ and $P_0 \gcd(\pi' \times \tau, s)^{-1} M^*(\tau^*, 1-s)$ are holomorphic. Applying Lemma 9.1 to $\pi \times \tau$, the polynomial $P(q^{-s})$ of the lemma divides P_0. This and (9.1) show that equality (1.1) holds.

Now assume $l \geqslant n$. Let η be a unitary irreducible supercuspidal representation of GL_m for $m > l$ chosen by Corollary 5.3 (see also Example 4.1). Then if $\varepsilon = \mathrm{Ind}_{P_{m,n}}^{\mathrm{GL}_{m+n}}(\eta \otimes \tau)$ (which is irreducible), $\gcd(\pi \times \tau, s) \in \gcd(\pi \times \varepsilon, s)\mathbb{C}[q^{-s}, q^s]$.

Following Proposition 4.5 we may select η which also satisfies $\gcd(\pi' \times \eta, s) = \gcd(\pi' \times \eta^*, 1-s) = 1$. Corollary 5.3 also guarantees that $M_\eta(s) = \ell_{\eta \otimes \tau^*}(s) = \ell_{\tau^* \otimes \eta}(1-s) = 1$ whence $M_\varepsilon(s) \in M_\tau(s)\mathbb{C}[q^{-s}, q^s]$. The proof of the corollary shows that $M_{\eta \otimes \tau_1 \otimes \cdots \otimes \tau_a}(s) = M_{\tau_1 \otimes \cdots \otimes \tau_a}(s)$. Hence according to Theorem 1.2 applied to $\pi' \times \varepsilon$ and by our choice of η, for some $Q_0 \in \mathbb{C}[q^{-s}, q^s]$,

$$\gcd(\pi' \times \varepsilon, s) = G_{\pi' \times (\tau_1 \otimes \cdots \otimes \tau_a)}(s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) Q_0. \tag{9.2}$$

Similarly to the proof of Claim 3.3, we find

$$\gcd(\pi' \times \varepsilon^*, 1-s) \simeq G_{\pi' \times (\tau_1^* \otimes \cdots \otimes \tau_a^*)}(1-s) M_{\tau_1 \otimes \cdots \otimes \tau_a}(s) Q_0.$$

Thus the operators $Q_0 \gcd(\pi' \times \varepsilon^*, 1-s)^{-1} M^*(\varepsilon, s)$ and $Q_0 \gcd(\pi' \times \varepsilon, s)^{-1} M^*(\varepsilon^*, 1-s)$ are holomorphic. Since $k < l < m+n$, Lemma 9.1 is applicable to $\pi \times \varepsilon$ and $P(q^{-s})$ of the lemma divides Q_0. Therefore

$$\gcd(\pi \times \varepsilon, s) \in L(\sigma \times \varepsilon, s) \gcd(\pi' \times \varepsilon, s) L(\sigma^* \times \varepsilon, s) Q_0^{-1} \mathbb{C}[q^{-s}, q^s].$$

Since also $k < m$ and η is irreducible supercuspidal, $L(\sigma \times \eta, s) = L(\sigma^* \times \eta, s) = 1$. Hence, according to [JPS83, Theorem 3.1],

$$\gcd(\pi \times \varepsilon, s) \in L(\sigma \times \tau, s) \gcd(\pi' \times \varepsilon, s) L(\sigma^* \times \tau, s) Q_0^{-1} \mathbb{C}[q^{-s}, q^s].$$

Now using (9.2) the result follows. We summarize a corollary of the proof.

COROLLARY 9.1. *Let $\pi = \mathrm{Ind}_{P_k}^{G_l}(\sigma \otimes \pi')$ $(k < l)$ and suppose that for some $L^{-1}, \widetilde{L}^{-1}, P_0 \in \mathbb{C}[q^{-s}, q^s]$ it is true that $\gcd(\pi' \times \tau, s) = L M_\tau(s) P_0$ and $\gcd(\pi' \times \tau^*, 1-s) = \widetilde{L} M_\tau(s) P_0$. Then*

$$\gcd(\pi \times \tau, s) \in L \cdot L(\sigma \times \tau, s) L(\sigma^* \times \tau, s) M_\tau(s) \mathbb{C}[q^{-s}, q^s].$$

9.2 Proof of Theorem 1.3: case $k = l > n$

A stronger result, namely (1.1) with $M_\tau(s)$ instead of $M_{\tau_1 \otimes \cdots \otimes \tau_a}(s)$, is a direct consequence of the next lemma.

Lemma 9.2. *Assume that $k = l > n$. Let $P \in \mathbb{C}[X]$ with $P(0) = 1$ be of minimal degree such that the operators $P(q^{-s})M^*(\tau, s)$ and $P(q^{-s})M^*(\tau^*, 1 - s)$ are holomorphic. Then*

$$\gcd(\pi \times \tau, s) \in L(\sigma \times \tau, s)L(\sigma^* \times \tau, s)P(q^{-s})^{-1}\mathbb{C}[q^{-s}, q^s].$$

The lemma is proved in the same manner as Lemmas 8.1 and 9.1, utilizing the integral manipulations of [Kap10b]. The functional equation replacing $\pi' \times \tau$ is Shahidi's functional equation defining $\gamma(\tau, S^2, \psi, 2s - 1)$ (the local coefficient, see [Sha81, § 3]).

9.3 Proof of Theorem 1.3: case $k = l \leqslant n$

It seems difficult to prove this case via direct integral manipulations. We prove the following slightly stronger result. Let $\pi = \mathrm{Ind}_{P_l}^{G_l}(\sigma)$ where σ is a (generic) quotient of a representation $\theta = \mathrm{Ind}_{P_{l_1,\ldots,l_m}}^{\mathrm{GL}_l}(\sigma_1 \otimes \cdots \otimes \sigma_m)$ (if $m = 1$, $\theta = \sigma_1$) of Langlands' type, i.e. $\sigma_i = |\det|^{u_i}\sigma_i'$, σ_i' is tempered, $u_i \in \mathbb{R}$ and $u_1 > \cdots > u_m$. Since π is the generic quotient of $\mathrm{Ind}_{P_l}^{G_l}(\theta)$, then $\mathcal{W}(\pi, \psi_\gamma^{-1}) = \mathcal{W}(\mathrm{Ind}_{P_l}^{G_l}(\theta), \psi_\gamma^{-1})$, and hence for the purpose of the gcd we may replace σ with θ. Let

$$\mu = \mathrm{Ind}_{P_{l_1,\ldots,l_{m-1}}}^{\mathrm{GL}_{l-l_m}}(\sigma_1 \otimes \cdots \otimes \sigma_{m-1}).$$

Also let $\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_a}}^{\mathrm{GL}_n}(\tau_1 \otimes \cdots \otimes \tau_a)$ and $\tau' = \mathrm{Ind}_{P_{n_2,\ldots,n_a}}^{\mathrm{GL}_{n-n_1}}(\tau_2 \otimes \cdots \otimes \tau_a)$ be irreducible of Langlands' type. We show that

$$\gcd(\pi \times \tau, s) \in L(\theta \times \tau, s)L(\theta^* \times \tau, s)M_{\tau_1 \otimes \cdots \otimes \tau_a}(s)\mathbb{C}[q^{-s}, q^s]. \tag{9.3}$$

Now if σ is irreducible, then $\sigma \cong \theta$ and (1.1) follows (e.g. $L(\sigma \times \tau, s) = L(\theta \times \tau, s)$).

The following lemma is used to prove (9.3).

Lemma 9.3. *Let σ and τ be essentially tempered. Then*

$$\gcd(\pi \times \tau, s) \in L(\sigma \times \tau, s)L(\sigma^* \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^s].$$

Before proving the lemma, let us use it to derive (9.3). If $m > 1$, set $\pi' = \mathrm{Ind}_{P_{l_m}}^{G_{l_m}}(\sigma_m)$ and then $\pi = \mathrm{Ind}_{P_{l-l_m}}^{G_l}(\mu \otimes \pi')$. Assume that $a = 1$. The case of $m = 1$ is immediate. For $m > 1$ apply Lemma 9.3 to $\gcd(\pi' \times \tau, s)$, use Claim 3.3, Corollary 9.1 and note that, because θ and τ are of Langlands' type, $L(\theta \times \tau, s) = L(\mu \times \tau, s)L(\sigma_m \times \tau, s)$ [JPS83, Theorem 9.4].

Now suppose that $a > 1$. Using induction on a we find that, for some $Q', Q \in \mathbb{C}[q^{-s}, q^s]$,

$$\gcd(\pi \times \tau', s) = L(\theta \times \tau', s)L(\theta^* \times \tau', s)M_{\tau_2 \otimes \cdots \otimes \tau_a}(s)Q',$$
$$\gcd(\pi \times \tau_1, s) = L(\theta \times \tau_1, s)L(\theta^* \times \tau_1, s)M_{\tau_1}(s)Q.$$

By virtue of Claim 3.3 and using the multiplicativity of the γ-factors of $\mathrm{GL}_l \times \mathrm{GL}_n$ and (3.8), similar equalities hold for $\gcd(\pi \times (\tau')^*, 1 - s)$ and $\gcd(\pi \times \tau_1^*, 1 - s)$. Applying Lemma 8.1 to $\pi \times \tau$ with $\tau = \mathrm{Ind}_{P_{n_1,n-n_1}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau')$ and $f_s \in \xi(\tau, \mathrm{hol}, s)$, we see that $P_1(q^{-s})$ (of Lemma 8.1) divides Q' and $P_3(q^{-s})$ divides Q. Then $P_{\pi\times\tau}(q^{-s})$ divides $Q_0 = Q'\ell_{\tau_1\otimes(\tau')^*}(s)^{-1}\ell_{(\tau')^*\otimes\tau_1}(1 - s)^{-1}Q$. When we apply the lemma to $\pi \times \tau^*$ and $f_{1-s} \in \xi(\tau^*, \mathrm{hol}, 1 - s)$ we obtain that $P_{\pi\times\tau^*}(q^{s-1})$ also divides Q_0. Therefore

$$\gcd(\pi \times \tau, s) \in \gcd(\pi \times \tau_1, s)\gcd(\pi \times \tau', s)Q_0^{-1}\mathbb{C}[q^{-s}, q^s].$$

Now (9.3) follows using the multiplicativity of $L(\theta \times \tau, s)$ in τ [JPS83, Theorem 9.4].

Proof of Lemma 9.3. Assume first that τ is tempered. Let $\mathcal{U}$ be the set of all complex numbers u such that $\sigma_u = |\det|^u\sigma$ is tempered. The set $\mathcal{U}$ is just a vertical line in the plane. Let $\pi_u = \mathrm{Ind}_{P_l}^{G_l}(\sigma_u)$. Ignoring a discrete subset of $\mathcal{U}$, we can assume that π_u is tempered (we take tempered representations to be irreducible; see [Mui04, § 3] and [Wal03, Lemma III.2.3]). Since

$$\gamma(\pi_u \times \tau, \psi, s) = \omega_{\sigma_u}(-1)^n\omega_\tau(-1)^l\omega_\tau(2\gamma)^{-1}\gamma(\sigma_u \times \tau, \psi, s)\gamma(\sigma_u^* \times \tau, \psi, s)$$

and the γ-factors on the right-hand side are equal (up to $\mathbb{C}[q^{-s}, q^s]^*$) to Shahidi's γ-factors, we may apply Theorem 1.1 to $\pi_u \times \tau$ and obtain $\gcd(\pi_u \times \tau, s) \in L(\pi_u \times \tau, s)M_\tau(s)\mathbb{C}[q^{-s}, q^s]$, where $L(\pi_u \times \tau, s)$ is the L-function defined by Shahidi. Since π_u and τ are tempered, the results of [CS98, § 4] and (3.8) imply that

$$\gcd(\pi_u \times \tau, s) \in L(\sigma \times \tau, s + u)L(\sigma^* \times \tau, s - u)M_\tau(s)\mathbb{C}[q^{-s}, q^s].$$

If we could put $u = 0$, the result would follow, but this has to be justified. Currently, for each $u \in \mathcal{U}$ there is some $P_u \in \mathbb{C}[q^{-s}, q^s]$ such that

$$\gcd(\pi_u \times \tau, s) = L(\sigma \times \tau, s + u)L(\sigma^* \times \tau, s - u)M_\tau(s)P_u.$$

Write $M_\tau(s) = eM_\tau(s)'$, $P_u = e_uP'_u$ for $(M_\tau(s)')^{-1}, P'_u \in \mathbb{C}[q^{-s}]$ with constant terms equal to 1 and $e, e_u \in \mathbb{C}[q^{-s}, q^s]^*$. Since the other factors in this equation are inverses of polynomials in $\mathbb{C}[q^{-s}]$ with constant terms also equal to 1, we obtain $e \cdot e_u = 1$ and in particular e_u is independent of u. Since $\gcd(\pi_u \times \tau, s)$ is an inverse of a polynomial, any zero of P_u must be canceled by some other factor on the right-hand side. Therefore P'_u is uniquely determined by a finite product of factors appearing in either $L(\sigma \times \tau, s + u)$, $L(\sigma^* \times \tau, s - u)$ or $M_\tau(s)'$. Hence we may assume (perhaps passing to a smaller subset $\mathcal{U}' \subset \mathcal{U}$) that there exists $P \in \mathbb{C}[q^{\mp s}, q^{\mp u}]$ such that, for all $u \in \mathcal{U}$,

$$\gcd(\pi_u \times \tau, s) = L(\sigma \times \tau, s + u)L(\sigma^* \times \tau, s - u)M_\tau(s)P. \tag{9.4}$$

Note that $\mathcal{U}$ (still) contains some infinite sequence which converges to some point in the plane. Repeating this for $\pi_u \times \tau^*$ we may assume in addition that there is some $\widetilde{P} \in \mathbb{C}[q^{\mp s}, q^{\mp u}]$ satisfying, for all $u \in \mathcal{U}$,

$$\gcd(\pi_u \times \tau^*, 1 - s) = L(\sigma \times \tau^*, 1 - s + u)L(\sigma^* \times \tau^*, 1 - s - u)M_\tau(s)\widetilde{P}. \tag{9.5}$$

Fix $u \in \mathcal{U}$. According to the multiplicativity of $\gamma(\pi_u \times \tau, \psi, s)$, (3.7) and (3.8),

$$\begin{aligned}\epsilon(\sigma \times \tau, \psi, s + u)\epsilon(\sigma^* \times \tau, \psi, s - u)&\frac{L(\sigma^* \times \tau^*, 1 - s - u)L(\sigma \times \tau^*, 1 - s + u)}{L(\sigma \times \tau, s + u)L(\sigma^* \times \tau, s - u)}\\ &= \epsilon(\pi_u \times \tau, \psi, s)\frac{\gcd(\pi_u \times \tau^*, 1 - s)}{\gcd(\pi_u \times \tau, s)}.\end{aligned} \tag{9.6}$$

Here the factor $\omega_{\sigma_u}(-1)^n\omega_\tau(-1)^l\omega_\tau(2\gamma)^{-1}$ was omitted: it does not impact the argument because it is independent of u and s ($\omega_{\sigma_u}(-1) = \omega_\sigma(-1)$). Combining (9.4)–(9.6) we see that

$$\epsilon(\pi_u \times \tau, \psi, s) = \epsilon(\sigma \times \tau, \psi, s + u)\epsilon(\sigma^* \times \tau, \psi, s - u)P\widetilde{P}^{-1}. \tag{9.7}$$

Equality (3.6) implies that, for all $W_u \in \mathcal{W}(\pi_u, \psi_\gamma^{-1})$ and $f_s \in \xi(\tau, \mathrm{hol}, s)$,

$$\begin{aligned}c(l, \tau, \gamma, s)^{-1}\epsilon(\pi_u \times \tau, \psi, s)\gcd(\pi_u \times \tau, s)^{-1}\Psi(W_u, f_s, s)\\ = \gcd(\pi_u \times \tau^*, 1 - s)^{-1}\Psi(W_u, M^*(\tau, s)f_s, 1 - s).\end{aligned}$$

Plugging (9.4), (9.5) and (9.7) into this equation yields

$$c(l,\tau,\gamma,s)^{-1}\epsilon(\sigma\times\tau,\psi,s+u)\epsilon(\sigma^*\times\tau,\psi,s-u)$$
$$M_\tau(s)^{-1}L(\sigma\times\tau,s+u)^{-1}L(\sigma^*\times\tau,s-u)^{-1}\Psi(W_u,f_s,s)$$
$$=L(\sigma\times\tau^*,1-s+u)^{-1}L(\sigma^*\times\tau^*,1-s-u)^{-1}\Psi(W_u,M_\tau(s)^{-1}M^*(\tau,s)f_s,1-s). \quad (9.8)$$

Note that $M_\tau(s)^{-1}M^*(\tau,s)f_s\in\xi(\tau^*,\mathrm{hol},1-s)$. For an arbitrary $W\in\mathcal{W}(\pi,\psi_\gamma^{-1})$ take W_u such that $W_0=W$, by applying a Whittaker functional to a suitable section of $\xi_{P_l}^{G_l}(\sigma,\mathrm{std},u+\frac{1}{2})$. Let $C\subset\mathbb{C}$ be a compact subset containing 0, which intersects $\mathcal{U}$ in a set D_1 containing some infinite sequence that converges to some point in the plane. There exist constants $s_1,s_2>0$ depending only on C,σ and τ such that $\Sigma(W_u,f_s,s)$ (respectively $\Sigma(W_u,M_\tau(s)^{-1}M^*(\tau,s)f_s,1-s)$) represents $\Psi(W_u,f_s,s)$ (respectively $\Psi(W_u,M_\tau(s)^{-1}M^*(\tau,s)f_s,1-s)$) for all $u\in C$ and $\Re(s)>s_1$ (respectively $\Re(s)<-s_2$). Let $u\in D_1$. Then both sides of (9.8) are polynomials in $q^{\mp s}$ and as explained in § 8.2 this equality may be interpreted in $\Sigma(X)$,

$$A(u,X)\Sigma(W_u,f_s,s)=B(u,X)\Sigma(W_u,M_\tau(s)^{-1}M^*(\tau,s)f_s,1-s).$$

This is an equality of the form $\sum_{m\in\mathbb{Z}}a_m(u)X^m=\sum_{m\in\mathbb{Z}}b_m(u)X^m$, valid for all $u\in D_1$, with $a_m,b_m:C\to\mathbb{C}$ analytic (see the proof of Claim 8.5). Hence it is valid also for $u=0$. Since $\Sigma(W_0,f_s,s)$ represents $\Psi(W_0,f_s,s)=\Psi(W,f_s,s)$ and is a series with finitely many negative coefficients, as in Lemma 8.1 we find that $A(0,X)\Sigma(W_0,f_s,s)\in R(X)$ and conclude that

$$\Psi(W,f_s,s)\in L(\sigma\times\tau,s)L(\sigma^*\times\tau,s)M_\tau(s)\mathbb{C}[q^{-s},q^s].$$

A similar relation holds for $\Psi(W,M^*(\tau^*,1-s)f_{1-s},s)$, $f_{1-s}\in\xi(\tau^*,\mathrm{hol},1-s)$ and the result follows.

The case of an essentially tempered τ is reduced to the tempered, by considering some $v\in\mathbb{C}$ such that $\tau_v=|\det|^v\tau$ is tempered. Note that $\gcd(\pi\times\tau_v,s)=\gcd(\pi\times\tau,s+v)$ and $M_{\tau_v}(s)=M_\tau(s+v)$. □

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Eyal Kaplan kaplaney@post.tau.ac.il
School of Mathematical Sciences, Sackler Faculty of Exact Sciences,
Tel-Aviv University, Tel-Aviv 69978, Israel

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