A LARGE-SCALE PARTICLE SYSTEM WITH INDEPENDENT JUMPS AND DISTRIBUTED SYNCHRONIZATION

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Abstract

We study a system consisting of n particles, moving forward in jumps on the real line. Each particle can make both independent jumps, whose sizes have some distribution, and 'synchronization' jumps, which allow it to join a randomly chosen other particle if the latter happens to be ahead of it. The system state is the empirical distribution of particle locations. We consider the mean-field asymptotic regime where $n \to \infty$. We prove that v_n, the steady-state speed of advance of the particle system, converges, as $n \to \infty$, to a limit v_{**} which can easily be found from a *minimum speed selection principle*. Also we prove that as $n \to \infty$, the system dynamics converges to that of a deterministic mean-field limit (MFL). We show that the average speed of advance of any MFL is lower-bounded by v_{**}, and the speed of a 'benchmark' MFL, resulting from all particles initially being co-located, is equal to v_{**}. In the special case of exponentially distributed independent jump sizes, we prove that a traveling-wave MFL with speed v exists if and only if $v \geq v_{**}$, with v_{**} having a simple explicit form; we also show the existence of traveling waves for the modified systems with a left or right boundary moving at a constant speed v. We provide bounds on an MFL's average speed of advance, depending on the right tail exponent of its initial state. We conjecture that these results for exponential jump sizes extend to general jump sizes.

Keywords: Mean-field interaction; large-scale limit dynamics; average speed; traveling wave; minimum speed selection principle; branching random walk

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1. Introduction

1.1. Model, motivation, and (informally) main results

 We consider a system consisting of n particles, moving forward in jumps on the real line. A particle can jump either independently or via 'synchronizations' with other particles. Independent jumps of a particle occur at time points of an independent Poisson process of

constant rate $\lambda \geq 0$; the jump sizes are independent and identically distributed as a random variable $Z > 0$ with cumulative distribution function $J(\cdot)$ and finite mean, $\mathbb{E}Z < \infty$; without loss of generality we can and do assume $\mathbb{E}Z = 1$. Each particle also has another independent Poisson process (of synchronizations), of constant rate $\mu > 0$, at which points it chooses another particle uniformly at random, and makes a synchronization jump to the location of the chosen particle if the latter is ahead—to the right—of its own current location. The system state at a given time is the empirical distribution of particle locations. We focus on the mean-field asymptotic regime where n becomes large.

Our model is of the type in which particles move both autonomously (independent jumps) and according to a synchronization mechanism. (See [1, 11–13, 17–22] and references therein for previous models of this type.) Such models are motivated by distributed systems, in which agents need to both evolve and synchronize their states, and the synchronization is accomplished in a distributed fashion, via random peer-to-peer communication. The special feature of our model is that, even in the asymptotic limit, the autonomous movement of a particle is discontinuous (consists of random jumps). This brings up the question of how much the synchronization mechanism affects the overall speed of advance of the system, and of what constitutes a good trade-off between the 'self-propulsion' (independent jump) rate and the synchronization rate. In models of technological systems, such as parallel computing/simulation or wireless systems (cf. [11, 12, 17–22]), particles' locations correspond to agents' local 'clocks', which advance locally but need to be synchronized. Our specific system can also model a business environment, with multiple companies that can improve their products either via investments in themselves ('self-propulsion') or via acquisitions ('synchronization').

The main question that we address in this paper is very typical for this kind of system: as n becomes large, what is the average speed at which the system state (empirical distribution) moves forward? Specifically, we will be interested in two metrics of the speed of advance: the steady-state speed of advance v_n, especially $\lim_{n\to\infty} v_n$; and the average speed of the *mean-field limit* (MFL), which, informally speaking, is the deterministic limit of the system state dynamics as $n \to \infty$. Our main results are as follows:

- We prove that $v_n \to v_{**}$ as $n \to \infty$, where the value of v_{**} can easily be found from a *minimum speed selection principle*.

- We prove that as $n \to \infty$, the system dynamics converges to that of a deterministic MFL. We also prove this for two modified systems, with a left or right boundary moving at a constant speed v.

- We show that the average speed of advance of any MFL is lower-bounded by v_{**}, and the speed of a 'benchmark' MFL (BMFL), resulting from all particles initially being co-located, is equal to v_{**}.

- In the special case of exponentially distributed independent jump sizes, we obtain the following:

    - We prove that a traveling-wave MFL with speed v exists if and only if $v \geq v_{**}$, with v_{**} having a very simple explicit form. In addition, we show that a traveling wave with moving left boundary exists for any boundary speed $v > v_{**}$, and a traveling wave with moving right boundary exists for any boundary speed $v < v_{**}$.

    - Using monotonicity and the traveling waves as bounds, we obtain bounds on an MFL's average speed of advance, depending on the right tail exponent of the initial state.

We conjecture that the above results for exponential jump sizes extend to general jump sizes as well.

Using our results, in a large-scale system, one can easily optimize a trade-off between λ and μ (between self-propulsion and synchronization efforts) to achieve maximum speed. For example, one can maximize $v_{**} = v_{**}(\lambda, \mu)$ subject to $a\lambda + b\mu \leq 1$, where the constants $a > 0$ and $b > 0$ describe the unit rates at which self-propulsion and synchronization consume some common resource. We provide simulation results illustrating that this approach works well when the number of particles is large.

1.2. Previous work and discussion of our results

The literature on the mean-field asymptotic behavior of large-scale particle systems is vast. We will only describe the previous work that is most closely related; the reader will find further literature reviews in the works cited here.

An MFL is a function $(f_x(t), x \in \mathbb{R}, t \geq 0)$, where $(f_x(t), x \in \mathbb{R})$ is the cumulative distribution function of particle locations on the real line at time t, in the $n \to \infty$ limit. Thus, it describes the 'macro-behavior' of a particle system. (The term 'micro-behavior' often refers to the evolution of the system when the number n of particles is finite.)

It is well known that the classical Kolmogorov–Petrovskii–Piscounov (KPP) equation [15],

$$\frac{\partial}{\partial t} f_x(t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f_x(t) + F(f_x(t)), \tag{1.1}$$

describes MFLs (in our terminology) of many large-scale particle systems (cf. [13, 22] and references therein). In (1.1), $F(\cdot)$ is a non-positive function within a certain class (for example, $F(y) = -y(1-y)$ or $F(y) = -y(1-y^2)$). The seminal paper [15] studies Equation (1.1) and proves, in particular, that Equation (1.1) has traveling-wave solutions with any speed $v \geq v_{**}$, where the minimum speed v_{**} depends on function $F(\cdot)$; it is also proved that v_{**} is the average speed of the solution with initial condition being a Dirac distribution concentrated at 0 (we refer to such a solution as the benchmark MFL (BMFL)). It has since been found (cf. [4–6, 13] and references therein) that in many instances, a particle system having KPP as its MFL (or, in fact, a different asymptotic limit) is such that the steady-state particle system speed v_n satisfies $\lim_{n\to\infty} v_n = v_{**}$. In other words, the micro-behavior is such that the system 'selects the minimum speed of the traveling waves that can arise in macro-behavior'. This is sometimes referred to as a minimum speed selection principle (MSSP) [5, 6]. In this paper we show that the same basic phenomenon holds for our system, even though its macro-behavior (i.e., the MFL dynamics equation), even in the special case of exponential jumps, is *not within the KPP class*. Our proof of the traveling-wave existence results for exponential jumps, while it follows the general approach of KPP [15] (i.e., analysis of the phase portrait of a two-dimensional ordinary differential equation (ODE)), requires new observations—in particular, the use of parabolic (as opposed to linear) barriers.

The first paper to formally prove an MSSP for a particle system with macro-behavior within the KPP class is [4]. The recent paper [13] considers a particle system which is just like ours, but the independent movement of each particle is a Brownian motion (instead of forward jumps); the paper shows that the MFLs of the system are within the KPP class, and proves the MSSP for it. The MSSP is not limited to particle systems whose macro-behavior is within the KPP class. For example, the paper [7] proves the MSSP for a branching selection process whose MFL is described by an integro-differential equation with free left boundary; it proves, in particular, a traveling-wave existence result for all speeds greater than or equal to

a critical speed. We note that the auxiliary systems with moving left and right boundaries are substantially different from the moving free boundary arising in MFL dynamics in [7]: our boundaries are 'reflecting', rather than 'absorbing'.

The proofs of the fact that $\lim v_n = v_{**}$ in [7, 13] (see also references therein) rely on system monotonicity properties and on the relationship with a corresponding branching process, which is either a branching Brownian motion or a branching random walk (BRW). In fact, the critical speed v_{**} is the average speed of advance of the leading (rightmost) particle of the branching process. Our proof that $\limsup v_n = v_{**}$ is essentially the same as the proofs in [7, 13], except that the corresponding BRW in our case is different; our proof that $\liminf v_n = v_{**}$ is related to, but different from, the proof of the corresponding property in [7].

As we just mentioned, the critical speed v_{**} is the average speed of advance of the leading (rightmost) particle of the branching process. McKean [23] found that the BMFL for a KPP equation [15] is such that $(f_x(t), \ x \in \mathbb{R})$ is the distribution of the leading particle location for the associated branching Brownian motion with the initial particle location at 0; we need and provide an analogous characterization of the BMFL for our system in terms of the corresponding BRW. For a general discrete-time BRW, the average speed v_{**} of the leading particle (as well as that of the trailing particle) was found by Biggins [3] (who extended the results of Kingman [14]); the average speed v_{**} of the leading particle of the continuous-time BRW in our case follows from the results of [3].

The line of work in [9, 10, 16, 27] is concerned with the speed of the solutions (MFLs) of the KPP equation [15], depending on the initial condition or, more precisely, the exponent of the right tail of the initial condition. The key results show that any speed $v \geq v_{**}$ can be achieved. In [16] this is shown via direct analysis of the KPP solutions [15], while [9, 10, 27] use a Feynman–Kac representation of a solution and a large-deviations approach. We obtain analogous results for our system, under the additional assumption of exponential jumps, but our approach is completely different: we prove the existence of traveling waves, including waves with moving boundaries, and use these traveling waves as lower and upper bounds of MFLs. We note that our approach can be applied to the KPP solutions [15] as well, because the existence of traveling waves with moving boundaries for KPP can be obtained from [15] in the same way as we do for our model. In particular, we believe this gives alternative proofs of many of the MFL speed results in [9, 10, 16, 27], and it should be applicable to other models as well.

In terms of motivation, this paper is closely related to the work on models where a degree of 'synchronization' is desired, meaning that the collection of particles remains 'tight'; specifically, most particles remain within $O(1)$ distance of each other, uniformly in n. This includes the 'rank-based' model in [11, 12, 25, 26], where the instantaneous rate at which a particle jumps forward is a non-increasing function of the particle's location quantile within the empirical distribution of all of the particles' locations. The results of [11, 12, 25, 26] establish most of the properties of interest: convergence to an MFL as $n \to \infty$; convergence of an MFL to a traveling wave as time goes to infinity; and convergence of stationary distributions to a Dirac distribution concentrated on a traveling wave. Other models of distributed synchronization include those in, e.g., [1, 17–22]. The paper [1] considers a 'barycentric' model, where the instantaneous rate at which a particle jumps forward is a non-increasing function of the particle's location displacement from the mean of the empirical distribution of all of the particles' locations; it establishes (under certain conditions) the results on convergence to the MFL, and it finds traveling-wave forms in some special cases. From a technical point of view, our proofs of convergence to the MFL and its uniqueness closely follow the approach used in [1, 25].

The paper [26] employs an artificial system with moving 'reflecting' boundaries as a tool for analysis of the original system; this approach is one of the tools used in the present paper as well.

Finally, we note that the papers [5, 6] provided the first explanation of the phenomenon of 'slow convergence' of steady-state speeds, $v_n \to v_{**}$, for particle systems having KPP solutions as MFLs. The first formal proof of this phenomenon was given, for a closely related particle system, in [2]. Our simulations indicate that this phenomenon occurs in our system as well; proving this rigorously may be a subject of future work.

1.3. Outline of the rest of the paper

Section 2 gives basic notation used throughout the paper. In Section 3 we formally define the system process and describe its basic properties, while Section 4 specifically focuses on the important monotonicity properties. Section 5 presents our results on the convergence to—and properties of—mean-field limits (MFLs). Section 6 defines traveling waves. In Section 7 we formally define the critical speed v_{**} via a minimum speed selection principle (MSSP). In Section 8 we define the associated BRW, give a McKean-type characterization of the original system's BMFL in terms of this BRW, and show that v_{**} is equal to the average speed of the leading particle of the BRW; finally, we present the result that $\lim v_n = v_{**}$. In Section 9 we state the traveling-wave existence results in the special case of exponentially distributed jumps. In Section 10 we present the results on the MFL speeds: the lower bound v_{**}, which holds for general jump size distribution, and speed bounds for exponential-jumps case, which depend on the right tail exponent of the initial state. In Section 11 we state some further conjectures and present simulation results. Sections 12–20 contain proofs.

2. Basic notation

The set of real numbers is denoted by $\mathbb{R}$ and is viewed as the usual Euclidean space. As a measurable space, $\mathbb{R}$ is endowed with the Borel σ-algebra. For scalar functions $g(x)$ of a real x, we write $\|g\|_1 = \int_x |g(x)| dx$ for the L_1-norm; $g(x)$ is called c-Lipschitz if it is Lipschitz with constant $c \geq 0$. Let $\mathcal{C}_b$ be the set of continuous bounded functions on $\mathbb{R}$ which are constant outside a closed interval (with one constant value to the 'left' of the interval, and possibly another constant value to the 'right' of it.)

For functions $g(x)$ of a real x, we denote by $g(x+)$ and $g(x-)$ the right and left limits at x; a function $g(x)$ is *RCLL* if it is right-continuous and has left limits at each x.

A function g of x may be written as either $g(x)$ or g_x. The notation $d_x g(x, t)$ for a multivariate function $g(x,t)$, where $x \in \mathbb{R}$, means the differential in x.

Denote by $\mathcal{M}$ the set of scalar RCLL non-decreasing functions $f = (f(x), x \in \mathbb{R})$ that are (proper) probability distribution functions, i.e., such that $f(x) \in [0, 1]$, $\lim_{x\downarrow-\infty} f(x) = 0$, and $\lim_{x\uparrow\infty} f(x) = 1$. For elements $f \in \mathcal{M}$ we use the terms *distribution function* and *distribution* interchangeably. The space $\mathcal{M}$ is endowed with the Lévy–Prokhorov metric (cf. [8]) and the corresponding topology of weak convergence (which is equivalent to the convergence at every point of continuity of the limit); the weak convergence in $\mathcal{M}$ is denoted $\xrightarrow{w}$. Subspaces of $\mathcal{M}$ that we define later inherit its metric. The inverse (vth quantile) of $f \in \mathcal{M}$ is $f_v^{-1} \doteq \inf\{y \,|\, f_y \geq v\}$, $v \in [0, 1]$; $f_1^{-1} = \infty$ when $f_y < 1$ for all y; we will also use the notation $q_v(f) \doteq f_v^{-1}$. For a given $v \in (0, 1)$, set $\mathring{\mathcal{M}} = \{f \in \mathcal{M} \,|\, f_v^{-1} = 0\}$; the parameter v used in the definition of $\mathring{\mathcal{M}}$ will be clear from the context.

Unless explicitly specified otherwise, we use the following conventions regarding random elements and random processes. A measurable space is assumed to be equipped with the Borel σ-algebra induced by the metric, which is clear from the context. A random process $Y(t)$, $t \geq 0$, always takes values in a complete separable metric space (clear from the context) and has RCLL sample paths. For a random process $Y(t)$, $t \geq 0$, we denote by $Y(\infty)$ the random value of $Y(t)$ in a stationary regime (which will be clear from the context). The symbol $\Rightarrow$ signifies convergence of random elements in distribution; $\xrightarrow{\mathbb{P}}$ means convergence in probability. For a condition/event A, $\mathbf{I}\{A\} = 1$ if A holds, and $\mathbf{I}\{A\} = 0$ otherwise.

The space $D([0, \infty), \mathbb{R})$ (resp. $D([0, \infty), \mathcal{M})$) is the Skorokhod space of RCLL functions on $[0, \infty)$ taking values in $\mathbb{R}$ [resp. $\mathcal{M}$], with the corresponding Skorokhod (J_1) metric and topology (cf. [8]); the symbol $\xrightarrow{J_1}$ denotes convergence in these spaces.

For a distribution $f \in \mathcal{M}$ and scalar function $h(x)$, $x \in \mathbb{R}$, we define $fh \doteq \int_{\mathbb{R}} h(x) df_x$. In some cases it is important whether or not an integral includes the endpoints of the integration interval; $\int_{a+}^{b} h(x) df_x$ means that the integral excludes the left endpoint a, and $\int_{a}^{b+} h(x) df_x$ means that the integral includes the right endpoint b.

For scalar functions $h(x)$, $x \in \mathcal{X}$, with some domain $\mathcal{X}$, $\|h\| = \sup_{x \in \mathcal{X}} |h(x)|$ is the sup-norm. When G_k, G are operators mapping the space of such functions into itself, $\lim G_k h = Gh$ and $G_k h \to Gh$ mean uniform convergence: $\|G_k h - Gh\| \to 0$.

Suppose we have a Markov process with state space $\mathcal{X}$ and transition function $P^t(x, H)$, $t \geq 0$. Then P^t, as an operator, is $P^t h(x) \doteq \int_y P^t(x, dy) h(y)$, where h is a scalar function with domain $\mathcal{X}$; $I = P^0$ is the identity operator. The process (infinitesimal) generator B is

$$Bh \doteq \lim_{t \downarrow 0} \frac{1}{t} [P^t - I] h.$$

The function h is within the domain of the generator B if Bh is well-defined.

We denote by $\bar{J}(x) = 1 - J(x)$ the complementary cumulative distribution function of the jump size distribution. The Heaviside step function (or Dirac distribution concentrated at the point 0) is denoted by $\chi = (\chi_x = \mathbf{I}\{x \geq 0\}) \in \mathcal{M}$.

3. System process definition

We already noted that, without loss of generality, we may assume $\mathbb{E}Z = 1$; otherwise, this condition can be achieved by rescaling space. Note also that, without loss of generality, we can and do assume $\mu = 1$; otherwise, we can achieve this condition by rescaling time.

Let $f^n(t) = (f_x^n(t), \ x \in \mathbb{R})$ be the (random) empirical distribution of the particle locations at time t; that is, $f_x^n(t)$ is the fraction of particles located in $(-\infty, x]$ at time t. Clearly, $f^n(\cdot)$ is a Markov process with the state space $\mathcal{M}^{(n)} \subset \mathcal{M}$. Let us also consider a centered version of the process. Fix a number $\nu \in (0, 1)$, and let $\mathring{g}_x = g_{x+q_\nu(g)}$, $x \in \mathbb{R}$. That is, $\mathring{g}$ is a version of g that is centered at its νth quantile. Then $\mathring{f}^n(t)$—the centered version of $f^n(t)$—is also a Markov process, with state space $\mathring{\mathcal{M}}^{(n)} = \mathcal{M}^{(n)} \cap \mathring{\mathcal{M}}$. This process is regenerative, with the regenerations occurring at time points when all particles 'collapse' to one location. A regeneration cycle has finite mean; indeed, for any time interval of a fixed length $\varepsilon > 0$, with probability at least some $\delta = \delta(\varepsilon, n) > 0$, at the end of the interval all particles will be at a single location. Therefore the process is positive recurrent, with unique stationary distribution. Denote by $\mathring{f}^n(\infty)$ a random element with distribution equal to the stationary distribution of $\mathring{f}^n(\cdot)$.

Consider a version $f^n(\cdot)$ of the system process, such that the centered process $\mathring{f}^n(\cdot)$ is stationary. Specifically, consider the process $f^n(\cdot)$ with $f^n(0)$ equal in distribution to $\mathring{f}^n(\infty)$. Then, for each n, the steady-state average speed v_n of the particle system is finite and given by

$$v_n = \frac{1}{T}\mathbb{E}[q_\beta(f^n(t)) - q_\beta(f^n(0))],$$

where $\beta \in (0, 1)$ and $t > 0$ can be chosen arbitrarily. Also, v_n is the steady-state average speed of the center of mass, i.e. of the mean of the distribution $f^n(t)$:

$$v_n = \lambda + \mathbb{E}\int_{-\infty}^{\infty} d_y \mathring{f}_y^n(\infty) \int_y^{\infty} d_\xi \mathring{f}_\xi^n(\infty)(\xi - y).$$

Given the regenerative structure of the (centered) process, the steady-state speed v_n can also be defined in terms of probability-one convergence, regardless of the initial state. In particular, for any initial state of the system,

$$v_n = \lim_{t\to\infty} \frac{1}{t}[D^n(t) - D^n(0)], \quad \text{with probability 1},$$

where $D^n(t)$ is the location of the leading (rightmost) particle.

We remark that, trivially, $\lambda = 0$ implies $v_n = 0$, because then, with probability one, all particles will end up at a single location and will stop moving after that. Therefore, $v_n > 0$ if and only if $\lambda > 0$, and we have the trivial lower bound $v_n \geq \lambda$ for any n.

Recall that we assumed $\mu = 1$. If we do need expressions for a case when $\mu \neq 1$, they are obtained from the corresponding expressions for the $\mu = 1$ case. For example, if $v_n = v_n(\lambda, \mu)$ is the steady-state speed as a function of λ and $\mu > 0$, then $v_n(\lambda, \mu) = \mu v_n(\lambda/\mu, 1)$.

4. Monotonicity properties

The order relation (stochastic dominance) $g^{(1)} \preceq g^{(2)}$ between two distributions $g^{(1)}, g^{(2)} \in \mathcal{M}$ means $g_x^{(1)} \geq g_x^{(2)}$, $\forall x$.

The process $f^n(\cdot)$ possesses some simple monotonicity properties. If we have two versions of this process $f^n(\cdot)$, labeled $f^{n,(1)}(\cdot)$ and $f^{n,(2)}(\cdot)$, with fixed initial conditions $f^{n,(1)}(0) \preceq f^{n,(2)}(0)$, then these two processes can be coupled (constructed on a common probability space) so that $f^{n,(1)}(t) \preceq f^{n,(2)}(t)$ at all times $t \geq 0$. Indeed, it suffices to set a one-to-one correspondence between particles in the two systems, such that each particle in the second system in initially ahead of the corresponding particle in the first system, and consider a probability space such that each pair of corresponding particles has the same realizations of time instants and sizes of the independent jumps, as well as the same realizations of the time instants and 'targets' of the synchronization jumps. Then, clearly, each particle in the second system will remain ahead of its counterpart in the first system at all times.

This property easily generalizes to the case when the second ('larger') process has a larger number of particles, i.e. when we compare $f^{n,(1)}(\cdot)$ and $f^{k,(2)}(\cdot)$, where $n \leq k$. In this case, we will use the order relation $f^{n,(1)}(t) \preceq_l f^{k,(2)}(t)$, which denotes the situation when $f^{n,(1)}(t) \preceq f^{n(k),(2)}(t)$, where $f^{n(k),(2)}(t)$ is the empirical distribution of the leading n particles of the state $f^{k,(2)}(\cdot)$. Again, if $f^{n,(1)}(0) \preceq_l f^{k,(2)}(0)$, then the processes can be coupled so that $f^{n,(1)}(t) \preceq_l f^{k,(2)}(t)$ prevails at all times. This extension has been used in the literature in different contexts (cf. the proof of Theorem 2.3(4) in [13], and references therein), but is also easy to see directly.

Indeed, the synchronization jumps can equivalently be defined as follows: a particle gets a 'synchronization urge' as a Poisson process of rate μ (rescaled to 1), and it relocates another particle, chosen uniformly at random, to its own location if the chosen particle happens to be behind it. Given this interpretation, the coupling construction is straightforward.

Since the process is such that (a) for any initial state, $v_n = \lim_{t \to \infty} D^n(t)/t$ almost surely, where $D^n(t)$ is the location of the leading particle at time t, and (b) monotonicity with respect to $\preceq_l$ holds, we see that the steady-state speed v_n is non-decreasing in n:

$$v_n \leq v_{n+1}, \quad \forall n \geq 1. \tag{4.1}$$

(Again, this property was used in the proof of Theorem 2.3(4) in [13] for a different system, but it obviously holds for any system satisfying (a) and (b).)

The monotonicity with respect to $\preceq$ (and, more generally, $\preceq_l$) easily generalizes further in several directions. For example, it still holds if the 'larger' process $f^{k,(2)}(\cdot)$ has larger parameter λ, or if it is 'helped' by a left boundary (which can be static or moving in any way), or if the 'smaller' process $f^{n,(1)}(\cdot)$ is 'impeded' by a right boundary (which can be static or moving in any way), etc. We will not give formal statements for all these generalizations, and we will refer to the monotonicity properties described in this section simply as *monotonicity*.

5. Mean-field limits

In this section we state our results on the convergence to and characterization of MFLs. The results are for the original system, the system with fixed right boundary (which is primarily a tool for the analysis of the original system), and the two systems with moving right and left boundaries. We will use the same notation $f^n(\cdot)$ and $f(\cdot)$ for the process and an MFL, respectively, for each of the four systems. It should be clear that they do depend on the system, and when we use them later in the paper it will be clear from the context which system they refer to.

5.1. Original system

The following definition of a mean-field model describes what it is natural to expect an MFL to be. Theorem 1 then shows that MFLs this definition.

Definition 1. A function $f_x(t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, will be called a *mean-field model (MFM)* if it satisfies the following conditions:

(a) For any t, $f(t) = (f_x(t), x \in \mathbb{R}) \in \mathcal{M}$.

(b) For any x, $f_x(t)$ is non-increasing and $(\lambda + 1)$-Lipschitz in t.

(c) For any x, for any t where the partial derivative $(\partial/\partial t)f_x(t)$ exists (which is almost all t with respect to the Lebesgue measure, by the Lipschitz property), the equation

$$\frac{\partial}{\partial t}f_x(t) = -\lambda \int_{-\infty}^{x} d_y f_y(t)(1 - J(x - y)) - f_x(t)(1 - f_x(t)). \tag{5.1}$$

holds.

Denote by $L^{(n)}$ the generator of the process $f^n(\cdot)$. For any $h \in \mathcal{C}_b$, the function $f^n h$ of an element $f^n \in \mathcal{M}^{(n)}$ is within the domain of $L^{(n)}$ (where we use the fact that each function in $\mathcal{C}_b$ is constant outside a closed interval), and

$$L^{(n)}[f^n h] = \lambda \mathbb{E}_Z \int_{-\infty}^{\infty} d_y f_y^n(h(y + Z) - h(y)) + \int_{-\infty}^{\infty} d_y f_y^n \int_y^{\infty} d_\xi f_\xi^n(h(\xi) - h(y)),$$

where the expectation is over the distribution of the random jump size Z. We also formally define the 'limit' L of $L^{(n)}$ (for elements $f \in \mathcal{M}$) as

$$L[fh] = \lambda \mathbb{E}_Z \int_{-\infty}^{\infty} d_y f_y(h(y + Z) - h(y)) + \int_{-\infty}^{\infty} d_y f_y \int_y^{\infty} d_\xi f_\xi(h(\xi) - h(y)).$$

Theorem 1. *Suppose $f^n(0) \overset{w}{\to} f(0)$, where $\{f^n(0)\}$ is a deterministic sequence of elements $f^n(0) \in \mathcal{M}^{(n)}$, and $f(0) \in \mathcal{M}$. Then $f^n(\cdot) \Rightarrow f(\cdot)$ in $D([0, \infty), \mathcal{M})$, where $f(\cdot) \in D([0, \infty), \mathcal{M})$ is deterministic and uniquely determined by f(0). Moreover, $f(\cdot)$ is a continuous element of $D([0, \infty), \mathcal{M})$ and satisfies*

$$f(t)h - f(0)h - \int_0^t Lf(s)h ds = 0, \quad \forall h \in \mathcal{C}_b, \ \forall t \geq 0. \tag{5.2}$$

The trajectory $f(\cdot)$ is called the mean-field limit (MFL) *with initial state f(0). Furthermore, the MFL $f(\cdot)$ is the unique MFM (see Definition 1) with initial state f(0).*

The proof of Theorem 1 is in Section 13.

The MFL $f(\cdot)$ with initial state $f(0) = \chi$ (i.e., the Dirac distribution concentrated at a single point 0) will be called the *benchmark MFL (BMFL)*.

5.2. System with fixed right boundary

As an intermediate step towards the proof of Theorem 1, we will prove a special case of it, Theorem 2 below, which applies to the system with fixed right boundary, defined as follows.

Note that, for any fixed B, the evolution of $f_x^n(\cdot)$, $x \in (-\infty, B)$ in our original system is independent of the actual locations and/or evolution of particles in $[B, \infty)$. Therefore, if we are only interested in the evolution of $f_x^n(\cdot)$, $x \in (-\infty, B)$, we may as well assume that any particle in $[B, \infty)$ is located exactly at B (that is, $f_B^n(t) = 1$ at all times). In other words, we can assume that there is a fixed right boundary point B, and any particle which 'tries to jump over B' lands (absorbed) at exactly B. This is the modified system (with fixed right boundary) that we consider here. We see that the process $f^n(\cdot)$ for this system is a projection of the original process—it is such that $f^n(t) \in \mathcal{M}^B \doteq \{g \in \mathcal{M} \mid g_B = 1\}$ for all t.

Denote by $L^{(n),B}$ the generator of the process $f^n(\cdot)$ for this system. For any $h \in \mathcal{C}_b$, the function $f^n h$ of an element $f^n \in \mathcal{M}^{(n)} \cap \mathcal{M}^B$ is within the domain of $L^{(n),B}$, and

$$L^{(n),B}[f^n h] = \lambda \mathbb{E} \int_{-\infty}^{B} d_y f_y^n(h((y + Z) \vee B) - h(y)) + \int_{-\infty}^{B} d_y f_y^n \int_y^{B+} d_\xi f_\xi^n(h(\xi) - h(y)),$$

where the expectation is over the distribution of the random jump size Z. We also formally define the 'limit' L^B of $L^{(n),B}$ (for elements $f \in \mathcal{M} \cap \mathcal{M}^B$) as

$$L^B[fh] = \lambda \mathbb{E} \int_{-\infty}^{B} d_y f_y(h((y + Z) \vee B) - h(y)) + \int_{-\infty}^{B} d_y f_y \int_y^{B+} d_\xi f_\xi(h(\xi) - h(y)).$$

Observe the following. For a fixed $h \in \mathcal{C}_b$, if B is large enough so that $h(x)$ is constant for all $x \geq B$, then $L^{(n),B}[f^n h] = L^{(n)}[f^n h]$ and $L^B[f^n h] = L[f^n h]$, where $L^{(n)}$ and L are the operators we defined for the original system.

Theorem 2. *Suppose $f^n(0) \xrightarrow{w} f(0)$, where $\{f^n(0)\}$ is a deterministic sequence of elements $f^n(0) \in \mathcal{M}^{(n)} \cap \mathcal{M}^B$, and $f(0) \in \mathcal{M}^B$. Then $f^n(\cdot) \Rightarrow f(\cdot)$ in $D([0, \infty), \mathcal{M}^B)$, where $f(\cdot) \in D([0, \infty), \mathcal{M}^B)$ is deterministic and uniquely determined by $f(0)$. Moreover, $f(\cdot)$ is a continuous element of $D([0, \infty), \mathcal{M}^B)$ which satisfies*

$$f(t)h - f(0)h - \int_0^t L^B f(s) h \, ds = 0, \quad \forall h \in \mathcal{C}_b, \ \forall t \geq 0. \tag{5.3}$$

This trajectory $f(\cdot)$ is called the mean-field limit (MFL) *with initial state $f(0)$. Furthermore, the MFL $f(\cdot)$ is the unique MFM with initial state $f(0)$; the MFM here is defined as in Definition 1 except that (5.1) holds for $x < B$, and $f_x(t) \equiv 1$ for $x \geq B$ and all t.*

The proof of Theorem 2 is in Section 12.

5.3. System with moving right boundary

Consider now the system with the right boundary point B moving to the right at constant speed $v > 0$, i.e. $B = B_0 + vt$ for some constant B_0. Just as in the system with fixed right boundary, any particle which 'tries to jump over the boundary' lands exactly at its current location. However, particles landing at the right boundary are not 'absorbed' in it, as the boundary keeps moving right while the particles do not move until they jump again. If B is the boundary location at time t, then the state $f^n(t) \in \mathcal{M}^B$. Formally speaking, the state of the Markov process has an additional component corresponding to the real number B; somewhat abusing notation, we will not include this component in the state descriptor.

It is important to note that the process with moving right boundary is *not* a projection of the original process.

Denote by $L^{(n),\{r\}}$ the generator of the process $f^n(\cdot)$ for this system. For any $h \in \mathcal{C}_b$, the function $f^n h$ of an element f^n (or rather pair (f^n, B)) satisfying $f^n \in \mathcal{M}^{(n)} \cap \mathcal{M}^B$ is within the domain of $L^{(n),\{r\}}$, and

$$L^{(n),\{r\}}[f^n h] = \lambda \mathbb{E} \int_{-\infty}^B d_y f_y^n (h((y+Z) \vee B) - h(y)) + \int_{-\infty}^B d_y f_y^n \int_y^{B+} d_\xi f_\xi^n (h(\xi) - h(y)),$$

where the expectation is over the distribution of the random jump size Z. We also formally define the 'limit' $L^{\{r\}}$ of $L^{(n),\{r\}}$ (for pairs (f,B) satisfying $f \in \mathcal{M}^B$) as

$$L^{\{r\}}[fh] = \lambda \mathbb{E} \int_{-\infty}^B d_y f_y (h((y+Z) \vee B) - h(y)) + \int_{-\infty}^B d_y f_y \int_y^{B+} d_\xi f_\xi (h(\xi) - h(y)).$$

The operators $L^{(n),\{r\}}$ and $L^{\{r\}}$ are the same as the operators $L^{(n),B}$ and L^B, respectively, for the system with fixed right boundary at B. Here, however, B is part of the process state, rather than a fixed constant.

Theorem 3. *Let $v > 0$ and B_0 be fixed, and recall that the boundary is given by $B = B_0 + vt$. Suppose $f^n(0) \xrightarrow{w} f(0)$, where $\{f^n(0)\}$ is a deterministic sequence of elements $f^n(0) \in \mathcal{M}^{(n)} \cap \mathcal{M}^{B_0}$, and $f(0) \in \mathcal{M}^{B_0}$. Then $f^n(\cdot) \Rightarrow f(\cdot)$ in $D([0, \infty), \mathcal{M})$, where $f(\cdot) \in D([0, \infty), \mathcal{M})$ is*

deterministic and uniquely determined by f(0); $f^n(t) \in \mathcal{M}^{(n)} \cap \mathcal{M}^B$ and $f(t) \in \mathcal{M}^B$ for all t. Moreover, $f(\cdot)$ is a continuous element of $D([0, \infty), \mathcal{M})$ which satisfies

$$f(t)h - f(0)h - \int_0^t L^{\{r\}} f(s) h ds = 0, \quad \forall h \in \mathcal{C}_b, \ \forall t \geq 0. \tag{5.4}$$

This trajectory $f(\cdot)$ is called the mean-field limit (MFL) *with initial state f(0). Furthermore, MFL $f(\cdot)$ is the unique MFM with initial state f(0); the MFM here is defined as in Definition* 1 *except that* (5.1) *holds for $x < B(= B_0 + vt)$.*

The proof of Theorem 3 is in Section 14.

5.4. System with moving left boundary

Consider yet another modification of our original system, namely the system with left boundary A moving right at constant speed $v > 0$, i.e. $A = A_0 + vt$ for some constant A_0. In this system there are no particles to the left of A—as the boundary moves right it 'drags forward' any particle that it encounters, and each such particle stays at the (moving) boundary until it jumps forward. The corresponding process is *not* a projection of our original process.

By the definition of the process, there are no particles located to the left of the moving boundary A. However, for the purposes of analysis it will be convenient to adopt an equivalent view of the process which allows particles to be to the left of A. Specifically, assume that particles make synchronization jumps as in the original system, but when a particle located at y makes an independent jump, it first instantly moves to the point $y \vee A$ and then jumps. Clearly, this process is such that the evolution in time of $(f_x^n(t), x \geq A)$ is exactly the same as it would be if the moving left boundary dragged the particles that it encountered with it. So this is how we will define the process $f^n(\cdot)$ for this system. Then $f^n(t) \in \mathcal{M}^{(n)}$ for all t, and the boundary location A is implicitly a part of the definition of the process.

Denote by $L^{(n),\{l\}}$ the generator of the process $f^n(\cdot)$. For any $h \in \mathcal{C}_b$, the function $f^n h$ of an element f^n (or rather pair (f^n, A)) satisfying $f^n \in \mathcal{M}^{(n)}$ is within the domain of $L^{(n),\{r\}}$, and

$$L^{(n),\{l\}}[f^n h] = \lambda \mathbb{E} \int_{-\infty}^{\infty} d_y f_y^n (h((y \vee A) + Z) - h(y)) + \int_{-\infty}^{\infty} d_y f_y^n \int_y^{\infty} d_\xi f_\xi^n (h(\xi) - h(y)),$$

where the expectation is over the distribution of the random jump size Z. We also formally define the 'limit' $L^{\{l\}}$ of $L^{(n),\{l\}}$ (for pairs (f, A) with $f \in \mathcal{M}$) as

$$L^{\{l\}}[f^n h] = \lambda \mathbb{E} \int_{-\infty}^{\infty} d_y f_y (h((y \vee 0) + Z) - h(y)) + \int_{-\infty}^{\infty} d_y f_y \int_y^{\infty} d_\xi f_\xi (h(\xi) - h(y)).$$

Theorem 4. *Let $v > 0$ and A_0 be fixed, and recall that the boundary is given by $A = A_0 + vt$. Suppose $f^n(0) \xrightarrow{w} f(0)$, where $\{f^n(0)\}$ is a deterministic sequence of elements $f^n(0) \in \mathcal{M}^{(n)}$, and $f(0) \in \mathcal{M}$. Then $f^n(\cdot) \Rightarrow f(\cdot)$ in $D([0, \infty), \mathcal{M})$, where $f(\cdot) \in D([0, \infty), \mathcal{M})$ is deterministic and uniquely determined by f(0). Moreover, $f(\cdot)$ is a continuous element of $D([0, \infty), \mathcal{M})$ which satisfies*

$$f(t)h - f(0)h - \int_0^t L^{\{l\}} f(s) h ds = 0, \quad \forall h \in \mathcal{C}_b, \ \forall t \geq 0. \tag{5.5}$$

This trajectory f(·) is called the mean-field limit (MFL) *with initial state f(0). Furthermore, MFL f(·) is the unique MFM with initial state f(0); the MFM here is defined as in Definition* 1 *except that* (5.1) *is replaced by*

$$\frac{\partial}{\partial t} f_x(t) = -\lambda f_x(t) - f_x(t)(1 - f_x(t)), \ x \le A, \tag{5.6}$$

$$\frac{\partial}{\partial t} f_x(t) = -\lambda f_A(t)(1 - J(x - A)) - \lambda \int_A^x d_y f_y(t)(1 - J(x - y)) - f_x(t)(1 - f_x(t)), \ x > A. \tag{5.7}$$

The proof of Theorem 4 is in Section 15.

5.5. Monotonicity of MFLs

The monotonicity properties described in Section 4, as well as the uniqueness of the MFL for each initial state, imply the following monotonicity property of MFLs.

Lemma 1. *For two MFLs $f^{(1)}(\cdot)$ and $f^{(2)}(\cdot)$, $f^{(1)}(0) \preceq f^{(2)}(0)$ implies $f^{(1)}(t) \preceq f^{(2)}(t)$ for all $t \ge 0$.*

Just as the monotonicity properties of the process continue to hold for various generalizations (see Section 4), Lemma 1 holds under those generalizations as well.

6. MFLs that are traveling waves

An MFL (MFM) $f(\cdot)$ for our original system is a *traveling wave* if $f_x(t) = \phi_{x-vt}$ for some speed $v > 0$ and some $\phi = (\phi_x, x \in \mathbb{R}) \in \mathcal{M}$, in which case we call ϕ a *traveling-wave shape (TWS)*. Substituting $f_x(t) = \phi_{x-vt}$ into (5.1), we can easily see that a TWS, *if it exists*, must satisfy

$$v\phi_x' = \lambda \int_{-\infty}^x \phi_y'(1 - J(x - y))dy + \phi_x(1 - \phi_x) \tag{6.1}$$

for each x, and in fact the derivative ϕ_x' must be continuous in x.

Since a traveling wave is an MFL (which is a limit of particle system dynamics), we see that if a TWS with speed v exists, then necessarily $v \ge \lambda$.

In the special case when $\lambda = 0$, the integro-differential equation (6.1) becomes the logistic differential equation

$$v\psi_x' = \psi_x(1 - \psi_x). \tag{6.2}$$

In this case, a unique (up to a shift) TWS does exist for every speed $v > 0$. This follows, for example, from [22, Proposition 5.1], but of course it is well known: the relevant solution to (6.2) is

$$\psi_x = \frac{1}{1 + e^{-(1/v)(x+c)}}, \ x \in \mathbb{R}, \ \text{where } v > 0 \text{ and } c \in \mathbb{R} \text{ are parameters.} \tag{6.3}$$

(There is no contradiction here with the fact that for any *finite n*, $\lambda = 0$ implies that the steady-state speed $v_n = 0$. Traveling waves are MFLs, which means that we first take the $n \to \infty$ limit and then look at the time-evolution of the limit. So, if the time interval $[0, T]$ is fixed, n is very large, and the initial state $f^n(0)$ is close to ψ in (6.3), then the evolution of $f_x^n(t)$ in $[0, T]$ is close to the traveling wave ψ_{x-vt} moving at speed v. This does not contradict the fact that, eventually, all of the particles will assemble at the location of the leading particle and will stop moving, i.e. $v_n = 0$.)

In the special case of exponential jump size distribution, $J(x) = 1 - e^{-x}$, for any $\lambda \geq 0$, the integro-differential equation (6.1) becomes an ODE

$$v\phi''_x = (1 + \lambda - v - 2\phi_x)\phi'_x + \phi_x(1 - \phi_x). \tag{6.4}$$

Indeed, in this case (6.1) can be written as

$$v\phi'_x e^x = \lambda \int_{-\infty}^x \phi'_y e^y dy + \phi_x(1 - \phi_x)e^x,$$

from which, after differentiating in x, we obtain (6.4).

We will also consider MFLs that are traveling waves for the modified systems with moving right and left boundaries; $v > 0$ is the speed of the boundary. Clearly, in a system with a boundary moving at speed v, the speed of any traveling wave is also v. For both systems, for a given boundary speed $v > 0$, an MFL $f(\cdot)$ is a traveling wave (of speed v) if $f_x(t) = \phi_{x-vt}$ for some $\phi = (\phi_x, x \in \mathbb{R}) \in \mathcal{M}$, where ϕ is called a TWS for the corresponding system; without loss of generality we can assume that the initial location of the boundary is 0.

Substituting $f_x(t) = \phi_{x-vt}$ into (5.1), we see that a TWS ϕ for the system with moving right boundary, *if it exists*, must be continuous with $\phi_0 = 1$ and must satisfy (6.1) for each $x < 0$; in fact, the derivative ϕ'_x must be continuous in $(-\infty, 0]$ (if at $x = 0$ we consider the left derivative). In the special case of $J(x) = 1 - e^{-x}$, (6.1) reduces to (6.4).

Substituting $f_x(t) = \phi_{x-vt}$ into (5.7), we obtain that a TWS ϕ for the system with moving left boundary, *if it exists*, must be continuous for $x \in [0, \infty)$ and must satisfy

$$v\phi'_x = \lambda\phi_0(1 - J(x)) + \lambda \int_{0+}^x \phi'_y(1 - J(x - y))dy + \phi_x(1 - \phi_x) \tag{6.5}$$

for each $x > 0$; in fact, the derivative ϕ'_x must be continuous in $[0, \infty)$ (if at $x = 0$ we consider the right derivative). Note that the initial condition $v\phi'_0 = \lambda\phi_0 + \phi_0(1 - \phi)$ must hold. In the special case of $J(x) = 1 - e^{-x}$, (6.5) reduces to (6.4).

7. Definition of the critical speed v_{**} via an MSSP

In this section we formally define the critical value v_{**} of the speed, which plays a central role in our results and analysis. Assume $\lambda > 0$. Denote by $L(s) = \int_0^\infty e^{-sx} dJ(x)$ the Laplace transform of the jump distribution $J(\cdot)$. As a function of real s it is, of course, a convex non-negative non-increasing function. Denote by α the tail exponent of $J(\cdot)$:

$$\alpha \doteq \liminf_{x \to \infty} \frac{-\log(1 - J(x))}{x} = \sup\{\zeta \geq 0 \mid L(-\zeta) < \infty\} \in [0, \infty]. \tag{7.1}$$

Suppose $\alpha > 0$ (with the case $\alpha = \infty$ allowed). Then $L(s) < +\infty$ for $s > -\alpha$, $L(s) = +\infty$ for $s < -\alpha$, and $L(s) \uparrow L(-\alpha)$ as $s \downarrow -\alpha$; $L(-\alpha)$ may be finite or infinite, but it is necessarily infinite when $\alpha = \infty$.

The following heuristic argument, leading to (7.3), is analogous to the one in, e.g., [6], where it is applied to the KPP equation (1.1). This heuristic argument serves as a motivation for the *rigorous* definition of the speed value v_{**}. Assume that a TWS $\phi = (\phi_x, x \in \mathbb{R})$ with speed $v > 0$ exists, and that the front tail of ϕ decays exponentially: $1 - \phi_x \sim e^{-\zeta x}$ as $x \to \infty$, where $\zeta > 0$. Then, when x is large, Equation (6.1) 'becomes'

$$v\zeta e^{-\zeta x} = \lambda \int_{-\infty}^x \zeta e^{-\zeta y} \bar{J}(x - y)dy + e^{-\zeta x}. \tag{7.2}$$

In (7.2),

$$\int_{-\infty}^{x} e^{-\zeta y} \bar{J}(x-y)dy = e^{-\zeta x} \int_{0}^{\infty} e^{\zeta \xi} \bar{J}(\xi)d\xi = \frac{1}{\zeta} e^{-\zeta x}[L(-\zeta)-1],$$

so we obtain the following dependence of the speed v on $\zeta > 0$:

$$v(\zeta) = \frac{1}{\zeta}[\lambda L(-\zeta) - \lambda + 1]. \tag{7.3}$$

The relation (7.3), viewed formally, is a starting point for the following rigorous definitions and observations. The dependence $v(\zeta)$ is convex, because $(1/\zeta)[L(-\zeta)-1]$ is the Laplace transform of function $\bar{J}(\cdot)$ at the point $s = -\zeta$. Note also that $v(\zeta) \to +\infty$ as $\zeta \downarrow 0$. It is easy to see that the minimum value v_{**} of $v(\zeta)$ is attained at the unique positive finite value

$$\zeta_{**} \doteq \arg\min_{\zeta > 0} v(\zeta) \in (0, \alpha], \quad \text{so that} \quad v_{**} \doteq v(\zeta_{**}) = \min_{\zeta > 0} v(\zeta). \tag{7.4}$$

Because $\zeta_{**} \in (0, \alpha]$ and $L(-\zeta) > 1$ for $\zeta > 0$, note from (7.3) that

$$v_{**} > 1/\alpha. \tag{7.5}$$

If $\zeta_{**} < \alpha$, then it necessarily solves the equation

$$\lambda \zeta_{**} L'(-\zeta_{**}) + \lambda L(-\zeta_{**}) - \lambda + 1 = 0; \tag{7.6}$$

and if $L(-\alpha) = \infty$, then necessarily $\zeta_{**} < \alpha$. Note for future reference that the inverse function to $v(\zeta)$, $0 < \zeta \leq \zeta_{**}$, is a continuous strictly decreasing convex function $\zeta(v)$, $v_{**} \leq v < \infty$, with $\zeta(v) \downarrow 0$ as $v \uparrow \infty$.

In the special case $\bar{J}(x) = e^{-x}$, we have $\alpha = 1$, (7.3) becomes

$$v(\zeta) = \frac{\lambda}{1-\zeta} + \frac{1}{\zeta}, \tag{7.7}$$

and then

$$\zeta_{**} = \zeta_{*} \doteq \frac{1}{1+\sqrt{\lambda}}, \quad v_{**} = v_{*} \doteq (1+\sqrt{\lambda})^2.$$

The inverse function $\zeta(v)$, $v_{*} \leq v < \infty$, of the function $v(\zeta)$, $0 < \zeta \leq \zeta_{*}$, in this case is

$$\zeta(v) = \frac{(1+v-\lambda) - \sqrt{(1+v-\lambda)^2 - 4v}}{2v}. \tag{7.8}$$

Recall that the above definition of v_{**} assumes $\alpha > 0$. In view of (7.5), we adopt the convention that $v_{**} = +\infty$ when $\alpha = 0$. All results in this paper involving the critical speed v_{**} are valid for the case when $\alpha = 0$ and $v_{**} = +\infty$. They follow easily from the corresponding results for the case when $\alpha > 0$ and $v_{**} < \infty$, thanks to monotonicity (Section 4). Specifically, if $\alpha = 0$, then we can compare the actual process to an auxiliary one with the jump size distribution slightly changed so that it is stochastically smaller, with an arbitrarily small positive tail exponent α. The auxiliary process, which is stochastically dominated by the actual one, can have an arbitrarily large value of v_{**}. Thus, the corresponding results for the actual process hold with $v_{**} = \infty$.

8. Limit of steady-state speeds and the associated BRW

8.1. McKean-type characterization of the original system's BMFL

Theorem 5 (presented in this section) and its proof are analogous to the corresponding result and its proof in [23], for the system in which the independent movement of each particle is a Brownian motion.

Denote by $W(t)$, $t \geq 0$, a single-particle independent-jump (Markov) process, starting at some point y at initial time $t = 0$. Specifically, $W(t)$ is the location at time t of a particle that starts at 0 and makes independent and identically distributed jumps, distributed as $J(\cdot)$, at time points of a Poisson process of rate $\lambda > 0$.

Consider the following process, which we will call the *associated (with our particle system) branching random walk (BRW)*. The process starts with a single particle, located initially at 0. Its location $W_1(t)$ evolves as a single-particle independent-jump process. Any particle present in the system generates a new particle in the same location (i.e. splits into two) according to a unit-rate Poisson process. Each newly created particle continues as a single-particle independent-jump process. We label particles in the order of their creation, so that at time t their locations are $W_1(t), \ldots, W_{N(t)}(t)$, where $N(t)$ is the total number of particles. Let $D(t) \doteq \max_i W_i(t)$ be the location of the rightmost particle of the associated BRW.

Theorem 5. *The associated BRW is such that*

$$\mathbb{P}\{D(t) \leq x\} = f_x(t),$$

where f(t) is the BMFL.

The proof of Theorem 5 is in Section 16.

8.2. The value v_{**} as the limiting average speed of the leading particle of the associated BRW

The following proposition can be obtained from a general result on the average speed of progress of a leading particle of a BRW in *discrete time*, namely Theorem 3 in [3].

Proposition 1. *Consider the associated BRW, starting with one particle. Let D(t) be the location of the leading particle at time t. Then*

$$D(t)/t \to v_{**}, \quad \text{with probability one.} \tag{8.1}$$

Indeed, the process can viewed as follows. Each particle already present in the process 'waits' for an independent, exponentially distributed time with mean $1/(1 + \lambda)$, and then either jumps forward (with probability $\lambda/(1 + \lambda)$) or splits into two particles at the same location (with probability $1/(1 + \lambda)$). Suppose we discretize time, with the time step being some small $\delta > 0$. We obtain a process 'slower' than the original one if we assume that the time τ a particle 'waits' until the next event is geometric, with $\mathbb{P}\{\tau = j\delta\} = \varepsilon(1 - \varepsilon)^{j-1}$, $j = 1, 2, \ldots$, $\varepsilon = 1 - \exp(-(1 + \lambda)\delta)$. The particles present at time $n\delta$ represent the nth generation of the particles. The process is such that, after each time step, a particle has either one descendant in the same location (with probability $1 - \varepsilon$), or two descendants in the same location (with probability $\varepsilon/(1 + \lambda)$), or one descendant located at the current particle location $+ Z$. Theorem 3 in [3] can be applied to this discrete-time process to obtain the leading particle's average speed of progress, which is a lower bound on the speed of progress for the original process. To obtain a discrete-time process which is 'faster' than the original one, we assume that the time τ a

particle 'waits' until the next event is geometric, with $\mathbb{P}\{\tau = j\delta\} = \varepsilon(1 - \varepsilon)^j, j = 0, 1, 2, \ldots$. (A subtlety here is that $\tau = 0$ with non-zero probability, so that the distribution of the set of descendants after one time step, i.e. after time δ, has to account for that.) Applying Theorem 3 in [3] to this discrete-time process, we obtain an upper bound on the leading particle's average speed of progress for the original process. Letting $\delta \downarrow 0$, we can check that both bounds converge to exactly v_{**}.

8.3. Limit of steady-state speeds

Theorem 6. *For the steady-state speeds* v_n *we have* $v_n \leq v_{n+1} \leq v_{**}$ *for all* *n*, *and* $\lim_n v_n = v_{**}$.

The proof of Theorem 6 is in Section 17.

9. TWSs in the case of exponentially distributed independent jumps

Theorem 7. *Consider the special case* $J(x) = 1 - e^{-x}$ *and recall the notation* $v_* = (1 + \sqrt{\lambda})^2$. *Assume* $\lambda > 0$. *Then the following hold:*

    (i) A TWS ϕ *for the original system exists if and only if* $v \geq v_*$. *If* $v > v_*$, *then*

$$-\lim_{x \to \infty} [\log(1 - \phi_x)]/x = \zeta(v), \tag{9.1}$$

where $\zeta(v)$ *is defined in* (7.8).

    (ii) For any $v > v_*$ *there exists a TWS* ϕ *for the system with moving left boundary at speed v, and it is such that* (9.1) *holds.*

    (iii) For any $v < v_*$ *there exists a unique TWS* ϕ *for the system with moving right boundary at speed v.*

The proof of Theorem 7 is in Section 18.

10. MFL speed results

The following is a natural and standard notion of the average speed of a given MFL. We say that the average speed of an MFL $f(t)$ with a given initial state $f(0) \in \mathcal{M}$ is lower-bounded (upper-bounded) by v if, for any quantile $v \in (0, 1)$ of $f(t)$, its average speed of progress is lower-bounded (upper-bounded) by v; that is,

$$\liminf_{t \to \infty} (\limsup_{t \to \infty}) \frac{1}{t}[q_v(f(t)) - q_v(f(0))] \leq (\geq) v.$$

An equivalent definition of a lower (upper) speed bound v is that for any $\varepsilon > 0$,

$$\lim_{t \to \infty} f_{(v-\varepsilon)t}(t) = 0 \quad \left(\lim_{t \to \infty} f_{(v+\varepsilon)t}(t) = 1\right).$$

The speed v is the average speed of an MFL if it is both a lower and an upper bound.

Clearly an MFL's average speed depends on the initial state, and it can be arbitrarily large. (This follows from the existence of a traveling wave with $\lambda = 0$ and arbitrarily large speed v.)

As a corollary of Theorem 5 and Proposition 1 we obtain the following result.

Proposition 2. *The average speed of the BMFL is* v_{**}.

Proposition 2 in turn is used to prove (in Section 19) the following theorem.

Theorem 8. *The average speed of any MFL is lower-bounded by* v_{**}.

In cases when one can establish the existence of traveling waves, including traveling waves with moving boundaries—as we did in the case of exponential jump sizes—those traveling waves can be used as lower and/or upper bounds for MFLs to obtain more precise MFL speed bounds. Specifically, we can use the following fact, which is a corollary of Lemma 1.

Proposition 3. *Consider an MFL $f(\cdot)$. If there exists a traveling wave $\phi(\cdot)$ with speed v, and possibly with a lower (resp. upper) boundary with speed v, such that $f(0) \preceq \phi(0)$ (resp. $\phi(0) \preceq f(0)$), then $f(t) \preceq \phi(t)$ (resp. $\phi(t) \preceq f(t)$) at all times, and consequently the MFL speed is lower-bounded (resp. upper-bounded) by v.*

Theorem 9. *Assume $J(x) = 1 - e^{-x}$.*

(i) The average speed of the BMFL is v_*, and the average speed of any MFL is lower-bounded by v_*.

(ii) If the right tail exponent of an MFL $f(\cdot)$ with initial state f(0) is lower-bounded by $\zeta \leq \zeta_*$, i.e. $\liminf_{x\to\infty} -\log(1-f_x(0))/x \geq \zeta$, then the MFL's average speed is upper-bounded by $v(\zeta)$. Consequently, if $\liminf_{x\to\infty} -\log(1-f_x(0))/x \geq \zeta_*$, then the MFL's average speed is v_*.

(iii) If the right tail exponent of an MFL $f(\cdot)$ with initial state f(0) is upper-bounded by $\zeta \leq \zeta_*$, i.e. $\limsup_{x\to\infty} -\log(1-f_x(0))/x \leq \zeta$, then the MFL's average speed is lower-bounded by $v(\zeta)$.

The proof of Theorem 9 is in Section 20. This proof does *not* rely on Proposition 2, which in turn relies on the connection to the associated BRW and its properties, as described in Theorem 5 and Proposition 1. It relies only on Theorem 7 and Proposition 3. (This is why we include the statement (i) in Theorem 9, even though it is a special case of Proposition 2 and Theorem 8. We give an alternative proof of (i).)

The proof of Theorem 9 shows how MFL speed bounds can be obtained directly from the existence and properties of traveling waves. In our case these are given by Theorem 7. However, the proof is quite generic. For example, some of the MFL speed bounds for the KPP model under certain initial conditions, as obtained in [9, 10, 16, 27] via direct analysis or via Feynmann–Kac representations and large-deviations analysis, can be instead obtained in essentially the same way as our Theorem 9. Indeed, given the traveling-wave existence results for the KPP model [15], it is straightforward to obtain an analogue of our Theorem 7, namely the existence of traveling waves with moving left and right boundaries for speeds larger and smaller, respectively, than the critical speed. Then an analogue of our Theorem 9 follows, with essentially the same proof.

11. Conjectures, simulation results, and discussion

11.1. Conjectures

We have proved that the limit of the steady-state speeds, $\lim_{n\to\infty} v_n = v_{**}$, holds for our model with independent jump sizes having general distributions. However, we proved Theorem 7 and Theorem 9(ii)–(iii) only for exponential jumps. We conjecture that analogues of Theorem 7 and Theorem 9(ii)–(iii) in fact hold for generally distributed jump sizes as well. We also conjecture that stationary distributions of centered processes converge to the Dirac measure concentrated on a TWS. Formally, we put forward the following conjecture.

Conjecture 1. *Suppose the jump size distribution $J(\cdot)$ is such that $\alpha > 0$. Then the following hold:*

TABLE 1. Exponential jump sizes, $n = 10000$, $2\lambda + \mu = 1$. Steady-state speed v_n (simulated) and $v_{**} = \lim_n v_n$. (For exponential jump sizes, $v_{**} = (\sqrt{\lambda} + \sqrt{\mu})^2$.) The pair $(\lambda_{opt}, \mu_{opt})$ maximizes v_{**}.

λ	μ	v_n (simulation)	$v_{**} = (\sqrt{\lambda} + \sqrt{\mu})^2$
0.45	0.1	0.9321	0.974264069
0.4	0.2	1.0863	1.165685425
0.35	0.3	1.2104	1.29807407
0.3	0.4	1.2974	1.392820323
0.25	0.5	1.3236	1.457106781
0.2	0.6	1.3318	1.492820323
$\lambda_{opt} = 1/6$	$\mu_{opt} = 2/3$	1.3566	1.5
0.15	0.7	1.3071	1.49807407
0.1	0.8	1.2206	1.465685425
0.05	0.9	1.0567	1.374264069

(i) A unique TWS exists for any speed $v \geq v_{**}$, and a TWS does not exist for speeds $v < v_{**}$.

(ii) If the right tail exponent of an MFL $f(\cdot)$ with initial state $f(0)$ is lower-bounded by $\zeta \leq \zeta_{**}$, i.e. $\liminf_{x \to \infty} -\log(1 - f_x(0))/x \geq \zeta$, then the MFL's average speed is upper-bounded by $v(\zeta)$. Consequently, if $\liminf_{x \to \infty} -\log(1 - f_x(0))/x \geq \zeta_{**}$, then the MFL's average speed is v_{**}.

(iii) If the right tail exponent of an MFL $f(\cdot)$ with initial state $f(0)$ is upper-bounded by $\zeta \leq \zeta_{**}$, i.e. $\limsup_{x \to \infty} -\log(1 - f_x(0))/x \leq \zeta$, then the MFL's average speed is lower-bounded by $v(\zeta)$.

(iv) We have $\mathring{f}^n(\infty) \Rightarrow \phi^{**}$, where ϕ^{**} is the TWS for the speed v_{**}.

Proving Conjecture 1, or at least some parts of it, would be an interesting subject for future work.

11.2. Simulations and discussion

Before presenting our simulation results, we note that if the objective is to maximize the steady-state speed of progress of a large-scale system, then our Theorem 6 allows one to easily optimize the trade-off between λ and μ, if these quantities are chosen subject to some constraint(s). For example, one may want to optimize the trade-off between the levels of effort allocated to self-propulsion and synchronization, if both require the consumption of a common limited resource. Our simulations show good accuracy of the optimal setting based on Theorem 6.

Tables 1 and 2 show steady-state speeds v_n, obtained by simulation of a system with $n = 10000$ particles, under exponentially and uniformly (in $[0, 2)$) distributed jump sizes, respectively. (For each simulation run, there are $400 \times n$ attempted jumps in total, i.e. 400 per particle. This corresponds to simulation time $400 \times n/(\lambda + \mu)$. The first half of the simulation run is a warm-up.) The tables also show the values of v_{**}. (For the case of exponential jump sizes, $v_{**} = v_* = (\sqrt{\lambda} + \sqrt{\mu})^2$.)

The results for exponential jump sizes are in Table 1. We see that the actual steady-state speed v_n stays below v_{**}, as we know it should. We also observe that, even for a relatively large number of particles (10000), the difference $v_{10000} - v_{**}$ is still significant. This is not unexpected, because the convergence $v_n \uparrow v_{**}$ in other contexts is known to be rather slow [5, 6]. Formally showing that this is true for our system as well would be an interesting problem for further research. The results for the uniform jump sizes in Table 2 show the same patterns.

TABLE 2. Uniform[0, 2] jump sizes, $n = 10000$, $2\lambda + \mu = 1$. Steady-state speed v_n (simulated) and $v_{**} = \lim_n v_n$. The pair $(\lambda_{opt}, \mu_{opt})$ maximizes v_{**}.

λ	μ	v_n (simulation)	v_{**}
0.45	0.1	0.8176	0.844
0.4	0.2	0.9243	0.955
0.35	0.3	0.9704	1.0165
0.3	0.4	0.9871	1.0458
$\lambda_{opt} \approx 0.27$	$\mu_{opt} \approx 0.46$	0.9917	1.0505
0.25	0.5	0.9995	1.0486
0.2	0.6	0.9716	1.0262
0.15	0.7	0.919	0.9761
0.1	0.8	0.8209	0.8907
0.05	0.9	0.6751	0.7469

Suppose now that independent jumps and synchronizations by a particle consume some common resource (say, computing power or energy). To be specific, suppose the maximum rate at which the resource may be consumed by a particle is normalized to 1, and the amounts of the resource consumed by one independent jump and one synchronization are $a > 0$ and $b > 0$, respectively. Suppose the aim is to maximize the speed of the particle system. Assume the number of particles is large enough so that $v_n \approx v_{**}$. Then we obtain the optimization problem

$$\max v_{**} \text{ subject to } a\lambda + b\mu = 1. \tag{11.1}$$

In the case of exponentially distributed jumps, we have $v_{**} = (\sqrt{\lambda} + \sqrt{\mu})^2$, and the problem (11.1) has a simple explicit solution

$$\lambda_{opt} = \frac{1}{a + a^2/b}, \quad \mu_{opt} = \frac{1}{b + b^2/a}.$$

For a general jump size distribution, the optimal solution $(\lambda_{opt}, \mu_{opt})$ can easily be found numerically using (7.3) and (7.4).

In Tables 1 and 2, for the cases of exponential and uniform jumps, we vary (λ, μ) while satisfying the constraint $a\lambda + b\mu = 1$ with $a = 2$, $b = 1$. In both cases we show $(\lambda_{opt}, \mu_{opt})$. We see that, even though the values of v_n for $n = 10000$ are not necessarily very close to their limiting values v_{**} 'yet' (because this n is not large enough), the parameter setting $(\lambda_{opt}, \mu_{opt})$ obtained from (11.1) essentially maximizes v_n for this n. Thus, the setting $(\lambda_{opt}, \mu_{opt})$, which is easily computable, gives a practical rule of thumb for optimizing the trade-off between self-propulsion and synchronization in large-scale systems.

12. Proof of Theorem 2

12.1. Characterization of distributional limit

Theorem 10. *Suppose $f^n(0) \xrightarrow{w} f(0)$, as in Theorem 2. Then the sequence of processes $f^n(\cdot)$ is tight in the space $D([0, \infty), \mathcal{M}^B)$, and any sub-sequential distributional limit $f(\cdot)$ is such that, with probability one, the trajectory $f(\cdot)$ is a continuous element of $D([0, \infty), \mathcal{M}^B)$ which satisfies (5.3).*

Proof. The proof is easily obtained using the steps given in the proof of Theorem 3 in [25] (for a different particle system), namely Theorems 4–7 and Corollary 8 in [25]. (In turn, the proof of Theorem 3 in [25] closely follows the development in [1].) The only substantial difference is the verification of the condition (i) of [24], which in our notation has the following form: for any $T > 0$ and $\varepsilon > 0$, there exists $K > 0$ such that

$$\sup_n \mathbb{P}\left(\sup_{t \leq T}[f^n_{-K}(t) + 1 - f^n_K(t)] > \varepsilon\right) < \varepsilon. \tag{12.1}$$

However, for the process under consideration, with fixed right boundary at the point B, (12.1) is trivial—it suffices to choose K large enough so that $K > B$ and $f^n_{-K}(0) \leq \varepsilon/2$ uniformly in n. □

12.2. Equivalent characterization of solutions to (5.3) as MFMs

Theorem 11. *For any initial condition $f(0) \in \mathcal{M}^B$, a trajectory $f(\cdot) \in D([0, \infty), \mathcal{M}^B)$ satisfies* (5.3) *if and only if it is an MFM.*

Proof. This proof is very similar to the proof of Theorem 10 in [25].

'Only if' statement: We only need to consider points $x < B$. Let $h = (h(u), \ u \in \mathbb{R})$ be the step function $h(u) = \mathbf{I}(u \leq x)$, jumping from 1 to 0 at the point x. Let $h_\varepsilon, \varepsilon > 0$, be a continuous approximation of h that is linearly decreasing from 1 to 0 in $[x, x + \varepsilon]$. We see that

$$L^B[f(t)h_\varepsilon] \to L^B[f(t)h], \ \ \forall t.$$

Indeed, $|L^B[f(t)h_\varepsilon] - L^B[f(t)h]|$ is upper-bounded by $(\lambda + 1)$ times the probability that a random jump—either independent of the size Z or thanks to synchronization—of a particle randomly located according to the distribution $f(t)$ is such that the jump either originates or lands in $(x, x + \varepsilon)$; this probability vanishes as $\varepsilon \downarrow 0$. Also, since both $L^B[f(t)h_\varepsilon]$ and $L^B[f(t)h_\varepsilon]$ are within $[-(\lambda + 1), 0]$, for all t and ε, we have a universal bound $|L[f(t)h_\varepsilon] - L[f(t)h]| \leq \lambda + 1$. Then, for any fixed t, by taking the $\varepsilon \downarrow 0$ limit in $f(t)h_\varepsilon - f(0)h_\varepsilon - \int_0^t L^B[f(s)h_\varepsilon]ds = 0$, we obtain

$$f(t)h - f(0)h - \int_0^t L^B[f(s)h]ds = 0.$$

This means that $f(t)h = f_x(t)$ is absolutely continuous in t, with the derivative equal to $(\partial/\partial t)f_x(t) = L^B[f(t)h]$ almost everywhere (with respect to the Lebesgue measure) in t. In particular this implies that for any fixed y, the difference $f_y(t) - f_{y-}(t)$ is continuous in t (in fact, Lipschitz); this, in turn, means that, possible discontinuity points y of $f_y(t)$ 'cannot move' at time t. We can then conclude that, for any x, the derivative $(\partial/\partial t)f_x(t) = L^B[f(t)h]$ is in fact continuous in t. Therefore, $(\partial/\partial t)f_x(t) = L^B[f(t)h]$ at every t. It remains to observe that $L^B[f(t)h]$ is exactly the right-hand side of (5.1).

'If' statement: The definition of an MFM $f(\cdot)$ implies that (5.3) holds for the above-defined step function h for any $x < B$. Then we have (5.3) for any h, which is piecewise constant with a finite number of pieces; and the set of such functions h is tight within the space of test functions $h \in \mathcal{C}_b$, equipped with the uniform metric. Thus (5.3) holds for any $h \in \mathcal{C}_b$. □

12.3. MFM uniqueness

Theorem 12. *For any initial condition $f(0) \in \mathcal{M}^B$, an MFM $f(\cdot) \in D([0, \infty), \mathcal{M}^B)$ is unique.*

Proof. The existence of an MFM for a given initial state $f(0)$ follows from Theorems 10 and 11. For any MFM $f(\cdot)$ we know that $f_x(t)$, as a function of t, is Lipschitz, uniformly in x. Consider two MFM trajectories, $f(\cdot)$ and $g(\cdot)$, with the same initial state $f(0) = g(0)$. It is easy to check that for any x and any $t > 0$

$$\left| \frac{d}{dt} g_x(t) - \frac{d}{dt} f_x(t) \right| \le C \|g(t) - f(t)\| \tag{12.2}$$

for a fixed constant $C > 0$ (for example, for $C = 2\lambda + 1$). From (12.2) it is easy to see that

$$\frac{d}{dt} \|g(t) - f(t)\| \le C \|g(t) - f(t)\|, \tag{12.3}$$

as long as $\|g(t) - f(t)\| > 0$. But then, by Gronwall's inequality, $\|g(t) - f(t)\| \equiv 0$. This proves the uniqueness. $\qquad\square$

12.4. Conclusion of the proof of Theorem 2

Theorem 2 follows immediately from the results of Subsections 12.1–12.3.

13. Proof of Theorem 1

We now generalize Theorem 2 to Theorem 1. Recall that the evolution of $f_x^n(\cdot)$, $x \in (-\infty, B)$, is independent of the behavior of particles located in $[B, \infty)$. Theorem 2 holds for every B for the projection $f^{n,B}(\cdot) \in D([0, \infty), \mathcal{M}^B)$ of our original process $f^n(\cdot)$, and we have convergence to the corresponding unique continuous MFL (MFM) $f^B(\cdot)$. By uniqueness, these MFLs for different B must be consistent; that is, for any $B_1 < B_2$, any $x < B_1$, and any t we have $f_x^{B_1}(t) = f_x^{B_2}(t)$. So we can formally define a trajectory $f(t)$ by $f_x(t) \doteq \lim_{B \uparrow \infty} f_x^B(t)$. Let us show that this trajectory is the MFL satisfying the statement of Theorem 1.

We first show that $f(t) \in \mathcal{M}$ for any t, i.e. $f_\infty(t) = 1$. Recall that, for any x, $(d/dt)f_x(t)$ is non-positive, bounded, and Lipschitz continuous in t (uniformly in x). Then $\varepsilon(t) \doteq 1 - f_\infty(t)$ is non-decreasing and continuous, starting from 0 at time 0. Moreover, $(d/dt)\varepsilon(t)$ is bounded and Lipschitz continuous. Consider a sequence $c \uparrow \infty$ and the corresponding (non-decreasing) sequence of space-rescaled versions of $f(t)$,

$$f_x^{(c)}(t) = f_{cx}(t).$$

Clearly, $f^{(c)}(t)$ is the limiting trajectory of the process with rescaled initial state and rescaled jumps; and for any t the limit of $f_{cx}(t)$, as a function of x, is constant, equal to $1 - \varepsilon(t)$. Consider any fixed $x > 0$ and $t > 0$. As $c \to \infty$, $1 - f_x^{(c)}(t) \to \varepsilon(t)$ and $(d/dt)f_x^{(c)}(t) \to -\varepsilon(t)(1 - \varepsilon(t))$. For fixed $t < s$,

$$\varepsilon(s) - \varepsilon(t) = \lim_{c \to \infty} [-f_x^{(c)}(s) + f_x^{(c)}(t)] = -\lim_{c \to \infty} \int_t^s (d/d\xi) f_x^{(c)}(\xi) d\xi = \int_t^s \varepsilon(\xi)(1 - \varepsilon(\xi)) d\xi.$$

We conclude that $\varepsilon(t)$, as a function of t, must be a solution to the logistic equation $\varepsilon' = \varepsilon(1 - \varepsilon)$ with initial condition $\varepsilon(0) = 0$. Therefore, $\varepsilon(t) \equiv 0$, and then $f_\infty(t) \equiv 0$, i.e. $f(t) \in \mathcal{M}$.

Now, because we know that $f(t) \in \mathcal{M}$ for any t, $f(\cdot)$ is clearly an MFM: it satisfies the conditions of Definition 1 for any x (because we can always choose $B > x$). The condition (5.2) holds for any $h \in \mathcal{C}_b$, because we can choose B large enough so that $h(x)$ is constant in $[B, \infty)$; for such h, the operators $Lf(t)h = L^B f(t)h$, and then (5.2) follows from (5.3).

It remains to show the convergence $f^n(\cdot) \Rightarrow f(\cdot)$. Fix any $T > 0$, any $\varepsilon > 0$, and a sufficiently large $B = B(T) > 0$ so that $1 - f_B(T) \le \varepsilon$. For the modified system with the right boundary at B, we have convergence of $f^{n,B}(\cdot) \in D([0, T], \mathcal{M}^B)$ to a continuous $f^B(\cdot) \in D([0, T], \mathcal{M}^B)$. But since $\varepsilon > 0$ can be arbitrarily small, we also have convergence of the original process $f^n(\cdot) \in D([0, T], \mathcal{M})$ to a continuous $f(\cdot) \in D([0, T], \mathcal{M})$. This is true for arbitrary $T > 0$. Therefore, we also have convergence of the original process $f^n(\cdot) \in D([0, \infty), \mathcal{M})$ to a continuous $f(\cdot) \in D([0, \infty), \mathcal{M})$.   □

14. Proof of Theorem 3

The proof is essentially the same as that of Theorems 1 and 2, with straightforward adjustments. The condition (12.1) for this process is automatic.

We only comment on the proof of MFM uniqueness. Note that for any MFM $f(\cdot)$, for each x, the derivative $(d/dt)f_x(t)$ can have a discontinuity at time t such that $B = x$ (i.e. when the moving boundary 'hits' the point x). However, this does not affect the uniqueness proof in Section 12.3—we just need to note that (12.2) and (12.3) hold for almost any $t > 0$ with respect to the Lebesgue measure.   □

15. Proof of Theorem 4

This proof is also essentially the same as that of Theorems 1 and 2, with straightforward adjustments.

Again, we only comment on the proof of MFM uniqueness. An MFM for this system is such that, for a given x, the derivative $(d/dt)f_x(t)$ may have a discontinuity (a jump down) at any time t such that the jump distribution has an atom at $x - A$. If we consider two MFM trajectories, $f(\cdot)$ and $g(\cdot)$, with the same initial state $f(0) = g(0)$, as in the uniqueness proof in Section 12.3, then for each x, both $(d/dt)f_x(t)$ and $(d/dt)g_x(t)$ will have the same countable set of potential discontinuities in time. We also have that, at any t, $f_x(t)$ and $g_x(t)$, as functions of x, have the same set of discontinuity points. Given these facts, we see that the uniqueness proof in Section 12.3 applies—again, we just need to note that (12.2) and (12.3) hold for almost any $t > 0$ with respect to the Lebesgue measure.   □

16. Proof of Theorem 5

For the single-particle independent-jump (Markov) process $W(\cdot)$, denote by $P^t(x, H)$ its transition function and by P^t its transition operator,

$$P^t g_x = \int_x^\infty P^t(x, dy) g_y.$$

The generator of this process is

$$B g_x = \lambda \left[\int_0^\infty g_{x+\eta} d\bar{J}(\eta) - g_x \right]. \tag{16.1}$$

Any function g such that $1 - g \in \mathcal{M}$ (i.e. a complementary distribution function) is within the domain of the generator B, and for such functions, integration by parts in (16.1) gives

$$B g_x = \lambda \int_x^\infty \bar{J}(\eta - x) dg_\eta. \tag{16.2}$$

(The integral in (16.2) includes a possible atom of measure g_η at $\eta = x$. If g is continuous, this subtlety is irrelevant.)

Let us fix a function g such that $1 - g \in \mathcal{M}$, and consider the function

$$u_x(t) = \mathbb{E}[g_{x+W_1(t)} \cdot \ldots \cdot g_{x+W_{N(t)}(t)}]. \tag{16.3}$$

Clearly, $u_x(0) = g_x$. Note that if $g_x = \mathbf{I}\{x \le 0\} = \chi_{-x}$, then $u_x(t) = \mathbb{P}\{\max_i W_i(t) \le 0\}$. Considering two cases—depending on whether or not the first split of the first particles occurs in $[0,t]$—we obtain

$$u_x(t) = e^{-t}P^t g_x + \int_0^t e^{-s}P^s u_x^2(t-s)ds = e^{-t}P^t g_x + \int_0^t e^{-(t-\xi)}P^{t-\xi}u_x^2(\xi)d\xi,$$

$$e^t u_x(t) = P^t g_x + \int_0^t e^{\xi}P^{t-\xi}u_x^2(\xi)d\xi.$$

Differentiating in t, we obtain

$$e^t u_x(t) + e^t \frac{\partial}{\partial t}u_x(t) = BP^t g_x + e^t P^0 u_x^2(t) + \int_0^t e^{\xi}P^{t-\xi}Bu_x^2(\xi)d\xi;$$

then

$$\frac{\partial}{\partial t}u_x(t) = Bu_x(t) + u_x^2(t) - u_x(t),$$

and finally, recalling (16.2), we have

$$\frac{\partial}{\partial t}u_x(t) = \lambda \int_x^\infty \bar{J}(\eta - x)d_\eta u_\eta(t) + u_x^2(t) - u_x(t). \tag{16.4}$$

(It is easy to see from the definition of $u_x(t)$ that, for any $t > 0$, it is continuous in x. Therefore the subtlety mentioned immediately after (16.2) is irrelevant.)

Now let $g_x = \mathbf{I}\{x \le 0\}$. Recall that $u_x(t) = \mathbb{P}\{\max_i W_i(t) \le 0\}$, and then

$$u_{-x}(t) = \mathbb{P}\{\max_i W_i(t) \le x\}.$$

Substituting $f_x(t) = u_{-x}(t)$ in (16.4), we obtain exactly (5.1) with $f(0) = \chi$. $\qquad \square$

17. Proof of Theorem 6

The monotonicity, $v_n \le v_{n+1}$, has already been observed in (4.1).

Let us show that $v_n \le v_{**}$ for all n. The proof of this fact is the same as the proof of the analogous upper bound in Theorem 2.3(4) in [13], except for our process we need to use Proposition 1, instead of the analogous fact for the process in [13] (which follows directly from the results of [15, 23]). Here is the proof, for completeness. Consider the system with a finite number n of particles. Consider another, artificial, system starting with the same initial state with n particles, but such that each initial particle generates its own independent associated BRW as defined above. Clearly, the two systems can be coupled so that the artificial system dominates the actual one in the sense of $\preceq_l$, and in particular the location of its rightmost particle, $M(t)$, is always to the right of location of the rightmost particle, $M_n(t)$, in the actual

system. But Proposition 1 holds for each of the n independent associated BRWs. We infer that $\limsup_{t\to\infty} M_n(t)/t \le \limsup_{t\to\infty} M(t)/t = v_{**}$ with probability one, which implies $v_n \le v_{**}$.

It remains to show that $\liminf_n v_n \ge v_{**}$. Consider an arbitrary initial state of the process with n particles. Suppose without loss of generality that the leading particle is initially at 0, and let $D_n(t)$ be the location of the leading particle at time t. Consider a modified process such that all particles except the leading one are initially placed at $-\infty$. By monotonicity, the location $\tilde{D}_n(t)$ of the leading particle in the modified system is stochastically dominated by $D_n(t)$. The modified process is such that, in particular, at some points in time some particles located at $-\infty$ will jump forward to join particles in $[0, \infty)$; let us call the particles located in $[0, \infty)$ *regular*. Let us now fix a finite time interval $[0,T]$, and consider the limit of the process of regular particles in this time interval as $n \to \infty$. It is easy to construct a coupling such that, with probability one, the process of regular particles (of the modified system) converges to an associated BRW. (Indeed, when n is large, the rate at which each regular particle is being joined by a particle from $-\infty$ is close to 1. We omit the details, which are straightforward.) By Proposition 1, if T is sufficiently large, then the average speed of advance of the leading particle of the BRW in $[0,T]$ is close to v_{**}. (This is if $\alpha > 0$ and then $v_{**} < \infty$. If $\alpha = 0$ and then $v_{**} = \infty$, 'close to v_{**}' means 'arbitrarily large'.) Consequently, for all large n, the average speed of advance of $\tilde{D}_n(t)$ is close to v_{**}. This implies (we skip straightforward ε–δ formalities) that $\liminf_n v_n \ge v_{**}$.   □

18. Existence and properties of traveling waves in the case of exponentially distributed jump sizes; proof of Theorem 7

In this section we consider the case of exponential jump size distribution, $J(x) = 1 - e^{-x}$, and develop results leading to the proof of Theorem 7.

Recall that in this case a TWS $\phi = (\phi_x)$ must satisfy the ODE (6.4):

$$v\phi'' = (1 + \lambda - v - 2\phi)\phi' + \phi(1 - \phi). \tag{18.1}$$

A function $\phi = (\phi_x)$ is a TWS for our original system if and only if it is a *proper* solution of the ODE (18.1)—that is, if and only if $\phi_x \in [0, 1]$ for all $x \in \mathbb{R}$, (18.1) holds for all $x \in \mathbb{R}$, $\lim_{x\downarrow-\infty} \phi_x = 0$, and $\lim_{x\uparrow\infty} \phi_x = 1$.

It will be convenient to consider the ODE (18.1) in terms of the first-order ODE in phase space $(\phi, z = \phi')$:

$$\phi' = z, \ \ z' = -z + \frac{1 + \lambda - 2\phi}{v}z + \frac{\phi}{v}(1 - \phi). \tag{18.2}$$

In terms of (18.2), a proper solution is a solution which starts at the point $(0,0)$, ends at the point $(1, 0)$, and stays within the strip $\phi \in [0, 1]$.

The following are immediate observations from (18.2). Solutions to (18.2) (not necessarily proper solutions) have the following properties: a solution trajectory cannot hit the ϕ-axis for $\phi \in (0, 1)$; given any two solution trajectories, either they coincide for $\phi \in [0, 1]$ or one strictly dominates another for $\phi \in (0, 1)$.

18.1. Lower bound on speed v, for any proper solution

Consider the linearization of (18.2) at the point $(1, 0)$. If we set $v = 1 - \phi$, the linearization is

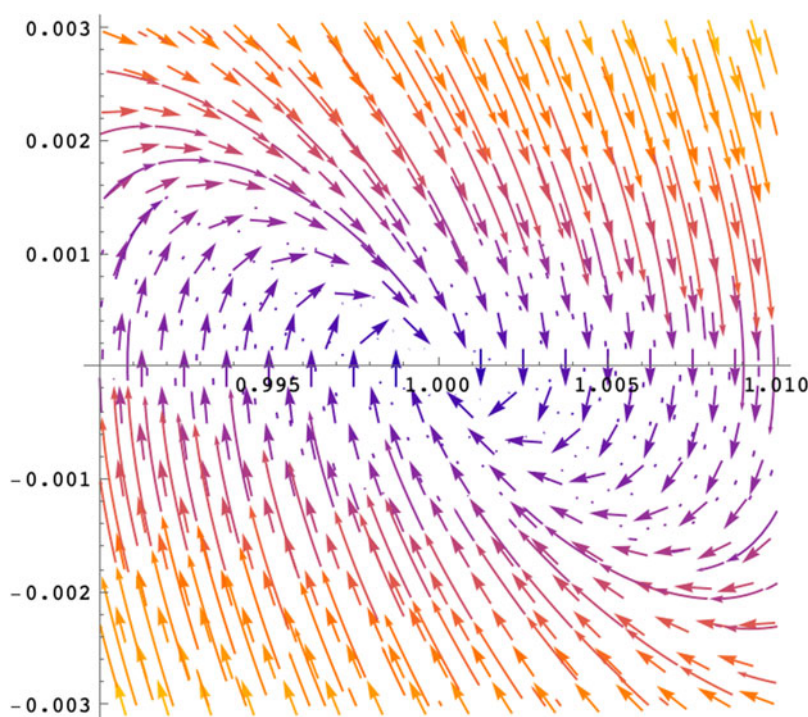$$v' = -z, \ \ z' = -z + \frac{\lambda - 1}{v}z + \frac{1}{v}v. \tag{18.3}$$

FIGURE 1. Vector field (ϕ', z') in the vicinity of $(1, 0)$; $\lambda = 4$, $v = 7 < v_* = 9$.

The eigenvalues of this linear system satisfy the characteristic equation

$$v\zeta^2 + (1 + v - \lambda)\zeta + 1 = 0. \tag{18.4}$$

We know (see Section 6) that for a proper solution to exist, we must have $v \geq \lambda$. Given that, $-(1 + v - \lambda) < 0$, and it is easy to check that the condition $(1 + v - \lambda)^2 - 4v \geq 0$ is equivalent to the condition $v \geq v_* \doteq (1 + \sqrt{\lambda})^2$. We see that both eigenvalues of (18.3) (roots of (18.4)) have negative real parts; the eigenvalues are two (different) adjoint complex numbers if and only if $v < (1 + \sqrt{\lambda})^2$, and they are both real otherwise. We then observe that when $v < (1 + \sqrt{\lambda})^2$, a proper solution does *not* exist, because any solution with endpoint at $(1, 0)$ converges to $(1, 0)$ by oscillating around it (as illustrated in Figure 1) and thus must 'hit' the line $\phi = 1$ at a point strictly above the point $(1, 0)$.

We conclude that if a proper solution of (18.2) exists, then necessarily $v \geq v_* = (1 + \sqrt{\lambda})^2$.

18.2. Proof of Theorem 7(i)

To complete the proof of Theorem 9(i), we need to show that a a proper solution of (18.2) exists for any $v \geq v_*$. We will prove that it exists for $v > v_*$, and then will obtain the existence for $v = v_*$ by continuity.

18.2.1. *Existence, uniqueness, and continuity of solutions to (18.2).* Here we consider solutions to (18.2) (not necessarily proper solutions) within the domain $\{0 \leq \phi \leq 1, z \geq 0\}$. Consider a fixed parameter $v \geq v_*$ and an initial condition (ϕ_0, z_0) for a solution to (18.2). (The initial condition can be viewed as a point which a solution trajectory must contain, because the
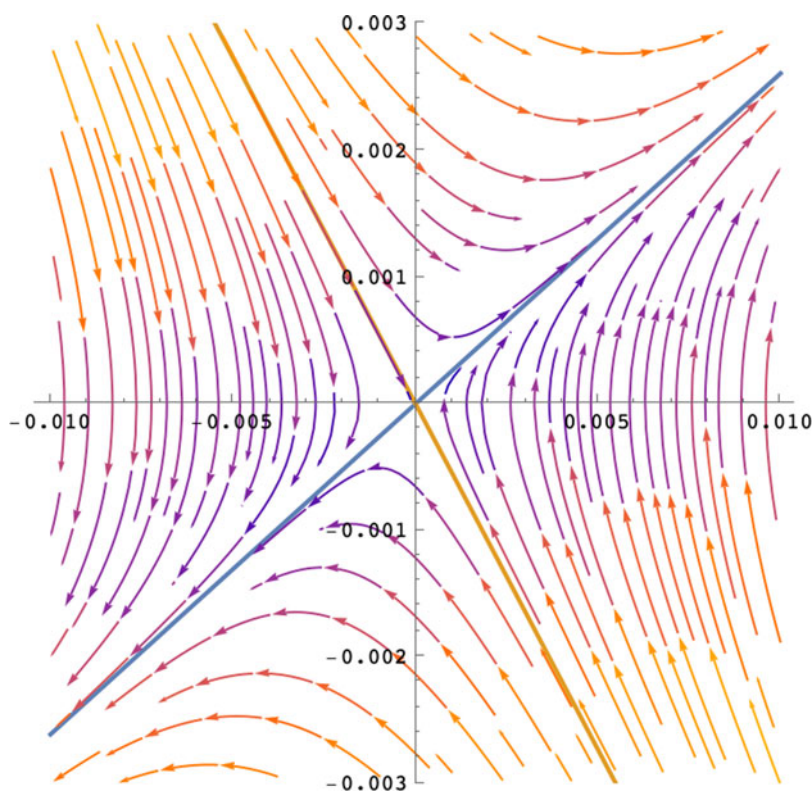
FIGURE 2. Vector field (ϕ', z') in the vicinity of $(0,0)$; $\lambda = 4$, $v = 10$, $v > v_*$.

trajectories satisfying (18.2) can be considered in both the forward and the reverse direction.) From the standard theory of ODEs, we see that the solution for a given triple (ϕ_0, z_0, v) exists and is unique as long as $(\phi_0, z_0) \neq (0, 0)$ and $(\phi_0, z_0) \neq (1, 0)$. Moreover, the solution is continuous in (ϕ_0, z_0, v) outside those two points. The following lemma shows that the existence, uniqueness, and continuity hold for the initial condition $(\phi_0, z_0) = (0, 0)$ as well, if we consider non-trivial solutions.

Lemma 2. *For a fixed $\lambda > 0$ and any speed $v \geq v_*$, there exists a unique non-trivial solution $(\phi, z(\phi))$ such that $z(\phi) > 0$ and $(\phi, z(\phi)) \to (0, 0)$ as $\phi \downarrow 0$. This solution is such that $dz/d\phi = \gamma$ at the initial $\phi = 0$, where*

$$\gamma = \frac{-(v - 1 - \lambda) + \sqrt{(v - 1 - \lambda)^2 + 4v}}{2v} > 0. \tag{18.5}$$

Moreover, the solution is continuous with respect to (ϕ_0, z_0, v).

Proof of Lemma 2. Recall that $v \geq v_* > 1 + \lambda$. Then, for the linearization of (18.2) at the point $(0,0)$, we always have two different real eigenvalues, one positive and one negative: $-\gamma_2 < 0 < \gamma$, where γ is given by (18.5). The corresponding eigenvectors are $(1, \gamma)$ and $(1, -\gamma_2)$; see Figure 2.

The proof of existence is constructive. Consider any sequence of solutions, each starting at a point $(\phi_0^{(k)}, z_0^{(k)}) \geq (0, 0)$ and corresponding to a speed $v^{(k)} \geq v_*$; the sequence is such that, as $n \to \infty$, $(\phi_0^{(k)}, z_0^{(k)}) \to (0, 0)$, $v^{(k)} \to v$, and $(\phi_0^{(k)}, z_0^{(k)}) \neq (0, 0)$ for all k. We view these solutions

as functions $z^{(k)} = z^{(k)}(\phi)$; this is possible because $\phi' = z > 0$ for $\phi \in (\phi_0, 1)$. Then any subsequent limit of this sequence of solutions must be a solution $z = z(\phi)$ (with speed parameter v) starting from $(0,0)$. Also, given the structure of the derivative vector field around $(0,0)$, it is easy to see that any such solution must satisfy $dz/d\phi = \gamma$ at $\phi = 0$. (Indeed, for $c > 0$, denote by $M_c = \{z = c\phi, \ \phi \geq 0\}$ the ray from the point $(0,0)$ in the direction $(1,c)$. For a small fixed $\varepsilon > 0$, denote by W the closed cone between the rays $M_{\gamma-\varepsilon}$ and $M_{\gamma+\varepsilon}$. If we look at the solution in the reverse direction, as it approaches the point $(0,0)$, it cannot be outside W arbitrarily close to $(0,0)$—otherwise it will hit the z-axis strictly above $(0,0)$.)

It remains to show the uniqueness of a solution for a given parameter v. (Then the continuity in (ϕ_0, z_0, v) will automatically follow.) Consider any two solutions to (18.2), viewed as functions $z = z(\phi)$ and $\hat{z} = \hat{z}(\phi)$. It is easy to see from (18.2) that

$$\frac{d}{d\phi}[\hat{z} - z] = \frac{\phi(1-\phi)}{v}\left[\frac{1}{\hat{z}} - \frac{1}{z}\right] = -\frac{\phi(1-\phi)}{v\hat{z}z}[\hat{z} - z].$$

Thus, the difference $\hat{z}(\phi) - z(\phi)$ cannot increase in ϕ, which implies uniqueness.

18.2.2. *Existence of a proper solution for $v \geq v_*$.*

Lemma 3. *A proper solution of* (18.2) *exists for any $v > v_*$. (Furthermore, by continuity of solutions in v, it exists for $v = v_*$ as well.)*

Proof of Lemma 3. Consider $C > 1$ (to be determined later). Consider a point (ϕ, z), with $\phi \in (0, 1)$ and $z \geq 0$. Let us compare the derivative vector (ϕ', z'), given by (18.2), to the tangent vector to the parabola $h(\phi) = C\phi(1-\phi)/v$ at the point ϕ, with the ϕ-component equal to ϕ'; the tangent vector is then $(\phi', \phi'C(1-2\phi)/v)$. The difference between the z-components of these two vectors is

$$z' - \phi'C(1-2\phi)/v = -\left(\frac{v - \lambda + (C-1)(1-2\phi)}{v} - \frac{1}{C}\right)z - \frac{1}{C}(z - h(\phi)).$$

If $(\phi, z) = (\phi, h(\phi))$, i.e. the point (ϕ, z) is on the parabola, this difference is

$$-\left(\frac{v - \lambda + (C-1)(1-2\phi)}{v} - \frac{1}{C}\right)z.$$

Let us consider the expression in the bracket (a linear function of ϕ). If we can prove that it is positive for $0 \leq \phi \leq 1$, then *the trajectories of our dynamic system will traverse the parabola $h = C\phi(1-\phi)/v$ entering the domain between that parabola and ϕ-axis.* (See Figure 3.) Let us check that this is the case. The coefficient in front of ϕ is negative, so it suffices to check the expression is positive at $\phi = 1$, i.e.

$$\frac{v - \lambda - (C-1)}{v} - \frac{1}{C} > 0, \ \text{i.e.,} \ \frac{v - \lambda + 1}{v} > \frac{C}{v} + \frac{1}{C}.$$

The condition $v > (1 + \sqrt{\lambda})^2$ is equivalent to $\sqrt{v} > 1 + \sqrt{\lambda}$, then to $(\sqrt{v} - 1)^2 > \lambda$, then to $v - \lambda + 1 > 2\sqrt{v}$. If we take $C = \sqrt{v}$, this gives exactly the last display.

Now, consider the solution $z(\phi)$ to (18.2) with initial condition $(0,0)$. The derivative $dz/d\phi$ at $\phi = 0$ is γ (in (18.5)), while the derivative of $h(\phi)$ (with $C = \sqrt{v}$) is $1/\sqrt{v}$. For $\lambda > 0$ and $v \geq v_*$, the inequality $1/\sqrt{v} > \gamma$ can be verified by simple algebra. This means that, for sufficiently small $\phi > 0$, the solution $z(\phi)$ is 'under' the parabola $h(\phi)$. But, as shown above, it cannot traverse $h(\phi)$. We also know that the solution $z(\phi)$ cannot hit the ϕ-axis while $\phi \in (0, 1)$. We conclude that $z(\phi) \leq h(\phi)$ for all $\phi \leq 1$, and $z(1) = 0$. In particular, $z(\phi)$ is a proper solution. (In fact, one can show that the trajectories also are transversal to the parabola $z = \phi(1-\phi)/v$.)
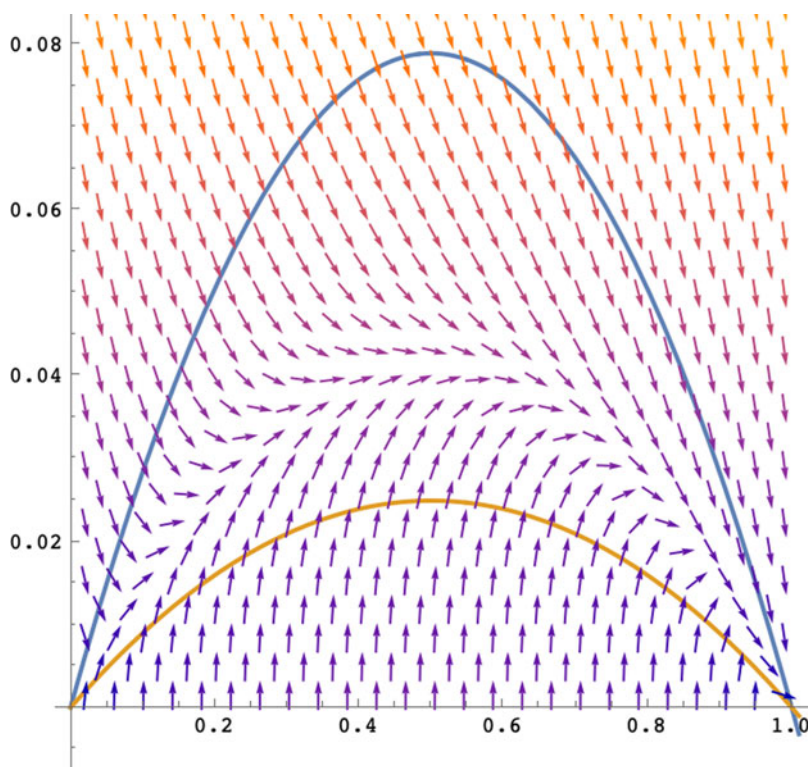
FIGURE 3. Vector field (ϕ', z'), for points inside the domain between the parabolas $z = C\phi(1 - \phi)/v$ and $z = \phi(1 - \phi)/v$; here $C = \sqrt{v}$, $\lambda = 4$, $v = 10$, $v > v_*$.

18.2.3. *Derivative of the proper solution at the point (1, 0).* To complete the proof of of Theorem 7(i), it remains to show that (9.1) holds when $v > v_*$. If $v > v_*$, the linearization of our dynamic system at the point (1, 0) has two strictly negative eigenvalues, $-\zeta_1 > -\zeta_2$,, being roots of (18.4), so that

$$\zeta_1 = \frac{(1 + v - \lambda) - \sqrt{(1 + v - \lambda)^2 - 4v}}{2v}, \quad \zeta_2 = \frac{(1 + v - \lambda) + \sqrt{(1 + v - \lambda)^2 - 4v}}{2v}.$$

Since the solution $z = z(\phi)$ for speed v is dominated by the parabola $\phi(1 - \phi)/\sqrt{v}$, we see that in the neighborhood of (1, 0) it is dominated by the line $z = (1 - \phi)/\sqrt{v}$. It is easy to see that $1/\sqrt{v} < \zeta_2$. (Indeed, the stronger inequality $1/\sqrt{v} < (1 + v - \lambda)/(2v)$ is equivalent to $v > v_*$.) Therefore, in the neighborhood of (1, 0) the solution $z = z(\phi)$ is separated from the eigendirection $z = \zeta_2(1 - \phi)$ by line $z = (1 - \phi)/\sqrt{v}$. But then the solution $z = z(\phi)$ must approach the point (1, 0) along the eigendirection $z = \zeta_1(1 - \phi)$, that is

$$\frac{dz}{d\phi}\bigg|_{\phi=1} = -\zeta_1.$$

It remains to notice that ζ_1 is exactly equal to $\zeta(v)$ in (7.8).   □

18.3. Existence of solutions giving TWS for systems with moving left or right boundary; proof of Theorem 7(ii)–(iii)

The following theorem is an extended version of Theorem 7(ii).

Theorem 13. *Suppose $\lambda > 0$ and $v > v_*$. For any sufficiently small $\phi_0 > 0$ and $z_0 = \phi_0(1 - \phi_0 + \lambda)/v$, the solution to (18.2) with initial condition (ϕ_0, z_0) ends at (1, 0). This solution corresponds to the unique (up to a shift) TWS $\phi = (\phi_x)$ for the system with moving left boundary at speed v, satisfying the conditions $\phi_{x_0} = \phi_0$, $\phi'_{x_0} = z_0$, where x_0 is the left boundary point of the TWS; moreover, the TWS ϕ satisfies (9.1).*

Proof of Theorem 13. Since $v > v_*$, there is a unique solution $z(\phi)$ to (18.2) with initial condition $(0,0)$, and it is proper. Moreover, this solution is 'under' the parabola $h(\phi) = \sqrt{v}\phi(1 - \phi)/v$, as shown in the proof of Lemma 3. Let us consider the solution $\hat{z}(\phi)$ to (18.2) with initial condition $(0, \varepsilon)$, for small $\varepsilon > 0$. By continuity of solutions to (18.2) in initial condition, we can choose ε small enough so that $\hat{z}(\phi)$ is close to $z(\phi)$ and therefore is under the $h(\phi)$ for some values $\phi \in (0, 1)$. We conclude (as in the proof of Lemma 3) that solution $\hat{z}(\phi)$ for $\phi \in [0, 1]$ ends at the point $(1, 0)$. We also know that $z(\phi) < \hat{z}(\phi)$ for all $\phi \in [0, 1)$. Recall that the derivative $dz/d\phi$ at $\phi = 0$ is equal to γ in (18.5). The dependence $z_0 = \phi_0(1 - \phi_0 + \lambda)/v$ on ϕ_0 has derivative $(1 + \lambda)/v$ at $\phi_0 = 0$. The inequality $(1 + \lambda)/v > \gamma$ is verified by simple algebra. We conclude the following: for all sufficiently small positive ϕ_0, the point (z_0, ϕ_0) lies strictly between the solutions $z(\phi)$ and $\hat{z}(\phi)$. Therefore, the unique solution to (18.2) with initial condition (z_0, ϕ_0) ends at the point $(1, 0)$.

Verification of (9.1) for the corresponding TWS ϕ is the same as in the proof of Theorem 7(i).

Proof of Theorem 7(iii). There is a unique solution to (18.2) with initial condition $(0,0)$. Since $v < v_*$, this solution cannot be proper. Therefore, the solution 'hits' the line $\phi = 1$ strictly above the ϕ-axis, i.e. at a point $(1, z_1)$ with $z_1 > 0$. The corresponding (unique up to a shift) distribution function $\phi = (\phi_x)$ is the unique TWS for the system with moving right boundary at speed v.

19. Proof of Theorem 8

Consider a fixed MFL $f(\cdot)$. Let us fix an arbitrarily small $v > 0$. Without loss of generality, assume that $f_0(0) \geq v$ and $f_{0-}(0) \leq v$. Recall that the trajectory $f(\cdot)$ is the unique distributional limit of a sequence of processes $f^n(\cdot)$ with deterministic initial states such that $f^n(0) \xrightarrow{w} f(0)$. We can and will choose a sequence of initial states such that, for all n, vn of the leftmost particles are located in $(-\infty, 0]$ (let us call them 'left particles') and then the remaining $(1 - v)n$ particles are located in $[0, \infty)$ (let us call them 'right particles').

Now, for each n, consider a lower-bounding process $\hat{f}^n(\cdot)$ constructed as follows. The initial state of $\hat{f}^n(\cdot)$ is such that the vn 'left' particles' locations are the same as in the original process, i.e. $\hat{f}^n_x(0) = f^n_x(0)$ for $x < 0$, while the $(1 - v)n$ 'right' particles are located exactly at 0. The evolution of $\hat{f}^n(\cdot)$ is such that the 'left' particles are 'frozen'—they do not jump at all; the 'right' particles jump as usual, making both independent and synchronization jumps. Clearly, $\hat{f}^n(\cdot)$ and $f^n(\cdot)$ can be coupled so that $\hat{f}^n(t) \preceq f^n(t)$ at all times; then $\hat{f}(t) \preceq f(t)$ for the corresponding MFLs. Now, observe that the evolution of $\hat{f}^n(\cdot)$ is such that the left particles can simply be ignored, and all of the right particles are initially at 0. Next observe that the process $\hat{f}^n(\cdot)$ is such that the evolution of the set of $(1 - v)n$ right particles can be viewed as that of a stand-alone

original system, except the synchronization rate of each particle is $(1 - \nu)$ instead of 1. (Recall that in the system under consideration we have $\mu = 1$, without loss of generality.) Therefore, the MFL $\hat{f}(\cdot)$ has the following form: $\hat{f}_x(t) = f_x(0)(t)$ for $x < 0$, and

$$\hat{f}_x(t) = \nu + (1 - \nu)\tilde{f}_x(t), \quad \text{for } x > 0, \tag{19.1}$$

where $\tilde{f}(\cdot)$ is the BMFL of the system with independent jump rate λ and synchronization rate $1 - \nu$. For a sufficiently small $\nu > 0$, the speed v'_{**} of $\tilde{f}(\cdot)$ is arbitrarily close to v_{**}. (In particular, if $\alpha = 0$, then $v'_{**} = v_{**} = \infty$.) We conclude that any quantile $\beta > \nu$ of $f(\cdot)$ advances at average speed at least v'_{**}. Since $\nu > 0$ can be chosen arbitrarily small, any quantile $\beta > 0$ of $f(\cdot)$ advances at average speed at least v_{**}. $\qquad\square$

20. Proof of Theorem 10

(i) Obviously, this property is a special case of the more general properties given in Proposition 2 and Theorem 8. However, we now give a simple alternative proof, which does *not* rely on the connection to and properties of the associated BRW, described in Theorem 7 and Proposition 1. The proof relies only on Theorem 8 and monotonicity.

By Theorem 7(ii)–(iii), the BMFL can be lower- and upper-dominated by the following traveling waves: a traveling wave with moving upper boundary with speed v' slightly smaller that v_*, and a traveling wave with moving lower boundary with speed v'' slightly larger that v_*. Since v' and v'' can be arbitrarily close to v_*, the average speed of BMFL is v_*. To prove that v_* is a lower bound on any MFL's average speed, the proof of Theorem 10 applies.

(ii) By Theorem 7(ii) there exists a TWS ψ with moving left boundary and speed v' slightly greater than $v(\zeta)$. The right tail exponent of ψ is slightly smaller than ζ. We can always consider a version of this ψ, shifted sufficiently far to the right, so that $f(0) \preceq \psi$. By monotonicity, MFL $f(\cdot)$ is dominated by the traveling wave (with moving left boundary) with speed v', starting from ψ. Therefore, the average speed of the MFL is upper-bounded by v', which can chosen arbitrarily close to $v(\zeta)$.

(iii) When $\zeta = \zeta_*$, the claimed property follows from (i). Therefore, it suffices to consider the case $\zeta < \zeta_*$. Fix a small $\nu > 0$. If ν is small enough, by Theorem 7(iii) there exists a TWS ψ for the system with synchronization rate $\mu = 1 - \nu$, with speed v' slightly smaller than $v(\zeta)$, and with the right tail exponent of ψ slightly larger than ζ. Consider now the function $\tilde{\psi} = \nu + (1 - \nu)\psi$, and consider the version of it shifted sufficiently far to the left, so that $\tilde{\psi} \preceq f(0)$. This $\tilde{\psi}$ can be viewed as the initial state of an MFL for a system where the νn leftmost particles are located at $-\infty$ and are frozen (i.e., never make any jumps); then the corresponding MFL is a traveling wave with speed v', dominated by $f(\cdot)$. We see that any quantile $\beta > \nu$ of $f(t)$ advances at average speed at least v'. Since ν can be arbitrarily small and v' can be arbitrarily close to $v(\zeta)$, we obtain the claim. $\qquad\square$

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Competing interests

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