The final result in [1] concerns Figure 5. Again there are three equilateral triangles, but now \( D \), but now \( \angle BDF = -120^\circ \) (i.e. \( \angle BDF \) is a clockwise angle of \( 120^\circ \)). Show that \( ac = ab + bc \). This is best regarded as a result about areas. Rotate triangle \( DCB \) anticlockwise about \( D \) through \( 60^\circ \) to the triangle \( DEB' \). Since \( \angle DCB = \angle DEC = 60^\circ \), \( B' \) lies on \( EC \), and, since \( \angle BDB' = 60^\circ \) and \( \angle FDB = 120^\circ \), \( FDB' \) is a straight line; also \( EB' = CB = a \). Note that the area of a triangle with sides of lengths \( x \) and \( y \) including an angle of \( 60^\circ \) or \( 120^\circ \) is \( \sqrt{3}xy/4 \). The result now follows immediately since

\[
\text{area } FE'B' = \text{area } DEB' + \text{area } FED.
\]

**References**


**JOHN RIGBY**

*University of Wales Cardiff, Senghennydd Road, Cardiff CF2 4AG*

**83.34 Morley's other triangles**

I enjoyed Roy Barbara's ingenious approach to Morley's triangle [1]. By using the function \( f(t) = \sin t / \sin (t + \alpha) \), which increases continuously for \( -\alpha < t < (\pi - \alpha) \), with \( \alpha = A/3 \) for any triangle \( ABC \) he shows that Morley's triangle has three equal angles and deduces the side to be

\[
8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}.
\]

The alternative approach uses the cosine rule to find the side, which involves considerable manipulation.
It may not be so well known that there are four more equilateral triangles associated with ΔABC, which are illustrated in [2, p. 155]. The first of these triangles has vertices given by the points of intersection of the nearside trisectors of the exterior angles A, B and C. With \( \alpha = (2\pi + A)/3 \) Barbara's method can be used to show that this triangle is equilateral with side

\[
8R \sin \frac{\pi - A}{3} \sin \frac{\pi - B}{3} \sin \frac{\pi - C}{3}.
\]

It is easy to show that the sides of this triangle are parallel to the sides of Morley's triangle by considering inclinations to the sides of ΔABC. The second triangle has one vertex given by the point of intersection of the nearside trisectors of the exterior angles B and C. The other two vertices are given by the intersections of the outer trisectors of these exterior angles with the trisectors of A. By taking \( \alpha = (\pi - C)/3 \), this triangle can be shown to have one angle \( \pi/3 \), and its containing sides equal, and is hence equilateral. The side of this triangle is

\[
8R \sin \frac{\alpha}{3} \sin \frac{\pi - B}{3} \sin \frac{\pi - C}{3}.
\]

It is easy to show that two of the sides of this triangle are extensions of sides of the first triangle, by considering the angle summation at their common vertex. The other two sides are parallel. The last pair of equilateral triangles are constructed like the second triangle, but relative to the trisectors of B and C respectively.

References

ROBERT J. CLARKE
11 Lansdowne Court, Stourbridge DY9 0RL

83.35 The complex roots of a quadratic from a circle

The really appropriate tool for explorations of real roots of a quadratic equation has always seemed to me to be the usual geometrical approach with a parabola. However, there exists a procedure for finding these roots by means of a circle (see [1, 2, 3]). It is known as Carlyle's algorithm.

The algorithm can be stated in a modified form as follows:

1. Consider the quadratic equation \( x^2 + px + q = 0, p, q \in \mathbb{R} \).
2. Plot the points A(0, -1) and B(-p, -q).
3. Find the midpoint \( C \) of \( \overline{AB} \).
4. Draw the circle \( k \) with centre \( C \) and diameter \( 2r = \overline{AB} \).
5. Observe the points where the circle \( k \) crosses the x-axis; these are the real roots of the quadratic.