MODULI SPACE OF BRANCHED SUPERMINIMAL IMMERSIONS OF A COMPACT RIEMANN SURFACE INTO S^4

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Abstract

In this paper we describe the moduli spaces of degree d branched superminimal immersions of a compact Riemann surface of genus g into S^4. We prove that when $d \geq \max\{2g, g + 2\}$, such spaces have the structure of projectivized fibre products and are path-connected quasi-projective varieties of dimension $2d - g + 4$. This generalizes known results for spaces of harmonic 2-spheres in S^4.

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In the Calabi construction (see [C]) the space of harmonic maps (or, equivalently, the space of branched minimal immersions) of S^2 into S^{2n} decomposes into a union of moduli spaces H_k labelled by the harmonic degree $k \geq 0$. (This means that if $f \in H_k$, then $\text{Area}(f(S^2)) = \text{Energy}(f) = 4\pi k$.) When f is linearly full (that is, its image is not contained in a totally geodesic subsphere), then either f or $-f$ (that is, f followed by the antipodal map on S^{2n}) has a holomorphic horizontal lift to the twistor space of S^{2n} such that its image is a curve of degree k. In the case when $n = 2$, if f is totally geodesic, then both f and $-f$ have holomorphic horizontal lifts to the twistor space $\mathbb{CP}^3$, and their images in $\mathbb{CP}^3$ are linear projective lines. The study of the space of harmonic maps of S^2 into S^4 (modulo the antipodal map on S^4) thus reduces to the study of holomorphic horizontal curves (of genus 0) in $\mathbb{CP}^3$. In [L] the study of holomorphic horizontal curves of degree k and genus 0 in $\mathbb{CP}^3$ was reduced to the study of the moduli space $\mathcal{M}_k$ of pairs of meromorphic functions of degree k with the same ramification divisor.

Recall from Wirtinger's inequality that a holomorphic curve in a Kähler manifold is area minimizing in its homology class. Since the twistor projection $\mathbb{CP}^3 \to S^4$ is a

Riemannian submersion, the projection of a holomorphic horizontal curve in $\mathbb{CP}^3$ is automatically minimal in S^4. By general twistor theory, such maps are also harmonic (see [BR] and [ES] for example). For a fixed compact Riemann surface Σ of genus $g > 0$, we shall consider the special class of harmonic maps from Σ to S^4 which admit holomorphic horizontal liftings to $\mathbb{CP}^3$. Such maps are called 'branched superminimal immersions'. In 1982 Robert Bryant obtained the following result:

THEOREM ([Br]). *Every compact Riemann surface admits a superminimal immersion into S^4.*

In this paper, we shall study the moduli spaces of all such maps. We shall give a *direct* construction for holomorphic horizontal curves of genus g. Since every such curve in $\mathbb{CP}^3$ has a degree, we can classify the moduli spaces by the degree of the curves. For a fixed compact Riemann surface Σ of genus g, and a fixed degree d, we shall let $\mathscr{S}_{d,\Sigma}$ denote the moduli space of branched superminimal immersions of Σ into S^4 of degree d (equivalently, the moduli space of degree d holomorphic horizontal maps of Σ to $\mathbb{CP}^3$). We shall abuse language by using the terms 'curve in $\mathbb{CP}^3$' and 'holomorphic map of Σ to $\mathbb{CP}^3$' interchangeably.

In the first section, we recall the basic construction given in [L]. The second section is concerned with using fibre products to describe the moduli space. This is done by first fixing a line bundle of degree d over a fixed Riemann surface Σ of genus g. In Section 3, we let the line bundle vary over the Jacobian variety of degree d line bundles over Σ and we obtain results on the dimension of the moduli space. Path-connectedness of the moduli space is discussed in Section 4. We obtain the following main result.

THEOREM 4.1. *For $d \geq \max\{2g, g+2\}$, $\mathscr{S}_{d,\Sigma}$ is a path-connected quasi-projective variety of dimension $2d - g + 4$.*

In Section 5 we discuss compactifications of the moduli space. We provide some simple examples of elements in the boundary of the moduli space.

1. Superminimal immersions and contact curves

Let us first recall the twistor fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \to S^4$. Observe that $\mathbb{H}^2$ has the structure of a quaternion module with right scalar multiplication. By taking the quotient of $\mathbb{H}^2 \smallsetminus \{0\}$ by the action of right scalar multiplication by nonzero quaternions we obtain $\mathbb{HP}^1$. The natural identification of $\mathbb{C}^4$ with $\mathbb{H}^2$ via $(z_0, z_1, z_2, z_3) \mapsto (z_0 + j z_1, z_2 + j z_3)$ gives us the formula for the twistor fibration over $S^4 \cong \mathbb{HP}^1$:

$$\mathbb{CP}^3 \ni [z_0, z_1, z_2, z_3] \mapsto [z_0 + j z_1, z_2 + j z_3] \in \mathbb{HP}^1.$$

The Fubini-Study metric on $\mathbb{CP}^3$ splits the tangent space $T\mathbb{CP}^3$ into vertical and horizontal components. The horizontal distribution $\mathcal{H}$ has complex codimension 1 and is a holomorphic sub-bundle of $T\mathbb{CP}^3$. This distribution is a holomorphic contact structure—it is the kernel of a contact 1-form θ on $\mathbb{CP}^3$ with values in a holomorphic line bundle $\mathcal{L}$, and it is encoded in the following exact sequence:

$$0 \longrightarrow \mathcal{H} \longrightarrow T\mathbb{CP}^3 \xrightarrow{\theta} \mathcal{L} \longrightarrow 0.$$

This contact form has a lifting to $\mathbb{C}^4$ given by

$$(1.1) \qquad \Omega = z_0\, dz_1 - z_1\, dz_0 + z_2\, dz_3 - z_3\, dz_2.$$

Since holomorphic horizontal curves are tangent to the distribution $\mathcal{H}$, we shall use the terms 'holomorphic horizontal curves' and 'contact curves' interchangeably.

We shall consider surfaces which arise as twistor projections of holomorphic curves tangent to the horizontal distribution in $\mathbb{CP}^3$. Such maps are known to be harmonic, and in fact branched minimal (see [Br], [La], [L], [ES] and [BR]).

DEFINITION 1.1. A harmonic map of a Riemann surface to S^4 is said to be *branched superminimal* if it is a projection of a horizontal holomorphic map into $\mathbb{CP}^3$.

In [L], such (generic) maps, modulo the 'contact involution' on $\mathbb{CP}^3$, were described in terms of pairs of meromorphic functions which satisfy a ramification condition. We summarize the construction of [L] below.

Consider the following two distinguished projective lines in $\mathbb{CP}^3$:

$$L_0 := \{[z_0, z_1, 0, 0]\}, \qquad L_1 := \{[0, 0, z_2, z_3]\}.$$

(These two distinguished lines project to the distinguished pair of antipodal points in S^4—the north and south poles.) We have a projection map

$$\psi_0 = (\pi_0, \pi_1) : \mathbb{CP}^3 \setminus (L_0 \cup L_1) \longrightarrow L_0 \times L_1 \cong \mathbb{CP}^1 \times \mathbb{CP}^1,$$

where π_0 and π_1 are respectively given by $x \mapsto \mathrm{span}(x, L_1) \cap L_0$ and $x \mapsto \mathrm{span}(x, L_0) \cap L_1$. It was proved in [L] that ψ_0 has contact fibres. Letting $\widehat{\mathbb{CP}}^3$ denote the blow-up of $\mathbb{CP}^3$ along $L_0 \cup L_1$, we obtain a well-defined map

$$(1.2) \qquad \psi : \widehat{\mathbb{CP}}^3 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1.$$

The map ψ has contact fibres if we equip $\widehat{\mathbb{CP}}^3$ with the pull-back of the contact structure on $\mathbb{CP}^3$. We obtain the following diagram:

$$(1.3)$$

where $\widetilde{\psi}$ is a contact $2:1$ branched covering [L].

Observe that a generic holomorphic curve $\gamma \subset \mathbb{CP}^3$ will miss a pair of skew lines, in particular, $\gamma \cap (L_0 \cup L_1) = \emptyset$. (This implies that its image in S^4 lies in the complement of the two poles.) Let γ be a generic holomorphic curve of degree d in $\mathbb{CP}^3$. We obtain 2 meromorphic functions associated to the curve γ:

$$f_i := \pi_i \mid_\gamma : \gamma \longrightarrow L_i, \quad i = 0, 1,$$

with

$$\deg f_0 = \#[(\text{span}(x, L_1) \smallsetminus L_1) \cap \gamma] = \#(\text{span}(x, L_1) \cap \gamma) = \deg(\gamma) = d,$$
$$\deg f_1 = \#[(\text{span}(y, L_0) \smallsetminus L_0) \cap \gamma] = \#(\text{span}(y, L_0) \cap \gamma) = \deg(\gamma) = d,$$

where x is any point in L_0 and y is any point in L_1. Thus, a generic holomorphic curve of degree d in $\mathbb{CP}^3$ furnishes us with two meromorphic functions of degree d.

Let us examine a nongeneric case where γ intersects one of the lines, say L_1, in k points (counting multiplicity). Let $\tilde{\gamma}$ denote the proper transform of γ. We let π_i denote the i-th factor projection map of (1.2) and let $f_i := \pi_i \mid_{\tilde{\gamma}}$. Then,

$$\begin{aligned}
\deg f_0 &= \#[(\text{span}(x, L_1) \smallsetminus L_1) \cap \gamma] \\
&= \#(\text{span}(x, L_1) \cap \gamma) - \#(\gamma \cap L_1) \\
&= \deg(\gamma) - k = d - k,
\end{aligned}$$

where x is any point in L_0.

If in addition $\gamma : \Sigma \to \mathbb{CP}^3$ is a contact curve, then its proper transform $\tilde{\gamma}$ is a contact curve in $\widehat{\mathbb{CP}}^3$ and we hence obtain a map $\Sigma \to \mathbb{CP}^1 \times \mathbb{CP}^1$ (that is, two meromorphic functions on Σ) which has a contact lift to $\mathbb{P}T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ by (1.3). The contact condition on the curve implies that the two meromorphic functions have the same ramification divisor.

Note that any nonconstant holomorphic map $\sigma : \Sigma \to \mathbb{CP}^1 \times \mathbb{CP}^1$ has a unique contact lift to $\mathbb{P}T^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ given by the Gauss lift $G(\sigma) = [\sigma \wedge \sigma']$. A lift of $G(\sigma)$ to $\widehat{\mathbb{CP}}^3$ is a contact curve whose image in $\mathbb{CP}^3$ is nongeneric in the sense that it intersects the two distinguished lines. However, if $\sigma = (g_0, g_1)$ is of bidegree (d, d) where g_0 and g_1 have the same ramification divisor, then $G(\sigma)$ lifts to a contact curve of degree d in $\mathbb{CP}^3$ which avoids the two distinguished lines. This enables us to describe the moduli space of generic branched superminimal surfaces in S^4 of area $4\pi d$ in terms of pairs of meromorphic functions with the same ramification divisor. Given such a pair of meromorphic functions, the construction given in [L] does not give an explicit description of the contact curve in $\mathbb{CP}^3$—it only associates to such a pair of meromorphic functions a pair of contact curves via the map $\widetilde{\psi}$ in (1.3). In the next section, we shall show how one may obtain an explicit description given such holomorphic data.

2. Fibre products, Gaussian maps and Grassmannians

Henceforth, we shall let Σ be a fixed compact Riemann surface of genus g. Let $\gamma :$ $\Sigma \to \mathbb{CP}^3$ be a holomorphic curve of degree d. Then there is a line bundle $L \to \Sigma$ of degree d such that $\gamma(z) = [s_0(z), s_1(z), s_2(z), s_3(z)]$, where $s_0, s_1, s_2, s_3 \in H^0(\Sigma, L)$. Note that the linear system $\langle s_0, s_1, s_2, s_3 \rangle$ has no base points. If the pencils $\langle s_0, s_1 \rangle$ and $\langle s_2, s_3 \rangle$ are base-point-free, then the two meromorphic functions (thought of as holomorphic maps to $\mathbb{CP}^1$) arising from γ are given by

$$f_0 = [s_0, s_1], \qquad f_1 = [s_2, s_3].$$

It follows from (1.1) that γ is a contact curve if

$$(2.1) \qquad\qquad s_0 s_1' - s_1 s_0' + s_2 s_3' - s_3 s_2' = 0.$$

For the rest of this section let $L \to \Sigma$ be a fixed line bundle of degree d. Let $W_L = H^0(\Sigma, L)$ and let W_L^k denote the k-fold product of W_L. We wish to find four holomorphic sections of L: $s_0, s_1, s_2, s_3 \in W_L$, such that (2.1) holds and where $\langle s_0, s_1, s_2, s_3 \rangle$ has no base points. Consider the map

$$(2.2) \qquad \text{Ram}: \quad W_L^2 \longrightarrow H^0(\Sigma, KL^2) \quad \text{given by} \quad (s, t) \longmapsto s\,dt - t\,ds,$$

where K is the canonical line bundle over Σ.

REMARK. Observe that $\text{Ram}(s, t) = -\text{Ram}(t, s)$. Thus, Ram is well defined on simple bivectors: $s \wedge t \mapsto s\,dt - t\,ds$. This extends linearly to a map

$$\Lambda : \bigwedge^2 W_L \longrightarrow H^0(\Sigma, KL^2) \quad \text{defined by}$$

$$(2.3) \qquad \sum_{i<j} a_{ij} s_i \wedge s_j \longmapsto \sum_{i<j} a_{ij} \{s_i\,ds_j - s_j\,ds_i\}.$$

This map is well-known and it is called the *Gaussian map* ([W1], [W2]). It is called a *Wahl map* when $L = K$.

We can rewrite (2.1) as $\text{Ram}(s_0, s_1) + \text{Ram}(s_2, s_3) = 0$, that is,

$$\text{Ram}(s_0, s_1) = -\text{Ram}(s_2, s_3).$$

In the case where the two pencils described above are base-point-free, this implies that the corresponding two functions, f_0 and f_1, have the same ramification divisor. Note that the ramification divisor of the function $f_0 = [s_0, s_1]$ is just the divisor $(s_0\,ds_1 - s_1\,ds_0) \in \mathbb{P}H^0(\Sigma, KL^2)$.

Given a diagram

$$X \times X$$
$$\downarrow {\scriptstyle \phi \times \psi}$$
$$Y \xrightarrow{\;\;\Delta\;\;} Y \times Y,$$

where Δ is the diagonal inclusion, recall that the fibre product $X \times_Y X$ is just the restriction to the diagonal of the product $\phi \times \psi : X \times X \to Y \times Y$, that is, $X \times_Y X = \Delta^*(X \times X)$. From the diagram:

(2.4)

$$W_L^2 \times W_L^2 = W_L^4$$
$$\downarrow {\scriptstyle \text{Ram} \times (- \text{Ram})}$$
$$H^0(\Sigma, KL^2) \xrightarrow{\;\;\Delta\;\;} H^0(\Sigma, KL^2) \times H^0(\Sigma, KL^2),$$

where Δ is the diagonal inclusion, consider the projectivized fibre product

(2.5)
$$\mathcal{M}_{L,d} := \mathbb{P}(W_L^2 \times_{H^0(\Sigma, KL^2)} W_L^2) \subset \mathbb{P}(W_L^4).$$

REMARK. We adopt the following notation in this paper: given a vector space V, we let $\mathbb{P}V$ denote the projective space of lines through the origin in V.

Let $\mathscr{C}_{L,d} := \{[s_0, s_1, s_2, s_3] \in \mathbb{P}(W_L^4) \mid \langle s_0, s_1, s_2, s_3 \rangle \text{ has base-points}\}$. Observe that each element of $\mathscr{C}_{L,d}$ corresponds to a parametrized curve in $\mathbb{CP}^3$ of degree strictly less than d.

REMARK. $\mathscr{C}_{L,d}$ is a rather complicated subset of $\mathbb{P}(W_L^4)$. It would be interesting to determine the structure of this space.

It follows from (2.1) that branched superminimal immersions of Σ into S^4 of area $4\pi d$ and arising from the line bundle L are parametrized by the quasi-projective variety

(2.6)
$$\mathscr{S}_{L,d} := \mathcal{M}_{L,d} \smallsetminus \mathscr{C}_{L,d}.$$

REMARK. Given $[s_0, s_1, s_2, s_3] \in \mathscr{S}_{L,d}$, we allow one of the pair of pencils (but not both), $\langle s_0, s_1 \rangle$ or $\langle s_2, s_3 \rangle$, to have base-points. At such a base point, the corresponding curve $[s_0, s_1, s_2, s_3]$ in $\mathbb{CP}^3$ intersects one of the lines L_0 or L_1, hence the corresponding associated meromorphic function has degree less than d. Thus, in the genus zero case, $\mathscr{S}_{\mathcal{O}(d)}$ contains elements which lie in the boundary of the moduli space described in [L].

Let $\mathscr{B}_L := \{(s, t) \in W_L^2 \mid \mathrm{Ram}(s, t) = 0\}$. Observe that $(s, t) \in \mathscr{B}_L$ if and only if $s = \lambda t$ for some constant $\lambda \in \mathbb{C}$, that is, $s \wedge t = 0$. Thus we can write

$$(2.7) \qquad \mathscr{B}_L = \{(a\sigma, b\sigma) \in W_L^2 \mid \sigma \in W_L, \ [a, b] \in \mathbb{CP}^1\}.$$

It follows from (2.4) and (2.7) that the fibre above $0 \in H^0(\Sigma, KL^2)$ in the fibre product $W_L^2 \times_{H^0(\Sigma, KL^2)} W_L^2$ is just the set

$$(2.8) \qquad \mathscr{B}_L \times \mathscr{B}_L = \{(a\sigma, b\sigma, c\tau, d\tau) \mid (\sigma, \tau) \in W_L^2, \ [a, b], [c, d] \in \mathbb{CP}^1\}.$$

Let us now consider the set $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L)$. Fix $([a, b], [c, d]) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ and let ℓ denote the line in $\mathbb{CP}^3$ spanned by $[a, b, 0, 0] \in L_0$ and $[0, 0, c, d] \in L_1$. Note that ℓ is horizontal since the map ψ in (1.2) has contact fibres. Also, observe that ℓ projects to a totally geodesic 2-sphere S_ℓ which passes through the north and south poles of S^4. Now fix $(\sigma, \tau) \in W_L^2$ such that $\langle \sigma, \tau \rangle$ is base-point-free and set $\phi = [(a\sigma, b\sigma, c\tau, d\tau)] \in \mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L)$. Then $\phi = [\sigma(a, b, 0, 0) + \tau(0, 0, c, d)]$ and hence $\phi(\Sigma) \subset \ell$, that is, $\phi : \Sigma \to \ell$ is a branched covering map. Thus, the space $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \setminus \mathscr{C}_{L,d}$ parametrizes a family of contact curves in $\mathbb{CP}^3$ which project to totally geodesic 2-spheres in S^4 which pass through the north and south poles of S^4. Fixing the line ℓ in $\mathbb{CP}^3$ amounts to fixing a totally geodesic 2-sphere S_ℓ and the composition $\pi \circ \phi$ is just a branched covering map $\Sigma \to S_\ell$.

Note that for $d > 2g - 2$, $\dim W_L = \dim H^0(\Sigma, L) = d - g + 1$ and thus from (2.7) and (2.8) we have $\dim \mathscr{B}_L = d - g + 2$, and hence

$$(2.9) \qquad \dim(\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L)) = 2d - 2g + 3.$$

REMARK. In the genus zero case, since elements of $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L)$ are curves in $\mathbb{CP}^3$ which intersect $L_0 \cup L_1$, they are elements in the *boundary* of the moduli space described in [L].

Now consider the case when $\mathrm{Ram}(s, t) \neq 0$, that is, $s \wedge t \neq 0$. (Note that for a function $f = [s, t]$, we have $s \wedge t \neq 0$.) Since $[s, t] \in \mathbb{P}(W_L^2 \setminus \mathscr{B}_L)$ implies that $s \wedge t \neq 0$, we may consider $[s, t]$ as a projectivized ordered 2-frame in W_L. Then, the *projectivized Stiefel manifold* of ordered 2-frames in W_L, $\mathbb{P}V(2, W_L)$, is just the set $\mathbb{P}(W_L^2 \setminus \mathscr{B}_L)$. Now, consider the action

$$\mathrm{PGL}(2, \mathbb{C}) \times \mathbb{P}V(2, W_L) \longrightarrow \mathbb{P}V(2, W_L) \quad \text{defined by}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [s, t] \right) \longmapsto [as + bt, cs + dt].$$

Let $[s \wedge t]$ denote the 2-plane in W_L spanned by s and t. Observe that the principal $\mathrm{PGL}(2, \mathbb{C})$-bundle over the Grassmannian $G(2, W_L)$ is given by

$$\mathbb{P}V(2, W_L) \xrightarrow{p} G(2, W_L) \quad \text{where} \quad [s, t] \mapsto [s \wedge t].$$

Given a line bundle $\mathscr{L} \to \Sigma$ and a holomorphic section $\alpha \in H^0(\Sigma, \mathscr{L})$, we let (α) denote the divisor of α. We have an induced map

$$(2.10) \qquad (\text{Ram}) : \mathbb{P}V(2, W_L) = \mathbb{P}(W_L^2 \smallsetminus \mathscr{B}_L) \longrightarrow \mathbb{P}H^0(\Sigma, KL^2)$$

defined by sending a point $[s, t]$ to the divisor $(s\, dt - t\, ds)$. A simple computation shows that the ramification map (2.10) factors through the Grassmannian, that is, the following diagram commutes:

$$(2.11)$$

$$\begin{CD} \mathbb{P}V(2, W_L) @= \mathbb{P}V(2, W_L) \\ @V{P}VV @VV{(\text{Ram})}V \\ G(2, W_L) @>{[\text{Ram}]}>> \mathbb{P}H^0(\Sigma, KL^2), \end{CD}$$

where the map [Ram] is given by $[s \wedge t] \mapsto (s\, dt - t\, ds)$. (This is independent of the choice of spanning vectors of the 2-plane.) Consider the Plücker embedding $G(2, W_L) \hookrightarrow \mathbb{P}(\bigwedge^2 W_L)$. Recall equation (2.3):

$$(2.12) \qquad \Lambda : \bigwedge^2 W_L \longrightarrow H^0(\Sigma, KL^2).$$

Projectivizing (2.12), we obtain a rational map:

$$(2.13) \qquad \bar{\Lambda} : \mathbb{P}\bigwedge^2 W_L \dashrightarrow \mathbb{P}H^0(\Sigma, KL^2),$$

which when restricted to the image of the Grassmannian under the Plücker embedding (the projectivized simple bivectors) is just the map [Ram] in (2.11). Observe that since $\Lambda(v \wedge w) = 0$ implies that $v \wedge w = 0$, the base locus of the map $\bar{\Lambda}$ does not intersect the Plücker image of the Grassmannian. We next show that $\bar{\Lambda}$ restricted to the Plücker image of the Grassmannian is a finite map onto its image. We let $G(2, W_L)$ denote the Plücker image of the Grassmannian in $\mathbb{P}\bigwedge^2 W_L$.

LEMMA 2.1. *Let L be a line bundle of degree d over a compact Riemann surface Σ of genus g. Let $W_L = H^0(\Sigma, L)$, and suppose that $\dim W_L \geq 2$. Then the map $\bar{\Lambda}$ restricted to $G(2, W_L)$ is a finite map onto its image.*

PROOF. Let $\mu : A \to B$ be a linear map between two vector spaces and let Z denote its kernel. Then $A = Z \oplus A/Z$ and hence, up to a choice of coordinates, μ is just the second factor projection in A. The corresponding rational map from $\mathbb{P}A$ to $\mathbb{P}B$ is then nothing but the projection map in $\mathbb{P}A$ from the base locus $\mathbb{P}Z$ (up to a choice of coordinates). Thus the rational map $\bar{\Lambda}$ in (2.13) can be considered a projection map from the base locus. Since the Grassmannian does not intersect the base locus, we see that the fibre of the projection map containing a point x in the complement of the base

locus is just the span of x with the base locus. Thus, by definition of the projection map, the base locus has codimension one in the fibre. Observe that the intersection of this fibre with the Grassmannian cannot have positive dimension since otherwise it will contain a projective line which will intersect the base locus. This implies that $\bar{\Lambda}$ restricted to the Grassmannian is a finite map. $\square$

The above lemma is in fact a special case of a more general result:

PROPOSITION 2.2. *Let G denote the Grassmannian of k-planes in $\mathbb{C}^m$ and let f : $G \to \mathbb{CP}^n$ be a nontrivial holomorphic map. Then f is a finite map onto its image.*

PROOF. Consider the ample line bundle $\mathscr{L}$ on G which gives the Plücker embedding of G into $\mathbb{P}(\bigwedge^k \mathbb{C}^m)$. Observe that $\mathscr{L}$ is a generator of $\mathrm{Pic}(G)$. Then $f^*(\mathscr{O}(1)) = \mathscr{L}^N$ for some integer N. Since f is non-constant, $N \neq 0$. Since $\mathscr{O}(1)$ is ample, we can pull back nontrivial sections to $f^*(\mathscr{O}(1))$ and hence N is nonnegative. Therefore $N > 0$ and $\mathscr{L}^N$ is ample. Suppose f has a positive dimensional fibre, say F. Then $\mathscr{L}^N|_F$ is trivial, contradicting ampleness. Thus, f is finite. $\square$

PROPOSITION 2.3. *Let L be a line bundle of degree d over a compact Riemann surface Σ of genus g. Let $W_L = H^0(\Sigma, L)$, and suppose that $\dim W_L \geq 3$. Then the map $[\mathrm{Ram}] : G(2, W_L) \to [\mathrm{Ram}](G(2, W_L)) \subset \mathbb{P}H^0(\Sigma, KL^2)$ is proper. Moreover, the number of points in a general fibre is N for some positive integer N.*

PROOF. We first show that $[\mathrm{Ram}]$ is nonconstant. Let s_0, s_1 and s_2 be three linearly independent sections in W_L. Suppose that $s_0\, ds_1 - s_1\, ds_0 = s_0\, ds_2 - s_2\, ds_0$. This implies that $s_0\, ds_3 - s_3\, ds_0 = 0$ where $s_3 = s_1 - s_2$, and hence $[s_0 \wedge s_3] \notin G(2, W_L)$, that is, $s_0 \wedge s_3 = 0$. In other words, s_0, s_1 and s_2 are not linearly independent, a contradiction.

By Proposition 2.2, $[\mathrm{Ram}]^* \mathscr{O}(1) = \mathscr{L}^N$ for some positive integer N. By [F, Proposition 10.2] the general fibre has N points (counting multiplicity). $\square$

REMARK 2.4. By upper semi-continuity, special fibres may contain more than N points. However, if the image $[\mathrm{Ram}](G(2, W_L))$ is smooth, then $[\mathrm{Ram}]$ is flat and hence every fibre has N points. Observe also that N depends on L.

REMARK. If $\Sigma = S^2$, then $\dim W_L = d + 1$, $\dim \mathbb{P}H^0(KL^2) = 2d - 2 = \dim G(2, W_L)$. Thus, $[\mathrm{Ram}] : G(2, W_L) \to \mathbb{P}H^0(KL^2)$ is a holomorphic map between two complex manifolds of the same dimension and is hence a branched covering map. It was shown in [L] that the degree of this map is precisely the degree of $G(2, W_L)$ in $\mathbb{P}\bigwedge^2 W_L$ under the Plücker embedding. Thus every fibre of $[\mathrm{Ram}]$ has $(2d - 2)!/[d!(d - 1)!]$ points, counting multiplicity.

Recall that if $\deg L = d > 2g - 2$, then $\dim W_L = d - g + 1$ and so the requirement that $\dim W_L \geq 3$ implies that $d \geq g + 2$. Henceforth, we shall suppose that $d \geq \max\{2g - 1, g + 2\}$. Consider the diagram

$$G(2, W_L) \times G(2, W_L)$$

$$\downarrow {\scriptstyle [\text{Ram}] \times [\text{Ram}]}$$

$$\mathbb{P}H^0(\Sigma, KL^2) \xrightarrow{\ \Delta\ } \mathbb{P}H^0(\Sigma, KL^2) \times \mathbb{P}H^0(\Sigma, KL^2),$$

where Δ is the diagonal inclusion. By the finiteness result of Proposition 2.3, we see that the dimension of the fibre product is given by

$$\dim(G(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} G(2, W_L)) = \dim(G(2, W_L)) = 2(d - g - 1).$$

Observe that the fibre product $\mathbb{P}V(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} \mathbb{P}V(2, W_L)$ is just the total space of the $(\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C}))$-bundle over the fibre product $G(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} G(2, W_L)$. Let

$$([s, t], [u, v]) \in \mathbb{P}V(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} \mathbb{P}V(2, W_L).$$

Since

$$[\text{Ram}]([s \wedge t]) = [\text{Ram}]([u \wedge v]) \in \mathbb{P}H^0(\Sigma, KL^2),$$

there exists a *unique* $\lambda^2 \in \mathbb{C} \setminus \{0\}$ such that

$$\text{Ram}(s, t) = -\lambda^2 \, \text{Ram}(u, v) = -\text{Ram}(\pm\lambda u, \pm\lambda v),$$

thereby giving us a pair $[s, t, \pm\lambda u, \pm\lambda v] \in \mathcal{M}_{L,d}$. This gives us a 2:1 correspondence

$$\mathcal{M}_{L,d} \setminus \mathbb{P}(\mathcal{B}_L \times \mathcal{B}_L) \longrightarrow \mathbb{P}V(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} \mathbb{P}V(2, W_L) \quad \text{given by}$$

$$[s, t, \pm\lambda u, \pm\lambda v] \longmapsto ([s, t], [u, v]).$$

This 2:1 correspondence is directly related to the 2:1 branched covering map $\tilde{\psi}$ in (1.3), and the contact involution discussed in [L].

REMARK 2.5. Let $G = (\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))/\pm 1$, where $1 = (I, I)$.
1. G acts on $\mathcal{M}_{L,d}$ and leaves the set $\mathbb{P}(\mathcal{B}_L \times \mathcal{B}_L)$ invariant.
2. G acts freely on $\mathcal{M}_{L,d} \setminus \mathbb{P}(\mathcal{B}_L \times \mathcal{B}_L)$. In fact

$$\mathcal{M}_{L,d} \setminus \mathbb{P}(\mathcal{B}_L \times \mathcal{B}_L) \longrightarrow G(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} G(2, W_L)$$

given by

$$[s_0, s_1, s_2, s_3] \longmapsto ([s_0 \wedge s_1], [s_2 \wedge s_3])$$

is a principal G-bundle.

3. $G \longrightarrow PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ is a 2:1 covering map and thus so is

$$\mathscr{M}_{L,d} \smallsetminus \mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \longrightarrow \mathbb{P}V(2, W_L) \times_{\mathbb{P}H^0(\Sigma, KL^2)} \mathbb{P}V(2, W_L).$$

Thus,

$$(2.14) \qquad \dim \mathscr{S}_{L,d} = \dim \mathscr{M}_{L,d} = 2(d - g - 1) + 6 = 2(d - g + 2)$$

since $\dim G = 6$. Observe from (2.9) that $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L)$ is of codimension 1 in $\mathscr{M}_{L,d}$.

3. The moduli space via bundles over the Jacobian variety

Let J_d denote the Jacobian variety of equivalence classes of degree d line bundles over Σ (also denoted $\mathrm{Pic}^d(\Sigma)$). For each $j \in J_d$, let L_j denote the line bundle corresponding to j, and let $W_j = H^0(\Sigma, L_j)$.

REMARK 3.1. When $d > 2g - 2$, $\dim W_j = d - g + 1$ for all $j \in J_d$. Henceforth, we shall assume that $d \geq \max\{2g - 1, g + 2\}$.

Recall that there exists a line bundle, $\mathscr{P} \longrightarrow \Sigma \times J_d$, called the Poincaré line bundle, with the property that $\mathscr{P}|_{\Sigma \times \{j\}} \cong L_j$ (see [ACGH] for example). Let $\pi : \Sigma \times J_d \to J_d$ denote the second factor projection. Consider the direct image sheaf $\pi_*\mathscr{P} \to J_d$. Observe that

$$\pi_*\mathscr{P}|_j \cong H^0(\Sigma \times \{j\}, \mathscr{P}|_{\Sigma \times \{j\}}) \cong H^0(\Sigma, L_j) = W_j.$$

As a set, $\pi_*\mathscr{P} = \bigcup_{j \in J_d} W_j$. By Remark 3.1, since $d > 2g - 2$, each fibre is of dimension $d - g + 1$. In fact, $\pi_*\mathscr{P}$ has the structure of a complex vector bundle of rank $d - g + 1$ and it is called a Picard bundle. For simplicity, we let $\mathscr{W} = \pi_*\mathscr{P}$. Given a vector bundle ξ, we shall let $\mathscr{P}\xi$ denote the projective bundle of lines through the origin of the fibres of ξ. We consider the following associated bundles and sub-bundles over J_d :

1. $\mathscr{W}^k$, the Whitney sum of k copies of $\mathscr{W}$;
2. the projective bundle $\mathscr{P}\mathscr{W}^k$;
3. the rank $d - g + 2$ sub-bundle, $\mathscr{B}$, of $\mathscr{W}^2$, with fibre isomorphic to $\mathscr{B}_{L_j}$;
4. the projective bundle $\mathscr{P}(\mathscr{B} \oplus \mathscr{B})$, of rank $2d - 2g + 3$ with fibre isomorphic to $\mathbb{P}(\mathscr{B}_{L_j} \times \mathscr{B}_{L_j})$;
5. the bundle $\bigwedge^2 \mathscr{W}$ $(= \bigcup_{j \in J_d} \bigwedge^2 W_j$ as a set);
6. the Grassmann bundle of 2-planes, $\mathscr{G}$, with fibre isomorphic to $G(2, W_j)$;
7. the projective Stiefel bundle, $\mathscr{P}\mathscr{V}$ (this is the principal $PGL(2, \mathbb{C})$-bundle over $\mathscr{G}$).

Now consider the Jacobian variety J_{d+g-1}. For $j \in J_{d+g-1}$, the corresponding line bundle can be expressed as $\mathscr{L}_j = K^{1/2} \otimes L_j$, where L_j is a line bundle of degree d, and K is the canonical line bundle over Σ. Let $\mathscr{P}_1 \longrightarrow \Sigma \times J_{d+g-1}$ denote the Poincaré line bundle with the property that $\mathscr{P}_1|_{\Sigma \times \{j\}} \cong \mathscr{L}_j = K^{1/2} \otimes L_j$. Let $\pi : \Sigma \times J_{d+g-1} \to J_{d+g-1}$ denote the second factor projection and consider the direct image $\pi_*(\mathscr{P}_1 \otimes \mathscr{P}_1) \to J_{d+g-1}$. Observe that

$$\pi_*(\mathscr{P}_1 \otimes \mathscr{P}_1)|_j \cong H^0(\Sigma \times \{j\}, (\mathscr{P}_1 \otimes \mathscr{P}_1)|_{\Sigma \times \{j\}}) \cong H^0(\Sigma, K \otimes L_j^2).$$

Note that $\dim H^0(\Sigma, K \otimes L_j^2) = (2d + 2g - 2) - g + 1 = 2d + g - 1$ for each $j \in J_d$. Let $f : J_d \to J_{d+g-1}$ be defined by $L_j \mapsto K^{1/2} \otimes L_j$. Let $\mathscr{K} \to J_d$ denote the complex vector bundle of rank $2d + g - 1$ defined by $f^*(\pi_*(\mathscr{P}_1 \otimes \mathscr{P}_1))$. We have a bundle map

(3.1)
$$\begin{array}{ccc} \mathscr{W}^2 & \xrightarrow{\ \text{Ram}\ } & \mathscr{K} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ J_d & \xrightarrow{\ \text{Id}\ } & J_d, \end{array}$$

where the top arrow is defined by $(s, t) \mapsto \text{Ram}(s, t)$. Note that the inverse image under Ram of the zero section in $\mathscr{K}$ is just the bundle $\mathscr{B} \to J_d$. Consider the diagonal inclusion of $\mathscr{K}$ into the Whitney sum $\mathscr{K} \oplus \mathscr{K}$:

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\ \Delta\ } & \mathscr{K} \oplus \mathscr{K} \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi'} \\ J_d & \xrightarrow{\ \text{Id}\ } & J_d. \end{array}$$

Next, from the bundle maps over J_d:

(3.2)
$$\begin{array}{ccc} \mathscr{W}^2 \oplus \mathscr{W}^2 & = & \mathscr{W}^4 \\ & & \downarrow{\scriptstyle\text{Ram}\,\oplus\{-\,\text{Ram}\}} \\ \mathscr{K} & \xrightarrow{\ \Delta\ } & \mathscr{K} \oplus \mathscr{K}, \end{array}$$

where the maps on the base J_d are identity maps, consider the projectivized fibre product

$$\mathscr{M}_{d,\Sigma} := \mathbb{P}(\mathscr{W}^2 \times_{\mathscr{K}} \mathscr{W}^2) \subset \mathbb{P}\mathscr{W}^4,$$

that is, the fibrewise projectivization of the fibre product $\mathscr{W}^2 \times_{\mathscr{K}} \mathscr{W}^2 \subset \mathscr{W}^4$ (over J_d). Notice that the restriction of $\mathscr{M}_{d,\Sigma}$ to the zero section in $\mathscr{K}$ is just the bundle $\mathbb{P}(\mathscr{B} \times \mathscr{B}) \to J_d$. Recall that

$$\mathscr{C}_{L_j,d} := \{[s_0, s_1, s_2, s_3] \in \mathbb{P}(W_j^4) \mid \langle s_0, s_1, s_2, s_3 \rangle \text{ has base-points}\}.$$

Let $\mathscr{C}_{d,\Sigma} := \bigcup_{j \in J_d} \mathscr{C}_{L_j,d} \subset \mathbb{P}\mathscr{W}^4$.

REMARK 3.2. Branched superminimal immersions of Σ into S^4 of area $4\pi d$ are parametrized by the quasi-projective variety

$$\mathscr{S}_{d,\Sigma} := \mathscr{M}_{d,\Sigma} \smallsetminus \mathscr{C}_{d,\Sigma}.$$

Observe that the projective Stiefel bundle over J_d is just the bundle with fibre $\mathbb{P}(W_j^2 \smallsetminus \mathscr{B}_{L_j})$. Projectivizing the bundle $\mathscr{K}$ fibrewise, we have an associated $(\mathbb{P}^{2d+g-2})$-bundle, $\mathscr{P}\mathscr{K} \xrightarrow{\pi'} J_d$. The maps (Ram) and [Ram] extend to bundle maps:

$$
\begin{array}{ccc}
\mathscr{P}\mathscr{V} & \xrightarrow{(\text{Ram})} & \mathscr{P}\mathscr{K} \\
\pi \downarrow & & \downarrow \pi' \\
J_d & \xrightarrow{\text{Id}} & J_d
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathscr{G} & \xrightarrow{[\text{Ram}]} & \mathscr{P}\mathscr{K} \\
\pi \downarrow & & \downarrow \pi' \\
J_d & \xrightarrow{\text{Id}} & J_d .
\end{array}
$$

Consider the diagonal inclusion of $\mathscr{P}\mathscr{K}$ into the Whitney sum $\mathscr{P}\mathscr{K} \oplus \mathscr{P}\mathscr{K}$:

$$
\begin{array}{ccc}
\mathscr{P}\mathscr{K} & \xrightarrow{\Delta} & \mathscr{P}\mathscr{K} \oplus \mathscr{P}\mathscr{K} \\
\pi' \downarrow & & \downarrow \pi' \\
J_d & \xrightarrow{\text{Id}} & J_d .
\end{array}
$$

Now, from the diagrams:

$$
\begin{array}{ccc}
\mathscr{G} \oplus \mathscr{G} & & \\
\downarrow {[\text{Ram}]\oplus[\text{Ram}]} & & \\
\mathscr{P}\mathscr{K} \xrightarrow{\Delta} \mathscr{P}\mathscr{K} \oplus \mathscr{P}\mathscr{K} & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
& & \mathscr{P}\mathscr{V} \oplus \mathscr{P}\mathscr{V} \\
& & \downarrow {(\text{Ram})\oplus(\text{Ram})} \\
& & \mathscr{P}\mathscr{K} \xrightarrow{\Delta} \mathscr{P}\mathscr{K} \oplus \mathscr{P}\mathscr{K} ,
\end{array}
$$

where the maps on the base J_d are identity maps, consider the fibre products $\mathscr{G} \times_{\mathscr{P}\mathscr{K}} \mathscr{G}$ and $\mathscr{P}\mathscr{V} \times_{\mathscr{P}\mathscr{K}} \mathscr{P}\mathscr{V}$. Since $\dim J_d = g$ we see that

$$\dim \mathscr{G} \times_{\mathscr{P}\mathscr{K}} \mathscr{G} = \dim \mathscr{G} = (2d - 2g - 2) + g = 2d - g - 2$$

and so $\dim \mathscr{P}\mathscr{V} \times_{\mathscr{P}\mathscr{K}} \mathscr{P}\mathscr{V} = 2d - g - 2 + 6 = 2d - g + 4$. Thus,

$$(3.3) \qquad \dim \mathscr{S}_{d,\Sigma} = \dim \mathscr{M}_{d,\Sigma} = \dim \mathscr{P}\mathscr{V} \times_{\mathscr{P}\mathscr{K}} \mathscr{P}\mathscr{V} = 2d - g + 4.$$

4. Path-connectedness

Let us first fix a line bundle L of degree d over Σ. We claim that $\mathscr{S}_{L,d}$ is path-connected.

We first show that any point in $\mathscr{S}_{L,d}$ can be connected by a path to a point in the path-connected space $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$. Let $[s_0, s_1, s_2, s_3] \in \mathscr{S}_{L,d}$. Suppose that $\langle s_0, s_2 \rangle$ has a base-point, say p. Since $\langle s_0, s_1, s_2, s_3 \rangle$ is base-point-free, then either $s_1(p) \neq 0$ or $s_3(p) \neq 0$. Without loss of generality, suppose that $s_1(p) \neq 0$. Consider a path $C \subset \mathscr{S}_{L,d}$ defined by

$$C(t) := [s_0 + ts_1, s_1, s_2, s_3].$$

Then p is not a base-point of $\langle s_0 + ts_1, s_2 \rangle$ for $t \neq 0$. Thus, we may assume that $\langle s_0, s_2 \rangle$ is base-point-free. Consider a path $\gamma \subset \mathscr{S}_{L,d}$ defined by

$$\gamma(t) := [s_0, (1-t)s_0 + ts_1, s_2, (1-t)s_2 + ts_3]$$

with $\gamma(1) = [s_0, s_1, s_2, s_3]$ and $\gamma(0) = [s_0, s_0, s_2, s_2] \in \mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$. We now show that $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$ is path-connected. Observe that any point $[a\sigma_0, b\sigma_0, c\sigma_2, d\sigma_2] \in \mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$ is path-connected to the point $[\sigma_0, \sigma_0, \sigma_2, \sigma_2]$ via a path in $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$. Let $[s_0, s_0, s_2, s_2]$ be another point in $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$. For a given $p \in \Sigma$, let $W_p = \{s \in W_L \mid s(p) = 0\}$. Observe that the map $W_L \to \mathbb{C}$ given by $s \mapsto s(p)$ is a linear map.

REMARK. If $d \geq 2g$, then the complete linear system $\mathbb{P} W_L$ has no base-points and thus W_p is a complex hyperplane in W_L. Since it is a real codimension two subspace of W_L, it cannot disconnect W_L. Henceforth, we shall assume that $d \geq 2g$.

Let $A = \{p \in \Sigma \mid s_2(p) = 0\}$ and consider the finite union of complex hyperplanes in W_L: $\mathscr{A} = \bigcup_{p \in A} W_p$. Since $\mathscr{A}$ is of real codimension two, there is a path in $W_L \smallsetminus \mathscr{A}$ which connects s_0 and σ_0. Thus, there is a path in $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$ which connects $[s_0, s_0, s_2, s_2]$ to $[\sigma_0, \sigma_0, s_2, s_2]$. Now let $A' = \{p \in \Sigma \mid \sigma_0(p) = 0\}$ and let $\mathscr{A}' = \bigcup_{p \in A'} W_p$. By a similar argument, we can join s_2 to σ_2 via a path in $W_L \smallsetminus \mathscr{A}'$, and thus there is a path in $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$ which connects the points $[\sigma_0, \sigma_0, s_2, s_2]$ and $[\sigma_0, \sigma_0, \sigma_2, \sigma_2]$. This proves that the space $\mathbb{P}(\mathscr{B}_L \times \mathscr{B}_L) \smallsetminus \mathscr{C}_{L,d}$ is path-connected.

THEOREM 4.1. For $d \geq \max\{2g, g+2\}$, $\mathscr{S}_{d,\Sigma}$ is a path-connected quasi-projective variety of dimension $2d - g + 4$.

PROOF. We have shown that $\mathscr{S}_{L,d}$ is path-connected for each line bundle L of degree $d \geq \max\{2g, g+2\}$. Path-connectedness of $\mathscr{S}_{d,\Sigma}$ follows from the fact that J_d is a connected Abelian variety. Quasi-projectiveness follows since projective bundles over projective varieties are projective. The dimension comes from (3.3). $\qquad\square$

5. Compactifying

Recall that $\mathscr{S}_{d,\Sigma} := \mathscr{M}_{d,\Sigma} \smallsetminus \mathscr{C}_{d,\Sigma}$, where $\mathscr{M}_{d,\Sigma} := \mathscr{P}(\mathscr{W}^2 \times_{\mathscr{X}} \mathscr{W}^2) \subset \mathscr{P}\mathscr{W}^4$. Since $\mathscr{M}_{d,\Sigma}$ is compact, it is an obvious compactification of $\mathscr{S}_{d,\Sigma}$. Unfortunately, elements of $\mathscr{C}_{d,\Sigma}$ correspond to parametrized curves of degree strictly less than d. Thus, from the geometric viewpoint, $\mathscr{M}_{d,\Sigma}$ is not a satisfactory compactification of the moduli space, $\mathscr{S}_{d,\Sigma}$, of branched superminimal immersions of Σ into S^4 with area $4\pi d$. In this section, we shall consider the genus zero case, that is, when $\Sigma = \mathbb{CP}^1$. (Hence, J_d has just one element—$\mathscr{O}(d)$.) We shall let $\mathscr{S}_d = \mathscr{S}_{d,\mathbb{CP}^1}$.

Since we are considering spaces of maps, it is appropriate to study the corresponding spaces of graphs. Recall that the graph of a map $f : M \to N$ is the subset $\Gamma_f = \{(x, f(x)) \mid x \in M\} \subset M \times N$. In our case, $M = \mathbb{CP}^1$ and $N = \mathbb{CP}^3$. Let $\pi_1 : \mathbb{CP}^1 \times \mathbb{CP}^3 \to \mathbb{CP}^1$ and $\pi_2 : \mathbb{CP}^1 \times \mathbb{CP}^3 \to \mathbb{CP}^3$ denote the first and second factor projection maps respectively. Let C be a curve in $\mathbb{CP}^1 \times \mathbb{CP}^3$. Since $H_2(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$, we have a well-defined notion of homology degree for $\pi_i(C)$, $i = 1, 2$. We say that C has bidegree (m, n) if $\deg(\pi_1(C)) = m$ and $\deg(\pi_2(C)) = n$. Thus, each element $s \in \mathscr{S}_d$ corresponds to a *smooth* curve $\Gamma_s \subset \mathbb{CP}^1 \times \mathbb{CP}^3$ of bidegree $(1, d)$. Let $\mathscr{H}_{1,d}$ denote the Hilbert scheme of rational curves of bidegree $(1, d)$ in $\mathbb{CP}^1 \times \mathbb{CP}^3$ and let $\mathscr{G}_{1,d} \subset \mathscr{H}_{1,d}$ denote the component of the Hilbert scheme containing the graphs of degree d maps of $\mathbb{CP}^1$ to $\mathbb{CP}^3$. We have a map $\Gamma : \mathscr{S}_d \to \mathscr{G}_{1,d}$ defined by $s \mapsto \Gamma_s$. The desired compactification is then given by the closure of $\Gamma(\mathscr{S}_d)$ in $\mathscr{G}_{1,d}$. We shall call the elements of the boundary 'limit graphs'. The question is: what do the limit graphs look like? This is a rather difficult question. Nevertheless, we expect a limit graph to be a curve, not necessarily irreducible, of bidegree $(1, d)$.

Let z_0, z_1 and x_0, x_1, x_2, x_3 denote the homogeneous variables for $\mathbb{CP}^1$ and $\mathbb{CP}^3$ respectively. Let S_z^n and S_x^n denote the space of homogeneous polynomials of degree n in the variables z_0, z_1, and the variables x_0, x_1, x_2, x_3 respectively. Since $H^0(\mathbb{CP}^1, \mathscr{O}(d)) = S_z^d$, any parametrized rational curve of degree d in $\mathbb{CP}^3$ is of the form $s = [s_0, s_1, s_2, s_3]$ where $s_i \in S_z^d$ for $i = 0, \ldots, 3$, and $\gcd(s_0, s_1, s_2, s_3) = 1$. (Elements in $\mathscr{S}_d$ have the additional property that $\mathrm{Ram}(s_0, s_1) = -\mathrm{Ram}(s_2, s_3)$.) The graph $\Gamma_s \subset \mathbb{CP}^1 \times \mathbb{CP}^3$ is defined by 6 bihomogeneous equations:

$$(5.1) \qquad F_{i,j} := s_i x_j - s_j x_i = 0, \quad \text{where } i < j, \quad i = 0, 1, 2, \quad j = 1, 2, 3.$$

Observe that the $F_{i,j}$'s are elements of $S_z^d S_x^1$, the tensor product of S_z^d and S_x^1. From (5.1) we see that in general Γ_s is not a complete intersection. This makes the study of the boundary elements much more difficult. An obvious way of compactifying the space of graphs is by dropping the gcd condition. (This corresponds to the inclusion of $\mathscr{C}_d$ in our space.) One approach to understanding limit graphs is to consider 1-parameter families. More precisely, consider a curve φ_t in $\mathscr{M}_d$ such that $\varphi_0 = c \in \mathscr{C}_d$. Suppose that $\varphi_t \smallsetminus \{c\} \subset \mathscr{S}_d$. Then $\varphi_t \smallsetminus \{c\}$ parametrizes a flat family of curves in $\mathbb{CP}^1 \times \mathbb{CP}^3$.

Since flatness extends uniquely across a puncture, we obtain a unique limit graph associated to the curve φ_t. Let us first consider the following simple example.

EXAMPLE 5.1. Let $\varphi_t = [z_0, z_0, z_0 + tz_1, z_0 + tz_1]$. Note that $\varphi_t \in \mathscr{S}_1$ for $t \neq 0$ and $\varphi_0 \in \mathscr{C}_1$. The equations for Γ_{φ_t} are obtained by setting each of the following six polynomials equal to zero:

$$
\begin{array}{ll}
F_{0,1}^t := z_0(x_1 - x_0), & F_{0,2}^t := z_0(x_2 - x_0) - tz_1 x_0, \\
(5.2) \qquad F_{0,3}^t := z_0(x_3 - x_0) - tz_1 x_0, & F_{1,2}^t := z_0(x_2 - x_1) - tz_1 x_1, \\
F_{1,3}^t := z_0(x_3 - x_1) - tz_1 x_1, & F_{2,3}^t := (z_0 + tz_1)(x_3 - x_2).
\end{array}
$$

Set $S = \mathbb{C}[x_0, x_1, x_2, x_3, z_0, z_1]$. For each constant t, let $I_t = (F_{i,j}^t) \subset S$ denote the ideal generated by the 6 bihomogeneous polynomials defined in (5.2). Then the graph described by the system of equations obtained by setting the polynomials in (5.2) equal to zero is just the variety $V(I_t)$ for $t \neq 0$. It is tempting to say that the limit as t approaches 0 of $\Gamma_{\varphi_t} = V(I_t)$ is the variety $V(I_0)$. However, if we simply set $t = 0$ in the system of equations obtained from (5.2), we obtain

$$
(5.3) \qquad V(I_0) = \mathbb{CP}^1 \times \{[1, 1, 1, 1]\} \cup \{[0, 1]\} \times \mathbb{CP}^3 \subset \mathbb{CP}^1 \times \mathbb{CP}^3.
$$

Observe that the $\{[0, 1]\} \times \mathbb{CP}^3$ term above is of dimension three. Since this notion of a limit of curves obtained via a limit of equations does not preserve dimension, it is not satisfactory from the geometric viewpoint. We have to look for an alternative approach for taking limits. We would like to complete the picture in (5.3) by 'filling in' a curve in the second factor of the $\{[0, 1]\} \times \mathbb{CP}^3$ term. To do this, we need to enlarge the ideal I_t via an algebraic process called *saturation*. We proceed as follows. By setting each of the polynomials in (5.2) equal to zero we obtain the following system of equations:

$$
\begin{array}{ll}
F_{0,1}^t(x_i - x_0) - F_{0,i}^t(x_1 - x_0) = tz_1 x_0(x_1 - x_0), & i = 2, 3; \\
F_{0,1}^t(x_i - x_1) - F_{1,i}^t(x_1 - x_0) = tz_1 x_1(x_1 - x_0), & i = 2, 3; \\
F_{1,i}^t(x_i - x_0) - F_{0,i}^t(x_i - x_1) = tz_1 x_i(x_1 - x_0), & i = 2, 3; \\
(5.4) \qquad F_{1,2}^t(x_3 - x_0) - F_{0,3}^t(x_2 - x_1) = tz_1[x_0 x_2 - x_1 x_3]; \\
F_{1,3}^t(x_2 - x_0) - F_{0,2}^t(x_3 - x_1) = tz_1[x_0 x_3 - x_1 x_2]; \\
F_{2,3}^t(x_1 - x_0) - F_{0,1}^t(x_3 - x_2) = tz_1(x_3 - x_2)(x_1 - x_0); \\
F_{2,3}^t(x_i - x_1) - F_{1,i}^t(x_3 - x_2) = tz_1 x_i(x_3 - x_2), & i = 2, 3.
\end{array}
$$

Consider the polynomials on the right side of each equality of (5.4) with $t = 1$. Let these polynomials: $z_1 x_0(x_1 - x_0)$, $z_1 x_1(x_1 - x_0)$, $z_1 x_2(x_1 - x_0)$, $z_1 x_3(x_1 - x_0)$, $z_1(x_0 x_2 - x_1 x_3)$, $z_1(x_0 x_3 - x_1 x_2)$, $z_1 x_2(x_3 - x_2)$, $z_1 x_3(x_3 - x_2)$ and $z_1(x_3 - x_2)(x_1 - x_0)$,

together with the six generators of I_t be the generators of an enlarged ideal, call it J_t. It follows from the system of equations (5.4) that $J_t = I_t$ for $t \neq 0$, and thus we have equality of the corresponding varieties $V(J_t) = V(I_t)$ when $t \neq 0$. Note that two bihomogeneous ideals define the same subscheme of $\mathbb{CP}^1 \times \mathbb{CP}^3$ if and only if they induce the same dehomogenizations. Let $I'_t = (F^t_{i,j}, z_1(x_1 - x_0), z_1(x_3 - x_2))$. Observe that I'_t and J_t have the same dehomogenizations. From the $z_1 \neq 0$ chart, we see that the last two generators of I'_0 imply that $x_0 = x_1$ and $x_2 = x_3$, thus giving us a line L_1 in $\mathbb{CP}^3$ spanned by $[1, 1, 0, 0]$ and $[0, 0, 1, 1]$. We thus obtain the limit graph

$$V(I'_0) = \mathbb{CP}^1 \times \{[1, 1, 1, 1]\} \cup \{[0, 1]\} \times L_1 \subset \mathbb{CP}^1 \times \mathbb{CP}^3.$$

This is indeed a curve, albeit one with two irreducible components. The first component is of bidegree $(1, 0)$, corresponding to a graph of a constant map $z \mapsto [1, 1, 1, 1]$. The second component is a curve of bidegree $(0, 1)$, and is therefore not a graph of any map to $\mathbb{CP}^3$; rather it is a 'bubble' when we interpret the limit as t approaches 0 of the 1-parameter family of maps $[s_0, (1 - t)s_0 + ts_1, s_2, (1 - t)s_2 + ts_3]$ as the second factor projection map of the limit graph:

$$\mathbb{CP}^1 \times \{[1, 1, 1, 1]\} \cup \{[0, 1]\} \times L_1 \longmapsto \mathbb{CP}^3.$$

Observe that for each t the image of $\varphi_t : \mathbb{CP}^1 \to \mathbb{CP}^3$ lies in the line L_1 and thus it is not at all surprising that the image of the limit map is L_1.

Now let us consider what happens when we have a different curve emanating from the same point $\varphi_0 \in \mathscr{C}_1$.

EXAMPLE 5.2. Let $\psi_t = [z_0 + tz_1, z_0, z_0, z_0 + tz_1]$. Then $\psi_0 = \varphi_0$. Let L_2 denote the line in $\mathbb{CP}^3$ spanned by $[1, 0, 0, 1]$ and $[0, 1, 1, 0]$. Observe that for each t the image of $\psi_t : \mathbb{CP}^1 \to \mathbb{CP}^3$ lies in the line L_2. The six bihomogeneous polynomials corresponding to Γ_{ψ_t} are:

$$\begin{aligned}
& G^t_{0,1} := z_0(x_1 - x_0) + tz_1x_1, \ G^t_{0,2} := z_0(x_2 - x_0) + tz_1x_1, \\
(5.5) \quad & G^t_{0,3} := (z_0 + tz_1)(x_3 - x_0), \ G^t_{1,2} := z_0(x_2 - x_1), \\
& G^t_{1,3} := z_0(x_3 - x_1) - tz_1x_1, \ G^t_{2,3} := z_0(x_3 - x_2) - tz_1(x_2 - x_1).
\end{aligned}$$

Proceeding as in Example 5.1, we obtain $I'_t = (G^t_{i,j}, z_1(x_2 - x_1), z_1(x_3 - x_0))$ and thus the limit graph is

$$V(I'_0) = \mathbb{CP}^1 \times \{[1, 1, 1, 1]\} \cup \{[0, 1]\} \times L_2 \subset \mathbb{CP}^1 \times \mathbb{CP}^3.$$

Observe that this limit graph is distinct from that obtained in Example 5.1 thus illustrating the fact that limits of graphs depend on the choice of a one-parameter family.

Examples 5.1 and 5.2 generalize as follows. Let $s = [s_0, s_1, s_2, s_3] \in \mathcal{M}_d$ such that $\gcd(s_0, s_1, s_2, s_3) = D \in S_z^m$, for some $m > 0$. Then $s_i = D\sigma_i$, for some $\sigma_i \in S_z^{d-m}$, $i = 0, 1, 2, 3$. Setting

$$(5.6) \qquad F_{i,j}' = \sigma_i x_j - \sigma_j x_i, \quad \text{where } i, j \in \{0, 1, 2\} \text{ and } i < j,$$

we have six bihomogeneous polynomials:

$$(5.7) \qquad F_{i,j} = D(\sigma_i x_j - \sigma_j x_i) = DF_{i,j}', \text{ where } i, j \in \{0, 1, 2\} \text{ and } i < j.$$

The variety of the ideal generated by the polynomials in (5.6) is a graph of bidegree $(1, d - m)$ in $\mathbb{CP}^1 \times \mathbb{CP}^3$ corresponding to the map given by $\sigma = [\sigma_0, \sigma_1, \sigma_2, \sigma_3]$. Suppose that $p_1, \ldots, p_m$ are the roots of D. Then the variety of the ideal generated by the polynomials in (5.7) is

$$\Gamma_\sigma \cup \bigcup_{i=1}^m (\{p_i\} \times \mathbb{CP}^3) \subset \mathbb{CP}^1 \times \mathbb{CP}^3.$$

In order to complete this picture, we consider a curve φ_t in $\mathcal{M}_d$ such that $\varphi_0 = s \in \mathcal{C}_d$ and $\varphi \smallsetminus \{s\} \subset \mathcal{S}_d$. From the ideal $I_t = (F_{0,i}', F_{1,j}', F_{2,k}', F_{3,l}')$, we perform some algebra to obtain its saturation I_t'. The limit graph, $V(I_0')$, has at least 2 irreducible components. The first component is Γ_σ and the other components have total bidegree $(0, m)$. This process, however, is rather tedious—even in the simple case of a curve of bidegree $(1, 1)$ as seen in Examples 5.1 and 5.2.

REMARK. In [LV1], a simpler compactification problem was studied: the variety parametrizing graphs of holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$. (The graph of such a map is a complete intersection.) Here a global approach was taken: obtain the compactification via an explicit sequence of blow-ups along smooth centres. The bidegree (1,1) case is described in [LV1], and the bidegree (2,2) case in [LV2]. It is hoped that an understanding of this simpler compactification problem will be of use in the compactification problem for the moduli space of superminimal surfaces in S^4.

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