Can. J. Math., Vol. XXIV, No. 4, 1972, pp. 598–611

RESTRICTIVE SEMIGROUPS OF CONTINUOUS FUNCTIONS ON 0-DIMENSIONAL SPACES

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1. Introduction. Let X be a topological space and Y a nonempty subspace of X. $\Gamma(X, Y)$ denotes the semigroup under composition of all closed selfmaps of X which carry Y into Y, and is referred to as a restrictive semigroup of closed functions. Similarly, $S(X, Y)$ is the analogous semigroup of continuous selfmaps of X, and is referred to as a restrictive semigroup of continuous functions. It is immediate that each homeomorphism from X onto U which carries the subspace Y of X onto the subspace V of U induces an isomorphism between $\Gamma(X, Y)$ and $\Gamma(U, V)$, and also an isomorphism between $S(X, Y)$ and $S(U, V)$. Indeed, one need only map f onto $h \circ f \circ h^{-1}$. An isomorphism of this form is called representable. In [**5**, Theorem (3.1), p. 1223] it was shown that in most cases, each isomorphism from $\Gamma(X, Y)$ onto $\Gamma(U, V)$ is representable. The analogous problem was discussed for the semigroup $S(X, Y)$ and it was pointed out by means of an example that one could not hope to obtain the same result for these semigroups without some further restrictions. In this example X and U are both 0-dimensional (i.e., each has a basis of closed and open sets) and Hausdorff, and Y and V are proper dense subspaces of X and U, respectively. It is shown that there exists an isomorphism between $S(X, Y)$ and $S(U, V)$. However, it is not representable since X and U are not homeomorphic. The fact that Y and V are dense subsets seems to be crucial for constructing the example, for in this paper we prove the following.

MAIN THEOREM. *Let X and U be 0-dimensional, Hausdorff spaces and let Y and V be nonempty subsets of X and U, respectively. If Y is not dense in X and φ is an isomorphism from $S(X, Y)$ onto $S(U, V)$, then there exists a unique homeomorphism h from X onto U such that*

(i) $\varphi(f) = h \circ f \circ h^{-1}$ *for each* $f \in S(X, Y)$, *and*

(ii) $h[Y] = V$.

In § 2 we establish the lemmas used to prove this theorem. In § 3 we prove the theorem and give various applications of it. One such application is made to near-rings of continuous functions on topological groups, where we obtain a generalization of a result of Beidleman [1, Theorem (1.3), p. 982].

Received July 5, 1971 and in revised form, October 25, 1971. This paper constitutes part of the authors' doctoral dissertation written at the State University of New York at Buffalo under the direction of Professor K. D. Magill, Jr.

2. Minimal right ideals. We begin by introducing some notation that will be used throughout the paper. If f is a map from some set Z into a set W, by $R(f)$ is meant the range of f. We will always assume that Y is nonempty. If $y \in Y$ then the constant map which carries all of X onto the point y is denoted by $\langle y \rangle$. A set which is both open and closed is called clopen, and finally, we will adopt the convention of writing $S(X, p)$ for $S(X, \{p\})$.

Definition (2.1). Suppose X is 0-dimensional and Hausdorff. A pair (A, B) of subsets of X will be called a decomposition of X if A and B are nonempty disjoint clopen sets whose union is X.

LEMMA (2.2). *Suppose X is 0-dimensional and Hausdorff. Let F be a closed subset of X and let $K \subseteq X$ be a compact set such that $K \cap F = \emptyset$. Then there exists a decomposition (A, B) of X such that $K \subseteq A$ and $F \subseteq B$.*

LEMMA (2.3). *Suppose X is a 0-dimensional, Hausdorff space consisting of more than one point and Y is a subset of X. If F is a closed subset of X and $x \in X - F$, there exists a continuous map $g \in S(X, Y)$ and a point $z \in X$ such that $g[F] = \{z\}$ and $g(x) \neq z$.*

Proof. Suppose first that Y consists of a single point p. If $p \notin F$, then by Lemma (2.2) there exists a decomposition (A, B) of X such that $\{x, p\} \subseteq A$ and $F \subseteq B$. Choose $z \neq p$ and define $g[A] = \{p\}$ and $g[B] = \{z\}$. Then g is continuous and since $g(p) = p$, $g \in S(X, p)$. On the other hand if $p \in F$, then by Lemma (2.2) there exists a decomposition (A', B') of X such that $F \subseteq A'$ and $x \in B'$. In this case define $g[A'] = \{p\}$ and $g[B'] = \{x\}$. Since g is continuous and $g(p) = p$, $g \in S(X, p)$.

Suppose now that Y consists of more than one point. Then there exists a decomposition (A'', B'') of X such that $F \subseteq A''$ and $x \in B''$. Choose $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and define g such that $g[A''] = \{y_1\}$ and $g[B''] = \{y_2\}$. Since $g[X] = \{y_1, y_2\} \subseteq Y$ and g is continuous, $g \in S(X, Y)$.

Definition (2.4). Let X be a topological space and let Y be a subset of X. Then Y is said to be an admissible subset of X if for each closed subset F of X and each $x \in X - F$, there exists a map $f \in S(X, Y)$ and a point $z \in X$ such that $f[F] = \{z\}$ and $f(x) \neq z$.

From lemma (2.3) it follows that every subset of a 0-dimensional, Hausdorff space is admissible. The following lemma is an immediate consequence of Lemma (2.3).

LEMMA (2.5). *If Y is an admissible subset of a topological space X, then the collection of all sets of the form $f^{-1}(x)$, where $f \in S(X, Y)$ and $x \in X$, is a basis for the closed sets of X.*

Let X be a topological space and let Y be a subset of X. Then for each $y \in Y$ the constant map $\langle y \rangle \in S(X, Y)$ is a left zero. Conversely, if $f \in S(X, Y)$ is a left zero then it is constant. For, if $f \in S(X, Y)$ then $f[Y] \subseteq Y$ and this implies that there exists $y_1 \in R(f) \cap Y$ such that $f^{-1}(y_1) \cap Y \neq \emptyset$. Choose $y_2 \in f^{-1}(y_1) \cap Y$. Then $f(y_2) = y_1$ and $f \circ \langle y_2 \rangle = \langle y_1 \rangle$. Since f is a left zero, $f = f \circ \langle y_2 \rangle$ and hence, $f = \langle y_1 \rangle$. We conclude that $f \in S(X, Y)$ is a left zero if and only if it is constant.

Definition (2.6). If a semigroup S has a minimal ideal K, then K is called the kernel of S. The kernel of $S(X, Y)$ is written $K(X, Y)$.

It is an easy matter to prove that $K(X, Y) = \{\langle y \rangle : y \in Y\}$.

THEOREM (2.7). *Let X and U be topological spaces and let Y and V be nonempty admissible subsets of X and U, respectively. If φ is an isomorphism from $S(X, Y)$ onto $S(U, V)$, then there exists a unique homeomorphism k from Y onto V such that for each $f \in S(X, Y)$ and each $y \in Y$, $\varphi(f) \circ k(y) = k \circ f(y)$.*

Proof. There is a correspondence between points of Y and elements of $K(X, Y)$. Since φ maps $K(X, Y)$ onto $K(U, V)$ it follows that a bijection k from Y onto V can be defined such that for each $y \in Y$, $\varphi(\langle y \rangle) = \langle k(y) \rangle$.

Let $f \in S(X, Y)$ and $y \in Y$ be given. For any $u \in U$,

$$\begin{aligned}\varphi(f) \circ k(y) &= \varphi(f) \circ \langle k(y) \rangle(u) = \varphi(f) \circ \varphi(\langle y \rangle)(u) \\ &= \varphi(\langle f(y) \rangle)(u) = \langle k \circ f(y) \rangle(u) = k \circ f(y).\end{aligned}$$

Thus $\varphi(f) \circ k(y) = k \circ f(y)$. Using this we now show that k is a homeomorphism. From Lemma (2.5) it follows that the collection of all sets of the form $f^{-1}(x) \cap Y$, where $x \in X$ and $f \in S(X, Y)$, forms a basis for the closed subsets of Y. A similar statement holds for sets of the form $\varphi(f)^{-1}(u) \cap V$. Let $f \in S(X, Y)$ and $x \in X$ be given. Then the following statements are equivalent:

$z \in f^{-1}(x) \cap Y$;
$f(z) = x$ and $z \in Y$;
$f \circ \langle z \rangle = \langle x \rangle$ and $z \in Y$;
$\varphi(f) \circ \langle k(z) \rangle = \langle k(x) \rangle$ and $k(z) \in V$;
$\varphi(f)(k(z)) = k(x)$ and $k(z) \in V$;
$k(z) \in \varphi(f)^{-1}(k(x)) \cap V$.

Therefore, $k[f^{-1}(x) \cap Y] = \varphi(f)^{-1}(k(x)) \cap V$, and since the latter is a closed subset of V, k is a closed map. By use of a similar argument one proves that k is continuous. Thus k is a homeomorphism.

Now let k' be a homeomorphism from Y onto V such that $\varphi(f) \circ k'(y) = k' \circ f(y)$ for each $f \in S(X, Y)$ and each $y \in Y$. For each such $y \in Y$ we have,

$$k'(y) = k' \circ \langle y \rangle(y) = \varphi(\langle y \rangle) \circ k'(y) = \langle k(y) \rangle \circ k'(y) = k(y).$$

Since $y \in Y$ was arbitrary, $k' = k$ and k is unique.

Remark. Observe that a bijection from Y onto V can be defined without requiring that Y and V be admissible subsets.

The reader will note that according to Theorem (2.7), each isomorphism from $S(X, Y)$ onto $S(U, V)$ is representable when $Y = X$ and $V = U$. This result was obtained by Magill in [3]. Recall that X is an admissible subset of itself if and only if it is an S^*-space.

In the proof of Lemma (2.3) we used functions whose range consisted of precisely two points. These maps play a fundamental role in what follows so we introduce a special notation for them.

Definition (2.8). If f is a continuous map from a 0-dimensional, Hausdorff space W into a topological space Z we will write

$$f = [A, B; z_1, z_2]$$

if (A, B) is a decomposition of W, $f[A] = \{z_1\}$, and $f[B] = \{z_2\}$.

For the remainder of this paper we will assume that all spaces are 0-*dimensional, Hausdorff.*

Definition (2.9). If I is a right ideal in a semigroup S and b is a left zero in S, then I is said to be a b-minimal right ideal if

(i) $\{b\} \subsetneq I$, and

(ii) if J is a right ideal such that $\{b\} \subsetneq J \subseteq I$, then $J = I$.

A $\langle y \rangle$-minimal right ideal in $S(X, Y)$ will henceforth be called y-minimal.

Definition (2.10). Let Y be a nonempty subset of X, let $y \in Y$ and let $x \in X - \{y\}$. The subset

$$\{[A, B; y, x]: Y \subseteq A\} \cup \{\langle y \rangle\}$$

of $S(X, Y)$ is denoted by $I(y, x)$. If Y consists of more than one point, if y_1, $y_2 \in Y$ and $y_1 \neq y_2$, then the two element subset $\{\langle y_1 \rangle, \langle y_2 \rangle\}$ of $S(X, Y)$ is denoted by $J(y_1, y_2)$.

THEOREM (2.11). *Suppose X consists of more than one point, $Y = \{p\}$, and I is a right ideal of $S(X, p)$. Then I is a p-minimal right ideal if and only if $I = I(p, x)$ for some $x \in X$ such that $x \neq p$.*

Proof. Let $x \neq p$ be arbitrary. We will prove that $I(p, x)$ is a p-minimal right ideal. If $[A, B; p, x] \in I(p, x)$ and $f \in S(X, p)$, then $[A, B; p, x] \circ f$ is equal to $\langle p \rangle$ when $R(f) \subseteq A$ and is equal to $[f^{-1}(A), f^{-1}(B); p, x]$ when $R(f) \cap B \neq \emptyset$. Thus we see that $I(p, x)$ is a right ideal. We now show it is p-minimal. Since $x \neq p$ there exists a decomposition (A, B) of X such that $p \in A$ and $x \in B$. Then the map $[A, B; p, x]$ is an element of $I(p, x) - \{\langle p \rangle\}$ and therefore

$$\{\langle p \rangle\} \subsetneq I(p, x).$$

Next, suppose that J is a right ideal of $S(X, p)$ such that

$$\{\langle p\rangle\} \underset{\neq}{\subset} J \subseteq I(p, x),$$

and let

$$f = [C, D; p, x] \in I(p, x) - \{\langle p\rangle\}$$

and

$$f_1 = [E, F; p, x] \in J - \{\langle p\rangle\}$$

be given. Then $p \in E$ and (E, F) is a decomposition of X so we may choose $w \in F$. Since (C, D) is a decomposition of X and $p \in C$, the map $g = [C, D; p, w]$ is in $S(X, p)$. Moreover, $f = f_1 \circ g$. For if $z \in C$, then $g(z) = p$ and $f_1 \circ g(z) = f_1(p) = p$. And if $z \in D$, then $g(z) = w \in F$ and $f_1 \circ g(z) = f_1(w) = x$. Since $f_1 \in J$ and J is a right ideal, $f = f_1 \circ g \in J$, and hence $J = I(p, x)$. Thus for each $x \in X - \{p\}$, $I(p, x)$ is p-minimal in $S(X, p)$.

Now suppose that I is a p-minimal right ideal in $S(X, p)$. Then

$$\{\langle p\rangle\} \underset{\neq}{\subset} I$$

and since $\langle p\rangle$ is the only constant map in $S(X, p)$ there exists a function $f \in I$ having at least two points in its range. Therefore $R(f)$ contains p and a point $x \neq p$. Let $w \in f^{-1}(x)$ and let (A, B) be a decomposition of X such that $p \in A$ and $w \in B$. Then the map $f_2 = [A, B; p, w]$ is in $S(X, p)$, and

$$f \circ f_2 = [A, B; p, x] \in I \cap I(p, x) - \{\langle p\rangle\}.$$

Now $I \cap I(p, x)$ is a right ideal in $S(X, p)$ such that

$$\{\langle p\rangle\} \underset{\neq}{\subset} I \cap I(p, x) \subseteq I.$$

Since I is p-minimal, $I = I \cap I(p, x)$. Similarly, since $I(p, x)$ is p-minimal, $I(p, x) = I \cap I(p, x)$. Thus $I = I(p, x)$.

THEOREM (2.12). *Suppose Y is a nondense subset of X that consists of more than one point and $y \in Y$. Then a right ideal I in $S(X, Y)$ is y-minimal if and only if either*

(i) $I = I(y, x)$ *for some* $x \in X - \{y\}$, *or*

(ii) $I = J(y, y')$ *for some* $y' \in Y - \{y\}$.

Proof. In a direct manner one shows that $J(y, y')$ is a y-minimal right ideal and that $I(y, x)$ is a right ideal. We now show that $I(y, x)$ is y-minimal. Since Y is not dense in X there exists $x' \in X - \operatorname{cl} Y$ and a decomposition (A, B) of X such that $\operatorname{cl} Y \subseteq A$ and $x' \in B$. The function

$$[A, B; y, x] \in I(y, x) - \{\langle y\rangle\},$$

so we conclude that

$$\{\langle y\rangle\} \underset{\neq}{\subset} I(y, x).$$

Now let J be any right ideal of $S(X, Y)$ such that

$$\{\langle y\rangle\} \underset{\neq}{\subset} J \subseteq I(y, x).$$

We prove that $J = I(y, x)$. Let

$$f = [C, D; y, x] \in I(y, x) - \{\langle y\rangle\}$$

and

$$f_1 = [E, F; y, x] \in J - \{\langle y\rangle\}$$

be given. Since E and F are nonempty there exist points $y' \in Y \subseteq E$ and $x' \in F$. The map $g = [C, D; y', x']$ is continuous and since $y' \in Y$ and $Y \subseteq C$, $g \in S(X, Y)$. But $f = f_1 \circ g$, which implies $f \in J$, and thus $J = I(y, x)$.

Now let I be any y-minimal right ideal. The constant map $\langle y\rangle$ is in I, so if I contains a left zero $\langle y'\rangle$ different from $\langle y\rangle$, then $I = J(y, y')$. So suppose the only left zero in I is $\langle y\rangle$. Then for each $f \in I$, $f(z) = y$ for every $z \in Y$. For if $z \in Y$, $\langle z\rangle \in K(X, Y)$ and $f \circ \langle z\rangle$ is a left zero in I. Since $\langle y\rangle$ is the only left zero in I, $f \circ \langle z\rangle = \langle y\rangle$ and $f(z) = y$. Now since

$$\{\langle y\rangle\} \underset{\neq}{\subset} I,$$

there exists a map $f_2 \in I$ and a point $x \in R(f_2)$ such that $x \neq y$. Let $w \in f_2^{-1}(x)$ and let $g = [A, B; y, w]$ where (A, B) is a decomposition of X such that $\mathrm{cl}\, Y \subseteq A$ and $B \neq \emptyset$. Since $Y \subseteq A$ and $y \in Y$, $g \in S(X, Y)$. Moreover,

$$f_2 \circ g = [A, B; y, x] \in I \cap I(y, x) - \{\langle y\rangle\}.$$

Therefore

$$\{\langle y\rangle\} \underset{\neq}{\subset} I \cap I(y, x) \subseteq I.$$

Now $I \cap I(y, x)$ is a right ideal and since I is y-minimal, $I = I \cap I(y, x)$. Similarly, since $I(y, x)$ is y-minimal, $I(y, x) = I \cap I(y, x)$. Thus $I = I(y, x)$ and the proof is complete.

LEMMA (2.13). *Suppose that X consists of more than one point and Y is a nonempty subset of X. Then Y is not dense in X if and only if $S(X, Y)$ contains a right ideal of the form $I(y, x)$.*

Proof. If Y is not dense in X there exists a point $x \in X - \mathrm{cl}\, Y$ and a decomposition (A, B) of X such that $\mathrm{cl}\, Y \subseteq A$ and $x \in B$. Then for any $y \in Y$, the right ideal $I(y, x)$ exists.

Conversely, suppose $S(X, Y)$ contains a right ideal of the form $I(y, x)$. Then there exists a decomposition (A, B) of X such that $[A, B; y, x] \in I(y, x)$. Since $Y \subseteq A$ and B is nonempty it follows that Y is not dense in X.

From Theorems (2.11) and (2.12) it follows that to each $y \in Y$ there can be associated a family of y-minimal right ideals. If Y consists of more than one point and Y is not dense in X, there are two algebraically distinct types. One type, which is of the form $I(y, x)$, contains only one left zero whereas the other type, which is of the form $J(y, y')$, contains precisely two left zeros. Therefore any isomorphism between restrictive semigroups must carry a right ideal of the form $I(y, x)$ onto a right ideal of the same form.

LEMMA (2.14). *Let Y and V be nonempty subsets of X and U, respectively, and let φ be an isomorphism from $S(X, Y)$ onto $S(U, V)$. If $f_1, f_2 \in S(X, Y)$, then*

(i) $R(f_1) \subseteq \operatorname{cl} R(f_2)$ *if and only if* $R(\varphi(f_1)) \subseteq \operatorname{cl} R(\varphi(f_2))$, *and*

(ii) $\operatorname{cl} R(f_1) \subseteq \operatorname{cl} R(f_2)$ *if and only if* $\operatorname{cl} R(\varphi(f_1)) \subseteq \operatorname{cl} R(\varphi(f_2))$.

Proof. Suppose

$$R(\varphi(f_1)) \not\subseteq \operatorname{cl} R(\varphi(f_2)).$$

Then there exists a point $u \in R(\varphi(f_1)) - \operatorname{cl} R(\varphi(f_2))$. From Theorem (2.3) it follows that there exists a map $g \in S(X, Y)$ and a point $u' \in U$ such that $\varphi(g)\,[\operatorname{cl} R(\varphi(f_2))] = \{u'\}$ and $\varphi(g)\,(u) \neq u'$. Since $R(\varphi(f_2)) \cap V \neq \emptyset$ and $\varphi(g) \in S(U, V)$, it follows that $u' = v \in V$. Then $\varphi(g) \circ \varphi(f_2) = \langle v \rangle$, $\varphi(g) \circ \varphi(f_1) \neq \langle v \rangle$, and if $\varphi(\langle y \rangle) = \langle v \rangle$, then $g \circ f_2 = \langle y \rangle$ and $g \circ f_1 \neq \langle y \rangle$. Therefore there exists $x \in R(f_1)$ such that $g(x) \neq y$. Since $R(f_2) \subseteq g^{-1}(y)$ which is closed in X, it follows that $x \in R(f_1) - \operatorname{cl} R(f_2)$, and hence

$$R(f_1) \not\subseteq \operatorname{cl} R(f_2).$$

One shows the converse by using the same techniques with φ^{-1}. This completes the proof of (i).

For (ii), $\operatorname{cl} R(f_1) \subseteq \operatorname{cl} R(f_2)$ implies $R(f_1) \subseteq \operatorname{cl} R(f_2)$. Then from (i), $R(\varphi(f_1)) \subseteq \operatorname{cl} R(\varphi(f_2))$ and hence $\operatorname{cl} R(\varphi(f_1)) \subseteq \operatorname{cl} R(\varphi(f_2))$. The converse is proved similarly.

LEMMA (2.15). *Let Y be a nondense subset of X that contains more than one point, let $f_1 = [A, B;\ y_1, x_1]$ and let $f_2 = [A', B';\ y_2, x_2]$ where $y_1, y_2 \in Y$, $Y \subseteq A \cap A'$, and $x_1, x_2 \in X - \{y_1, y_2\}$. Then $x_1 = x_2$ if and only if for each $f \in S(X, Y)$ such that $f(y_1) = f(y_2)$, $f \circ f_1 \in K(X, Y)$ implies $f \circ f_2 \in K(X, Y)$.*

Proof. Suppose $x_1 = x_2$ and $f \in S(X, Y)$ has the property that $f(y_1) = f(y_2) = y_0$. If $f \circ f_1 \in K(X, Y)$, then $f \circ f_1 = \langle y_0 \rangle$ and $f(x_1) = f(y_1) = y_0$. Since $x_1 = x_2$, $f(x_2) = y_0$, and since $f(y_2) = y_0$ it follows that $f \circ f_2 = \langle y_0 \rangle \in K(X, Y)$.

Now suppose $x_1 \neq x_2$ and choose $y_3, y_4 \in Y$ such that $y_3 \neq y_4$. Let (C, D) be a decomposition of X such that $\{y_1, y_2, x_1\} \subseteq C$ and $x_2 \in D$. Then $g = [C, D; y_3, y_4]$ is in $S(X, Y)$ since $R(g) \subseteq Y$. Moreover $g \circ f_1 = \langle y_3 \rangle \in K(X, Y)$ but $g \circ f_2 = [A', B'; y_3, y_4]$ is not in $K(X, Y)$.

https://doi.org/10.4153/CJM-1972-054-1 Published online by Cambridge University Press

In the above lemma, functions $f \in S(X, Y)$ such that $f(y_1) = f(y_2)$ are considered. Algebraically these functions are just those satisfying $f \circ \langle y_1 \rangle = f \circ \langle y_2 \rangle$.

LEMMA (2.16). *Let Y and V be nonempty nondense subsets of X and U, respectively, and let φ be an isomorphism from $S(X, Y)$ onto $S(U, V)$. Let*

$$f_1 = [A, B; y_1, x_1]$$

and

$$f_2 = [A, B; y_2, x_2]$$

be maps in $S(X, Y)$ where $x_1, x_2 \in X$, $Y \subseteq A$, $x_1 \neq y_1$ and $x_2 \neq y_2$. If $\varphi(\langle y_1 \rangle) = \langle v_1 \rangle$ and $\varphi(\langle y_2 \rangle) = \langle v_2 \rangle$, then

(i) $$\varphi(f_1) = [A', B'; v_1, u_1]$$

and

$$\varphi(f_2) = [A'', B''; v_2, u_2]$$

where $u_1, u_2 \in U$, $V \subseteq A' \cap A''$, $v_1 \neq u_1$ and $v_2 \neq u_2$, and

(ii) *$A' = A''$ and $B' = B''$.*

Proof. Since $Y \subseteq A$ and $y_1 \neq x_1$, $f_1 \in I(y_1, x_1)$. Since φ is an isomorphism and $\varphi(\langle y_1 \rangle) = \langle v_1 \rangle$, there exists a point $u_1 \in U$, necessarily different from v_1, such that φ maps the y_1-minimal right ideal $I(y_1, x_1)$ onto the v_1-minimal right ideal $I(v_1, u_1)$, and $V \subseteq A'$. Similarly, it follows for f_2 that $v_2 \neq u_2$, $u_2 \in U$, and $V \subseteq A''$.

Thus we need only show that (ii) holds. Let $g \in S(X, Y)$ be arbitrary. Then $f_1 \circ g = \langle y_1 \rangle$ if and only if $f_2 \circ g = \langle y_2 \rangle$. Therefore $\varphi(f_1) \circ \varphi(g) = \langle v_1 \rangle$ if and only if $\varphi(f_2) \circ \varphi(g) = \langle v_2 \rangle$. Let

$$\mathscr{A} = \{g \in S(X, Y): f_1 \circ g = \langle y_1 \rangle\}.$$

Then it is clear that

$$\mathscr{A} = \{g \in S(X, Y): f_2 \circ g = \langle y_2 \rangle\}.$$

We now prove that $A' = \bigcup \{R(\varphi(g)): g \in \mathscr{A}\}$.

First suppose that $u \in A'$. Let $v \in V$ and let $\varphi(g_1) = [A', B'; v, u]$. Since $v \in V \subseteq A'$,

$$\varphi(f_1) \circ \varphi(g_1) = [A', B'; v_1, u_1] \circ [A', B'; v, u] = \langle v_1 \rangle,$$

and therefore $f_1 \circ g_1 = \langle y_1 \rangle$. Hence $g_1 \in \mathscr{A}$ and $A' \subseteq \bigcup \{R(\varphi(g)): g \in \mathscr{A}\}$. Now suppose $u \in \bigcup \{R(\varphi(g)): g \in \mathscr{A}\}$. Then there exists $g_2 \in \mathscr{A}$ such that $u \in R(\varphi(g_2))$, and $f_1 \circ g_2 = \langle y_1 \rangle$. Therefore $\varphi(f_1) \circ \varphi(g_2) = \langle v_1 \rangle$ from which it follows that $R(\varphi(g_2)) \subseteq A'$. Thus $u \in A'$ and $A' = \bigcup \{R(\varphi(g)): g \in \mathscr{A}\}$. Similarly, it can be shown using f_2 that $A'' = \bigcup \{R(\varphi(g)): g \in \mathscr{A}\}$. Thus we conclude that $A' = A''$ and $B' = U - A' = U - A'' = B''$.

LEMMA (2.17). *Let Y and V be nondense subsets of X and U, respectively, and let $I(y_1, x_1)$ and $I(y_2, x_2)$ be y_1 and y_2-minimal right ideals, respectively, in $S(X, Y)$. Then $y_1 = x_2$ if and only if $\langle y_1 \rangle \notin I(y_2, x_2)$ and $R(\langle y_1 \rangle) \subseteq R(f)$ for each $f \in I(y_2, x_2) - K(X, Y)$.*

Proof. Suppose first that $y_1 = x_2$. Then since $y_2 \neq x_2$, it follows that $\langle y_1 \rangle \notin I(y_2, x_2)$ and $R(\langle y_1 \rangle) \subseteq R(f)$ for each $f \in I(y_2, x_2) - K(X, Y)$. Conversely, if $\langle y_1 \rangle \notin I(y_2, x_2)$ then $y_1 \neq y_2$. Therefore if $R(\langle y_1 \rangle) \subseteq R(f)$ for each $f \in I(y_2, x_2) - K(X, Y)$, we must necessarily have $y_1 = x_2$.

LEMMA (2.18). *Let Y and V be nondense subsets of X and U, respectively, and let φ be an isomorphism from $S(X, Y)$ onto $S(U, V)$. Let $y_1, y_2 \in Y$ and let $x \in X$ be such that $y_1 \neq x \neq y_2$. Then*

(i) *$\varphi[I(y_1, x)] = I(v_1, u_1)$ and $\varphi[I(y_2, x)] = I(v_2, u_2)$ for some $v_1, v_2 \in V$ and $u_1, u_2 \in U$, and*

(ii) *$u_1 = u_2$.*

Proof. Let $v_1, v_2 \in V$ be such that $\varphi(\langle y_1 \rangle) = \langle v_1 \rangle$ and $\varphi(\langle y_2 \rangle) = \langle v_2 \rangle$. Then by Theorems (2.11) and (2.12), φ maps the y_1-minimal right ideal $I(y_1, x)$ onto the v_1-minimal right ideal $I(v_1, u_1)$ for some $u_1 \in U$. Similarly, $\varphi[I(y_2, x)] = I(v_2, u_2)$ for some $u_2 \in U$. Now let

$$f_1 = [A_1, B_1; y_1, x] \in I(y_1, x)$$

and let

$$f_2 = [A_2, B_2; y_2, x] \in I(y_2, x).$$

We observe that there exist sets C_1, D_1, C_2, and D_2 such that

$$\varphi(f_1) = [C_1, D_1; v_1, u_1]$$

and

$$\varphi(f_2) = [C_2, D_2; v_2, u_2].$$

Now let $\varphi(f) \in S(U, V)$ have the property that $\varphi(f)(v_1) = \varphi(f)(v_2)$. Note that $\varphi(f)(v_1) = \varphi(f)(v_2)$ if and only if $\varphi(f) \circ \langle v_1 \rangle = \varphi(f) \circ \langle v_2 \rangle$, or what is the same, $\varphi(f) \circ \varphi(\langle y_1 \rangle) = \varphi(f) \circ \varphi(\langle y_2 \rangle)$. Therefore $f(y_1) = f(y_2)$. If $\varphi(f) \circ \varphi(f_1) \in K(U, V)$, then $f \circ f_1 \in K(X, Y)$ and by Lemma (2.15), $f \circ f_2 \in K(X, Y)$, which implies that $\varphi(f) \circ \varphi(f_2) \in K(U, V)$. From Lemma (2.17) it follows that $u_1, u_2 \in U - \{v_1, v_2\}$. Consequently, it follows from Lemma (2.15) that $u_1 = u_2$.

3. Representable isomorphisms.

THEOREM (3.1). *Let X and U be 0-dimensional, Hausdorff spaces and let Y be a nonempty nondense subset of X. If φ is an isomorphism from $S(X, Y)$ onto $S(U, V)$, then there exists a unique homeomorphism h from X onto U such that*

(i) *$\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(Y, X)$, and*

(ii) *$h[Y] = V$.*

Proof. Since Y is not dense in X it follows from Lemma (2.13) that V is not dense in U. We first consider the case where Y consists of more than one point. Then it follows from Theorem (2.7) that V consists of more than one point. The map h is defined as follows. Let $x \in X$ be given and choose $y \in Y$ different from x. Then by Theorem (2.12) φ maps $I(y, x)$ onto $I(v, u)$ for some $v \in V$

and $u \in U$. Because of Lemma (2.18) the point u is independent of the choice of y and we define the function h from X into U by $h(x) = u$. One shows in a straight forward manner that h is in fact a bijection.

By our definition of the function h we now have that for each $y \in Y$ there exists a (necessarily unique) point $v \in V$ such that $\varphi[I(y, x)] = I(v, h(x))$. We show next that $h(y) = v$. Note that $\varphi(\langle y \rangle) = \langle v \rangle$ since φ takes left zeros into left zeros. Let $y' \in Y$ with $y' \neq y$ and let $v' \in V$ be such that $\varphi[I(y', y)] = I(v', h(y))$. Now $R(\langle y \rangle) \subseteq R(f)$ for all $f \in I(y', y) - K(X, Y)$ but $\langle y \rangle \notin I(y', y)$. Then since $R(\varphi(\langle y \rangle)$ and $R(\varphi(f))$ are closed, it follows from Lemma (2.14) that

$$R(\langle v \rangle) = R(\varphi(\langle y \rangle)) \subseteq R(\varphi(f))$$

for all $\varphi(f) \in I(v', h(y)) - K(U, V)$. Also, $\langle v \rangle \notin I(v', h(y))$ so it follows from Lemma (2.17) that $v = h(y)$. Thus we have defined h in such a manner that for each $y \in Y$, $\varphi(\langle y \rangle) = \langle h(y) \rangle$ and $\varphi[I(y, x)] = I(h(y), h(x))$ for each y-minimal right ideal $I(y, x)$. It now follows in a straightforward manner that h carries Y bijectively onto V.

We now show that $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(X, Y)$. Let $x \in X$ be given. Then since Y consists of more than one point, there exists $y \in Y$ different from x. Let (A, B) be a decomposition of X such that $Y \subseteq A$ and let (C, D) be a decomposition of U such that

$$\varphi([A, B; y, x]) = [C, D; h(y), h(x)].$$

Then for a fixed $u \in D$,

$$\begin{aligned} \varphi(f) \circ h(x) &= \varphi(f) \circ [C, D; h(y), h(x)]\ (u) \\ &= \varphi(f) \circ \varphi([A, B; y, x])(u) \\ &= \varphi([A, B; f(y), f(x)])\ (u). \end{aligned}$$

If $f(x) = f(y)$, then

$$\varphi([A, B; f(y), f(x)])\ (u) = \varphi(\langle f(x) \rangle)\ (u) = \langle h \circ f(x) \rangle\ (u) = h \circ f(x).$$

On the other hand if $f(x) \neq f(y)$ then from (b) of lemma (2.16) it follows that

$$\varphi([A, B; f(y), f(x)])\ (u) = [C, D; h \circ f(y), h \circ f(x)]\ (u) = h \circ f(x)$$

since $u \in D$. Thus $\varphi(f) = h \circ f \circ h^{-1}$.

Next we show that h is a homeomorphism. According to Lemma (2.5), the collection of all sets of the form $f^{-1}(x)$, where $x \in X$ and $f \in S(X, Y)$, forms a basis for the closed sets of X. Since $h^{-1} \circ \varphi(f) = f \circ h^{-1}$ for each $f \in S(X, Y)$,

$$h[f^{-1}(x)] = (f \circ h^{-1})^{-1}\ (x) = (h^{-1} \circ \varphi(f))^{-1}\ (x) = \varphi(f)^{-1}(h(x)).$$

Since $\varphi(f)^{-1}(h(x))$ is a closed subset of U, it follows that $h[f^{-1}(x)]$ is closed in U, and we conclude that h is a closed map. Similarly, using the equality $\varphi(f) \circ h = h \circ f$ it can be proved that h is continuous, and hence that h is a homeomorphism.

https://doi.org/10.4153/CJM-1972-054-1 Published online by Cambridge University Press

We next show the function h is unique. Let k be any homeomorphism from X onto U which carries Y onto V such that $\varphi(f) = k \circ f \circ k^{-1}$ for each $f \in S(X, Y)$. Let $x \in X$ be given and let $y \in Y$ be such that $y \neq x$. Let (A, B) be a decomposition of X such that $Y \subseteq A$ and let (C, D) be a decomposition of U such that $\varphi([A, B; y, x]) = [C, D; h(y), h(x)]$. Then

$$\begin{aligned}[C, D; h(y), h(x)] &= \varphi([A, B; y, x]) \\ &= k \circ [A, B; y, x] \circ k^{-1} \\ &= [k(A), k(B); k(y), k(x)].\end{aligned}$$

Since k carries Y onto V and $Y \subseteq A$ we have $V \subseteq k(A)$, and since $V \subseteq C$ it follows that $C = k(A)$ and $D = k(B)$. Then we evaluate $[C, D; h(y), h(x)]$ and $[k(A), k(B); k(y), k(x)]$ at any point in D and obtain $h(x) = k(x)$. This proves that h is unique.

We now consider the case when Y consists of one point p. Then from Theorem (2.7) it follows that $V = \{q\}$ for some $q \in U$. The map h is defined as follows. Let $x \in X$ be given. If $x = p$ define $h(p) = q$. On the other hand if $x \neq p$, then from Theorem (2.11) it follows that φ maps $I(p, x)$ onto $I(q, u)$ for some $u \neq q$, and we define $h(x) = u$. One shows in a direct fashion that h is in fact a bijection from X onto U. We next show that $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(X, p)$. If $x = p$, then

$$\varphi(f) \circ h(p) = \varphi(f)\ (q) = q = h(p) = h \circ f(p).$$

So suppose $x \neq p$. Let (A, B) be a decomposition of X such that $p \in A$ and $x \in B$, and let (C, D) be a decomposition of U such that

$$\varphi([A, B; p, x]) = [C, D; h(p), h(x)].$$

For any fixed $u \in D$ we have,

$$\begin{aligned}\varphi(f) \circ h(x) &= \varphi(f) \circ [C, D; h(p), h(x)]\ (u) \\ &= \varphi(f) \circ \varphi([A, B; p, x])\ (u) \\ &= \varphi([A, B; p, f(x)])\ (u).\end{aligned}$$

Now if $f(x) = p$, then

$$\varphi([A, B; p, f(x)])\ (u) = \varphi(\langle p \rangle)(u) = q = h \circ f(x).$$

But if $f(x) \neq p$, then from Lemma (2.16) it follows that

$$\varphi([A, B; p, f(x)])(u) = [C, D; h(p), h \circ f(x)](u) = h \circ f(x)$$

since $u \in D$. Thus $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(X, p)$. The rest of the proof is analogous to that of the case when Y consists of more than one point, and therefore is not given.

As an immediate consequence of the theorem, we have the following.

COROLLARY (3.2). *Let X be a 0-dimensional, Hausdorff space and let Y be a nonempty subset of X. Then every automorphism of $S(X, Y)$ is inner.*

Definition (3.3). $\mathscr{H}(X, Y)$ is the group, under composition, of all homeomorphisms which map X onto X and carry Y onto Y. $\mathscr{A}(X, Y)$ denotes the group, under composition, of all automorphisms of $S(X, Y)$.

THEOREM (3.4). *Let X be a 0-dimensional, Hausdorff space and let Y be a nonempty nondense subset of X. Then $\mathscr{A}(X, Y)$ is isomorphic to $\mathscr{H}(X, Y)$.*

Proof. Define a map θ from $\mathscr{A}(X, Y)$ into $\mathscr{H}(X, Y)$ as follows. If $\varphi \in \mathscr{A}(X, Y)$, then according to Theorem (3.1) there is a unique homeomorphism $h \in \mathscr{H}(X, Y)$ such that $\varphi(f) = h \circ f \circ h^{-1}$, so define $\theta(\varphi) = h$. It is clear that θ is well-defined. If $h \in \mathscr{H}(X, Y)$ is arbitrary then the map φ defined by $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(X, Y)$ is an automorphism of $S(X, Y)$, and hence $\theta(\varphi) = h$. Moreover, if $\theta(\varphi) = \text{id}$, where $\text{id} \in \mathscr{H}(X, Y)$ is the identity map, then $\varphi(f) = \text{id} \circ f \circ \text{id} = f$ and φ is the identity of $\mathscr{A}(X, Y)$. Once we have shown that θ is a homomorphism, it will follow from what was just proved that θ is in fact an isomorphism. To show that θ is a homomorphism, suppose $\varphi_1, \varphi_2 \in \mathscr{A}(X, Y)$ and that $\theta(\varphi_1) = h_1$ and $\theta(\varphi_2) = h_2$. Then

$$\varphi_1 \circ \varphi_2(f) = \varphi_1(h_2 \circ f \circ h_2^{-1}) = (h_1 \circ h_2) \circ f \circ (h_1 \circ h_2)^{-1}$$

for each $f \in S(X, Y)$. This means that

$$\theta(\varphi_1 \circ \varphi_2) = h_1 \circ h_2 = \theta(\varphi_1) \circ \theta(\varphi_2).$$

Thus θ is an isomorphism and the proof is complete.

In the following theorem, $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of X.

THEOREM (3.5). *Let X and U be 0-dimensional, Hausdorff spaces and let Y be a nonempty subset of X. If Y is not dense in X and φ is an isomorphism from $S(X, Y)$ onto $S(U, V)$, then φ can be extended to an isomorphism φ^E which maps $S(X)$ onto $S(U)$.*

Proof. According to Theorem (3.1) there is a homeomorphism h from X onto U such that $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in S(X, Y)$. If $f \in S(X)$, define $\varphi^E(f) = h \circ f \circ h^{-1}$. It is easy to see that φ^E is an isomorphism and that it is an extension of φ.

From this theorem it follows in a straightforward way that $\mathscr{A}(X, Y)$ is isomorphic to a subgroup of $\mathscr{A}(X)$, the automorphism group of $S(X)$. It is proved by showing that the map θ from $\mathscr{A}(X, Y)$ into $\mathscr{A}(X)$ defined by $\theta(\varphi) = \varphi^E$ is an isomorphism.

We now turn our attention to the application of Theorem (3.1) to near-rings of functions on a topological group. Let G be an additive topological group and let H be a subgroup of G. $\mathscr{N}(G, H)$ denotes the set of all continuous selfmaps f of G such that $f[H] \subseteq H$. One proves in a direct way that $\mathscr{N}(G, H)$ forms a near-ring under pointwise addition and the usual composition of functions. We will use the term "topological isomorphism" to denote a map

https://doi.org/10.4153/CJM-1972-054-1 Published online by Cambridge University Press

between two topological groups which is both an isomorphism and a homeomorphism. The following theorem is analogous to a result which Magill proved for $\mathscr{N}(G) = \mathscr{N}(G, G)$ in [**4**].

THEOREM (3.6). *Let G and H be 0-dimensional, Hausdorff topological groups and let G' and H' be subgroups of G and H, respectively. If G' is not dense in G and φ is an isomorphism from $\mathscr{N}(G, G')$ onto $\mathscr{N}(H, H')$, then there exists a unique topological isomorphism h from G onto H such that*

(i) $\varphi(f) = h \circ f \circ h^{-1}$ *for each* $f \in \mathscr{N}(G, G')$, *and*

(ii) $h[G'] = H'$.

Proof. It follows from Theorem (3.1) that there exists a unique homeomorphism h from G onto H which satisfies (i) and (ii). We show that h is an isomorphism. We first assert that if $x, y \in G$ then there exist functions $f, g \in \mathscr{N}(G, G')$ and a point $z \in G$ such that $f(z) = x$ and $g(z) = y$. There are three cases to consider. First consider the case when $x, y \in G'$. Then the maps $f = \langle x \rangle$ and $g = \langle y \rangle$ have the desired property for any $z \in G$. Next suppose $y \in G'$ and $x \notin G'$. Let (A, B) be a decomposition of G such that $G' \subseteq A$, and choose $z \in B$. Then let $f = [A, B; y, x]$ and $g = \langle y \rangle$. We then have $f(z) = x$ and $g(z) = y$. A similar argument holds if $x \in G'$ and $y \notin G'$. Finally consider the case when $x, y \notin G'$. Let (A, B) be any decomposition of G such that $G' \subseteq A$ and let z be any point in B. Then for any $w \in G'$, we let $f = [A, B; w, x]$ and $g = [A, B; w, y]$ and observe that $f(z) = x$ and $g(z) = y$.

Now let $x, y \in G$ be given. We have just proved that there exist functions $f, g \in \mathscr{N}(G, G')$ and a point $z \in G$ such that $f(z) = x$ and $g(z) = y$. Then we have

$$\begin{aligned} h(x + y) &= h \circ (f + g)\ (z) \\ &= \varphi(f + g) \circ h(z) \\ &= (\varphi(f) + \varphi(g)) \circ h(z) \\ &= \varphi(f) \circ h(z) + \varphi(g) \circ h(z) \\ &= h \circ f(z) + h \circ g(z) \\ &= h(x) + h(y). \end{aligned}$$

Thus h is an isomorphism and the proof is complete.

In the following corollary, $\mathscr{N}_0(G)$ denotes the near-ring $\mathscr{N}(G, \{0\})$. This corollary generalizes the result of Beidleman [**1**].

COROLLARY (3.7). *Let G and H be 0-dimensional, Hausdorff topological groups. If φ is any isomorphism from $\mathscr{N}_0(G)$ onto $\mathscr{N}_0(H)$, then there exists a unique topological isomorphism h from G onto H such that $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in \mathscr{N}_0(G)$.*

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