MEAN CURVATURE AND SHAPE OPERATOR OF ISOMETRIC IMMERSIONS IN REAL-SPACE-FORMS

by BANG-YEN CHEN

(Received 30 August, 1944; revised 25 November, 1994)

1. Introduction. According to the well-known Nash's theorem, every Riemannian n-manifold admits an isometric immersion into the Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$. In general, there exist enormously many isometric immersions from a Riemannian manifold into Euclidean spaces if no restriction on the codimension is made. For a submanifold of a Riemannian manifold there are associated several extrinsic invariants beside its intrinsic invariants. Among the extrinsic invariants, the mean curvature function and shape operator are the most fundamental ones.

One of the very basic problems in submanifold theory is the following.

PROBLEM 1. *Find a simple relationship between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

Many famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's inequality, and Gauss-Bonnet's theorem among others, can be regarded as results in this respect.

In this paper we consider isometric immersions of Riemannian manifolds into real-space-forms *with arbitrary codimensions* unless mentioned otherwise. In §3 we establish a general sharp inequality between the main extrinsic invariant; being the squared mean curvature function, and the main intrinsic invariant; being the scalar curvature. By using this general and sharp inequality we obtain simple geometric characterizations for hyperspheres, spherical hypercylinders, among some others, in Euclidean space. We also utilize the inequality to obtain a simple geometric characterization of the isoparametric hypersurface $N^{2,n-2}:=H^2(-\frac{1}{2})\times S^{n-2}(1)$ imbedded in the hyperbolic space $H^{n+1}(-1)$ in a standard way. In the last part of this article we establish a sharp relationship between sectional curvature and the shape operator for submanifolds in real-space-forms.

2. Preliminaries. Let M^n be an n-dimensional submanifold of a real-space-form $R^m(\bar{c})$ of constant sectional curvature $\bar{c}$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M^n and $R^m(\bar{c})$, respectively. Then the Gauss and Weingarten formulas of M^n in $R^m(\bar{c})$ are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X, Y tangent to M^n and ξ normal to M^n, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold in $R^m(\bar{c})$. The second fundamental form and the shape operator are related by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle. \tag{2.3}$$

Moreover, the mean curvature vector H of the submanifold M^n is defined by $H = \frac{1}{n}$ trace h. A submanifold M^n is said to be *totally geodesic* at a point $p \in M^n$ if its second fundamental form vanishes at p. And it is said to be totally umbilical at p if there is a real number λ such that $h(X, Y) = \lambda\langle X, Y\rangle H$ for any $X, Y \in T_pM^n$.

Denote by R the Riemann curvature tensor of M^n. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) = (\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)\bar{c} \\ + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{2.4}$$

for vectors X, Y, Z, W tangent to M^n.

For the second fundamental form h, we define the covariant derivative $\bar{\nabla}h$ of h with respect to the connection in $TM^n \oplus T^\perp M^n$ by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.5}$$

The *equation of Codazzi* is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{2.6}$$

For a Riemannian n-manifold M^n, denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM^n$, $p \in M^n$. For an orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM^n, the *scalar curvature* τ and the *normalized scalar curvature* ρ are defined respectively by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \qquad \rho = \frac{2\tau}{n(n-1)}. \tag{2.7}$$

For each point $p \in M^n$, let

$$(\inf K)(p) = \inf\{K(\pi)\colon \text{plane sections } \pi \subset T_pM^n\}.$$

Then $\inf K$ is a well-defined function on M^n. We define δ_M to be the Riemannian invariant given by

$$\delta_M(p) = \tau(p) - \inf K(p). \tag{2.8}$$

We recall the following general result for later use.

THEOREM A ([2]). *Let M^n be an n-dimensional ($n \geq 3$) submanifold of a real-space-form $R^m(\bar{c})$ of constant sectional curvature $\bar{c}$. Then*

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}|H|^2 + (n+1)\bar{c}\right\}. \tag{2.9}$$

Equality holds if and only if, with respect to suitable orthonormal frame fields $e_1, \ldots, e_n$, $e_{n+1}, \ldots, e_m$, the shape operator takes the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & \mu - a & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix}, \tag{2.10}$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \dots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, r = n+2, \dots, m. \tag{2.11}$$

Furthermore, when the equality sign of (2.9) *holds at a point* $p \in M^n$, *we also have* $K(e_1 \wedge e_2) = \inf K$ *at point* p.

For further results on the Riemannian manifold δ, see **[3, 4]** and the references cited in **[3, 4]**.

3. Sharp estimate of squared mean curvature for product manifolds in real-space-forms and its applications. It is well-known that $|H|^2 \equiv 1$ for the standard isometric imbedding of the unit n-sphere $S^n(1)$ in $\mathbb{E}^{n+1}$. Therefore, it is natural to consider the following geometric question.

PROBLEM 2. *Let* $x: S^n(1) \to \mathbb{E}^m$ *be an arbitrary isometric immersion from a unit* n*-sphere into* $\mathbb{E}^m$. *Is it true in general that* $|H|^2 \geq 1$?

In this section we will establish a very general result which in particular provides an affirmative answer to this Problem. In order to do so, we need the following lemma.

LEMMA 1. *Let* $x: M^n \to R^m(\bar{c})$ *be an isometric immersion of a Riemannian* n*-manifold* M^n *with normalized scalar curvature* ρ *into an* m*-dimensional real-space-form* $R^m(\bar{c})$ *of constant sectional curvature* $\bar{c}$. *Then we have*

$$|H|^2 \geq \rho - \bar{c}, \tag{3.1}$$

equality holding at a point $p \in M^n$ *if and only if* p *is a totally umbilical point.*

Proof. Choose an orthonormal basis $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ at p such that e_{n+1} is parallel to the mean curvature vector and $e_1, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{3.2}$$

$$A_r = (h^r_{ij}), \qquad \sum_{i=1}^{n} h^r_{ii} = 0, \qquad r = n+2, \dots, m.$$

For any $i (1 \leq i \leq n)$, the equation of Gauss yields

$$S(e_i, e_i) = (n-1)\bar{c} + a_i \sum_{t \neq i} a_t - \sum_{r=n+2}^{n} \sum_{t=1}^{n} (h^r_{it})^2. \tag{3.3}$$

From (3.3) we find

$$n^2H^2 = 2\tau + \sum_i a_i^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - n(n-1)\bar{c}. \tag{3.4}$$

On the other hand, since

$$\sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j, \tag{3.5}$$

we get $\sum_i a_i^2 \geq n\,|H|^2$. Combining this with (3.4), we obtain

$$n(n-1)\,|H|^2 \geq 2\tau - n(n-1)\bar{c} + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \tag{3.6}$$

which implies inequality (3.1). If the equality sign of (3.1) holds at a point $p \in M^n$, then from (3.5) and (3.6) we get $A_r = 0$, $r = n+2, \ldots, m$ and $a_1 = \ldots = a_n$. Therefore, p is a totally umbilical point. The converse is trivial. □

By applying Lemma 1 we may establish the following sharp estimate of the squared mean curvature function for isometric immersions of Riemannian product manifolds in a real-space-form $R^m(\bar{c})$.

Theorem 1. *Let $M^n = N_1^{n_1} \times \ldots \times N_l^{n_l}$ be the Riemannian product of l Riemannian manifolds. Then, for any $m > n$ and any isometric immersion $f: M^n \to R^m(\bar{c})$, we have*

$$|H|^2 \geq \sum_{\alpha=1}^{l} \left(\frac{n_\alpha}{n}\right)^2 \rho_\alpha - \bar{c}, \tag{3.7}$$

where $\rho_1, \ldots, \rho_l$ are the normalized scalar curvature of $N_1^{n_1}, \ldots, N_l^{n_l}$, respectively.

Proof. Let $e_1^\alpha, \ldots, e_{n_\alpha}^\alpha$ denote an orthonormal basis for $N_\alpha^{n_\alpha}$. Then

$$e_1^1, \ldots, e_{n_1}^1, \ldots, e_1^l, \ldots, e_{n_l}^l$$

can be regarded as an orthonormal basis of M^n in a natural way. Let h be the second fundamental form of M^n in $R^m(\bar{c})$. For each $\alpha \in \{1, \ldots, l\}$, we put

$$\operatorname{trace} h^\alpha = \sum_{t=1}^{n_\alpha} h(e_t^\alpha, e_t^\alpha). \tag{3.8}$$

Then we have

$$nH = \sum_{\alpha=1}^{l} \operatorname{trace} h^\alpha. \tag{3.9}$$

Choose a point $(p_1, \ldots, p_l)$ in $N_1^{n_1} \times \ldots \times N_l^{n_l}$. For each $\alpha \in \{1, \ldots, l\}$, let ι_α denote the inclusion map from $N_\alpha^{n_\alpha}$ into $N_1^{n_1} \times \ldots \times N_l^{n_l}$ defined by $\iota_\alpha(x) = (p_1, \ldots, p_{\alpha-1}, x, p_{\alpha+1}, \ldots, p_l)$, $x \in N_\alpha^{n_\alpha}$. Let $f_\alpha = f \circ \iota_\alpha$ be the isometric immersion from $N_\alpha^{n_\alpha}$ into $R^m(\bar{c})$ given by the composition of f and ι_α. Since the inclusion map ι_α is a totally geodesic isometric imbedding, the mean curvature vector H_α of $f_\alpha : N_\alpha^{n_\alpha} \to R^m(\bar{c})$ is given by $H_\alpha = (\text{trace } h^\alpha)/n_\alpha$. Thus we obtain

$$H = \sum_{\alpha=1}^{l} \left(\frac{n_\alpha}{n}\right) H_\alpha. \tag{3.10}$$

For distinct α and β, the equation of Gauss yields

$$\langle h(e_i^\alpha, e_i^\alpha), h(e_j^\beta, e_j^\beta)\rangle = \langle h(e_i^\alpha, e_j^\beta), h(e_i^\alpha, e_j^\beta)\rangle - \bar{c}, \tag{3.11}$$

for unit vectors e_i^α, e_j^β tangent to $N_\alpha^{n_\alpha}$ and $N_\beta^{n_\beta}$, respectively. Thus, we find

$$n_\alpha n_\beta \langle H_\alpha, H_\beta\rangle = \sum_{i=1}^{n_\alpha} \sum_{j=1}^{n_\beta} |h(e_i^\alpha, e_j^\beta)|^2 - n_\alpha n_\beta \bar{c}. \tag{3.12}$$

From (3.7)–(3.12) we obtain

$$n^2 |H|^2 = \sum_{\alpha=1}^{l} n_\alpha^2 |H_\alpha|^2 + \sum_{\alpha\neq\beta} \sum_{i=1}^{n_\alpha} \sum_{j=1}^{n_\beta} |h(e_i^\alpha, e_j^\beta)|^2 - \sum_{\alpha\neq\beta} n_\alpha n_\beta \bar{c}. \tag{3.13}$$

On the other hand, Lemma 1 implies

$$|H_\alpha|^2 \geq \rho_\alpha - \bar{c}, \tag{3.14}$$

where the equality case holds if and only if the immersion f_α is a totally umbilical immersion.

From (3.13) and (3.14) we obtain inequality (3.7). □

Theorem 1 can be utilized to obtain some simple geometrical characterization theorems. For instance, for product Riemannian manifolds in a Euclidean m-space, we have the following.

THEOREM 2. *Let $M^n = N_1^{n_1} \times \ldots \times N_l^{n_l}$ be the Riemannian product of l Riemannian manifolds. Then, for any $m > n$ and any isometric immersion $f : M^n \to \mathbb{E}^m$, we have*

$$|H|^2 \geq \sum_{\alpha=1}^{l} \left(\frac{n_\alpha}{n}\right)^2 \rho_\alpha. \tag{3.15}$$

The equality case of (3.15) *holds identically if and only if* (1) $\rho_1, \ldots, \rho_l$ *are non-negative constants*, (2) *each component $N_\alpha^{n_\alpha}$ is isometric to an open portion of an n_α-sphere $S^{n_\alpha}(\rho_\alpha)$ or isometric to an open portion of an n_α-plane $\mathbb{E}^{n_\alpha}$*, and (3) *f is the product of l isometric immersions $f_\alpha : N_\alpha^{n_\alpha} \to \mathbb{E}^{m_\alpha}$, $\alpha = 1, \ldots, l$, where each f_α is given either by a totally umbilical isometric immersion of an n_α-sphere or by a totally geodesic isometric immersion of an n_α-plane.*

Proof. Inequality (3.15) is a special case of inequality (3.7) with $\bar{c} = 0$.

If the equality case of (3.15) holds, then, from (3.13) and (3.14), we know that the squared mean curvature function of each $f_\alpha : N_\alpha^{n_\alpha} \to \mathbb{E}^m$ satisfies $|H_\alpha|^2 = \rho_\alpha$. Thus, from the proof of Theorem 1, we know that the equality case of (3.15) implies that each $f_\alpha : N_\alpha^{n_\alpha} \to E^m$ is a totally umbilical isometric immersion. Thus, each $N_\alpha^{n_\alpha}$ is either isometric to an open portion of an n_α-sphere of positive sectional curvature, say c_α, or isometric to an open portion of an n_α-plane (cf. [**1**, page 50]). Thus, the normalized scalar curvature

ρ_α of $N_\alpha^{n_\alpha}$ is equal to c_α for the first case or zero for the second case. If the second case occurs, f_α is a totally geodesic immersion. Furthermore, from (3.13), we also know that the equality case of (3.15) implies $h(X, Y) = 0$, whenever X and Y are tangent to different components of the Riemannian product $M^n = N_1^{n_1} \times \ldots \times N_l^{n_l}$. Hence, by applying a Lemma of Moore, we conclude that the immersion f is indeed a product immersion of some totally umbilical and totally geodesic immersions.

The converse is easy to verify. □

The following simple geometrical characterization results follow immediately from Theorem 2.

Corollary 1. *Let M^n be an Einstein n-manifold with Ricci tensor $S = (n-1)cg$. Then for any isometric immersion of M^n into $\mathbb{E}^m$, we have*

$$|H|^2 \geq c. \tag{3.16}$$

The equality case holds identically if and only if either M^n is flat and it is immersed as an open portion of an affine n-subspace $\mathbb{E}^n$ of $\mathbb{E}^m$ or M^n is isometric to an open portion of an n-sphere of radius $\frac{1}{\sqrt{c}}$ and it is immersed as an open portion of an ordinary hypersphere in an affine $(n+1)$-subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.

Corollary 2. *Let $N^k(c)$ be a k-dimensional real-space-form with constant sectional curvature $c > 0$. Then, for any isometric immersion $f: M^n = N^k(c) \times \mathbb{E}^{n-k} \to \mathbb{E}^m$ of the Riemannian product M^n into $\mathbb{E}^m$, we have*

$$H^2 \geq \left(\frac{k}{n}\right)^2 c. \tag{3.17}$$

The equality case holds identically if and only if M^n is immersed as an open portion of an ordinary spherical hypercylinder: $S^k(c) \times \mathbb{R}^{n-k}$ in an affine $(n+1)$-subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.

Remark 3.1. The exact same proof of Theorem 2 shows that inequality (3.7) also holds for totally real submanifolds in a complex-space-form $\tilde{M}^m(4\bar{c})$ with constant holomorphic sectional curvature $4\bar{c}$ and in quaternion-space-form with constant quaternionic sectional curvature $4\bar{c}$.

4. Mean curvature of isometric immersions in hyperbolic space. For vectors X and Y in the Cartesian $(n+2)$-space $\mathbb{R}^{n+2}$, we put

$$g(X, Y) = -X^1Y^1 + \sum_{i=2}^{n+2} X^iY^i$$

and

$$H^{n+1}(-1) = \{x \in E^{n+2} : g(x, x) = -1 \text{ and } x_1 > 0\}.$$

Then $\mathbb{R}^{n+2}$ with the pseudo-Riemannian metric g is the $(n+2)$-dimensional Minkowski space-time, denoted by $\mathbb{R}_1^{n+2}$ and $H^{n+1}(-1)$ with the induced metric is a complete, simply-connected real-space-form with constant sectional curvature -1.

Let $N^{2,n-2} = H^2(-\frac{1}{2}) \times S^{n-2}(1)$ be the Riemannian product of the hyperbolic plane

$H^2(-\frac{1}{2})$ with constant curvature $-\frac{1}{2}$ and a unit $(n-2)$-sphere. $N^{2,n-2}$ admits a canonical isometric imbedding in $H^{n+1}(-1) \subset \mathbb{R}_1^{n+2}$ defined by

$$N^{2,n-2} = \{x : x_1^2 - x_2^2 - x_3^2 = 2, x_4^2 + x_5^2 + \ldots + x_{n+2}^2 = 1\} \tag{4.1}$$

which is called the standard imbedding of $N^{2,n-2}$. By a direct computation we know that $N^{2,n-2}$ is an isoparametric hypersurface of $H^{n+1}(-1)$ whose shape operator is given by

$$A_{n+1} = \frac{1}{\sqrt{2}} I_2 \oplus \sqrt{2} I_{n-2} \tag{4.2}$$

where I_k is the identity map.

The main purpose of this section is to prove the following.

THEOREM 3. *Let M^n be an open portion of $N^{2,n-2}$. Then, for any isometric immersion of M^n into the hyperbolic m-space $H^m(-1)$, we have*

$$|H|^2 \geq 2\left(\frac{n-1}{n}\right)^2, \tag{4.3}$$

equality holding if and only if, up to rigid motions of $H^m(-1)$, M^n is immersed as an open portion of the standard imbedded $N^{2,n-2}$ in a totally geodesic hyperbolic $(n+1)$-space $H^{n+1}(-1)$ of $H^m(-1)$.

Proof. Let M^n be an open portion of $N^{2,n-2}$ and $x : M^n \to H^m(-1)$ be an isometric immersion. Since the scalar curvature of M^n is given by $\tau = \dfrac{n^2 - 5n + 5}{2}$ and $\inf K = -\frac{1}{2}$, we have $\delta_M = \dfrac{(n-2)(n-3)}{2}$. Therefore, by applying Theorem A, we obtain inequality (4.3). (Inequality (4.3) also follows from Theorem 1.)

Now, assume that the equality sign of (4.3) holds identically, then, with respect to a suitable orthonormal local frame field $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$, the shape operator of M^n in $H^m(-1)$ takes the form: (2.10) and (2.11) according to Theorem A. Moreover, by Theorem A, we also have $\inf K = K(e_1 \wedge e_2)$.

Since the only plane section π in each tangent space of M^n with $K(\pi) = \inf K = -\frac{1}{2}$ is the plane section spanned by e_1, e_2, the vectors e_1, e_2 are tangent to the first component of $M^n \subset N^{2,n-2} = H^2(-\frac{1}{2}) \times S^{n-2}(1)$ and hence $e_3, \ldots, e_n$ are tangent to the second component of M^n. Since $K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = 0$, (2.10), (2.11) and the equation of Gauss imply

$$a\mu = 1, \qquad (\mu - a)\mu = 1$$

from which we obtain $\mu = 2a$.

On the other hand, because $K(e_1 \wedge e_3) = 0$, the equation of Gauss yields $a^2 = \frac{1}{2}$. Without loss of generality, we may assume $a = \frac{1}{\sqrt{2}}$. Hence we obtain

$$A_{n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{2} \end{pmatrix}, \tag{4.4}$$

Now, by using (2.11), (4.4) and the equation of Gauss, we have

$$-\frac{1}{2} = K(e_1 \wedge e_2) = \frac{1}{2} - 1 - \sum_{r=n+2}^{m} ((h_{11}^r)^2 + (h_{12}^r)^2). \tag{4.5}$$

Therefore, $A_r = 0$, $r = n+2, \dots, m$. Hence, the first normal space of M^n in $H^m(-1)$ is 1-dimensional at each point. Since the rank of A_{n+1} is n, the equation of Codazzi together with the special form of the second fundamental form imply that the first normal spaces form a 1-dimensional parallel subbundle of the normal bundle. Therefore, by a result of Erbacher [**5**], M^n is contained in a totally geodesic hyperbolic $(n+1)$-space $H^{n+1}(-1)$ of $H^m(-1)$. Now, by following a standard technique in hypersurface theory, we conclude that M^n is imbedded in $H^{n+1}(-1)$ in the standard way. The converse is trivial. □

REMARK 4.1. Theorem 3 has been announced in [**3**].

5. Sectional curvature and shape operator. The main purpose of this section is to obtain another solution to Problem 1 which establishes a sharp relationship between sectional curvature function K and the shape operator for submanifolds in real-space-forms.

THEOREM 4. *Let $x: M^n \to R^m(\bar{c})$ be an isometric immersion of a Riemannian n-manifold M^n into an m-dimensional real-space-form $R^m(\bar{c})$ of constant sectional curvature $\bar{c}$. If there exist a point $p \in M^n$ such that $c \equiv \inf K \neq \bar{c}$ at p, then the shape operator at the mean curvature vector H satisfies*

$$A_H > \frac{n-1}{n}(c - \bar{c}) I_n, \quad at \quad p, \tag{5.1}$$

where I_n is the identity map.

Proof. We assume that M^n is a submanifold in $R^m(\bar{c})$. If the sectional curvatures satisfy $\inf K = c > \bar{c}$ at p, Lemma 1 implies that the mean curvature vector H is nonzero at p. Choose an orthornormal basis $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ at p such that e_{n+1} is parallel to the mean curvature vector and $e_1, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \tag{5.2}$$

$$A_r = (h^r_{ij}), \qquad \sum_{i=1}^{n} h^r_{ii} = 0, \qquad r = n+2, \dots, m.$$

We put $u_{ij} = u_{ji} = a_i a_j$. From Gauss' equation we get

$$u_{ij} \geq c - \bar{c} + \sum_{r=n+2}^{m} (h^r_{ij})^2 - \sum_{r=n+2}^{m} h^r_{ii} h^r_{jj}, \qquad 1 \leq i \neq j \leq n, \tag{5.3}$$

We need the following lemmas.

LEMMA 2. *The following statements hold.*
(1) *For any fixed* $i \in \{1, \dots, n\}$, *we have* $\sum_{j \neq i} u_{ij} \geq (n-1)(c - \bar{c})$.
(2) $u_{ij} \neq 0$ *for* $i \neq j$.
(3) *For distinct* i, j, k, *we have* $a_i^2 = u_{ij} u_{ik} u_{jk}^{-1}$.

Proof. From (5.2) and (5.3), we get

$$\begin{aligned} \sum_{j \neq i} u_{ij} &\geq (n-1)(c - \bar{c}) + \sum_{r=n+2}^{m} \Big(\sum_{j \neq i} (h^r_{ij})^2 - h^r_{ii} \sum_{j \neq i} h^r_{jj} \Big) \\ &= (n-1)(c - \bar{c}) + \sum_{r=n+2}^{m} \sum_{i,j} (h^r_{ij})^2 \geq (n-1)(c - \bar{c}), \end{aligned}$$

which yields statement (1).

For statement (2), let us assume $u_{ij} = a_i a_j = 0$. If $a_i = 0$, then $u_{it} = 0$ for any $t \neq i$. Hence $\sum_{t \neq i} u_{it} = 0$ which contradicts statement (1).

Statement (3) follows from $u_{ij} u_{ik} = a_i^2 a_j a_k = a_i^2 u_{jk}$. □

For each subset B of $\{1, \dots, n\}$, denote by $\bar{B}$ the complement of B in $\{1, \dots, n\}$. Let S_k be the class of subsets of $\{1, \dots, n\}$ with k elements.

LEMMA 3. *For a fixed* k, $1 \leq k \leq \left[\dfrac{n}{2}\right]$, *and each* $B \in S_k$, *we have*

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq (n-k)k(c - \bar{c}).$$

Proof. Without loss of generality, we may assume $B = \{1, \dots, k\}$. From (5.3) we find

$$\begin{aligned} \sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} &\geq (n-k)k(c - \bar{c}) + \sum_{j=1}^{k} \sum_{t=k+1}^{n} \sum_{r=n+2}^{m} [(h^r_{jt})^2 - h^r_{jj} h^r_{tt}] \\ &= (n-k)k(c - \bar{c}) + \sum_{r=n+2}^{m} \left[\sum_{j=1}^{k} \sum_{t=k+1}^{n} (h^r_{jt})^2 + \sum_{j=1}^{k} (h^r_{jj})^2 \right], \end{aligned}$$

which implies the lemma. □

https://doi.org/10.1017/S001708950003130X Published online by Cambridge University Press

LEMMA 4. *For any* $1 \leq i \neq j \leq n$, *we have* $u_{ij} > 0$.

Proof. Assume $u_{1n} < 0$. Then, by statement (3) of Lemma 2, we get $u_{1i}u_{in} < 0$, for $1 < i < n$. Without loss of generality, we may assume

$$\begin{aligned} &u_{12}, \ldots, u_{1l}, u_{(l+1)n}, \ldots, u_{(n-1)n} > 0, \\ &u_{1(l+1)}, \ldots, u_{1n}, u_{2n}, \ldots, u_{ln} < 0, \end{aligned} \tag{5.4}$$

for some $\left[\frac{n+1}{2}\right] \leq l \leq n-1$.

If $l = n-1$, then $u_{1n} + u_{2n} + \ldots + u_{(n-1)n} < 0$ which contradicts statement (1) of Lemma 2. Thus, $l < n-1$. From statement (3) of Lemma 2 we get

$$a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \tag{5.5}$$

where $2 \leq i \leq l$ and $l+1 \leq t \leq n-1$. By (5.4) and (5.5), we obtain $u_{it} < 0$ which implies

$$\sum_{i=1}^{l} \sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.$$

This contradicts Lemma 3. □

Now, we return to the proof of Theorem 4. From Lemma 4, it follows that $a_1, \ldots, a_n$ are of the same sign. Therefore, the shape operator A_H is positive-definite. Now, from statement (1) of Lemma 2, we get

$$na_i |H| - a_i^2 = a_i \sum_{j \neq i} a_j \geq (n-1)(c - \bar{c}), \tag{5.6}$$

which implies (5.1). If $c < \bar{c}$ at p, (5.1) can be proved in a similar way. □

REMARK 5.1. Our estimate of the shape operator A_H in Theorem 4 is sharp. This can be seen from the following example.

Consider a hyper-ellipsoid in $\mathbb{E}^{n+1}$ defined by

$$ax_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1, \tag{5.7}$$

where $0 < a < 1$. The principal curvatures $a_1, \ldots, a_n$ of the hyper-ellipsoid are given by (cf. [**6**])

$$a_1 = \ldots = a_{n-1} = \frac{1}{(1 + a(a-1)x_1^2)^{\frac{1}{2}}}, \qquad a_n = \frac{a}{(1 + a(a-1)x_1^2)^{\frac{3}{2}}}. \tag{5.8}$$

Therefore, the sectional curvature function K satisfies

$$K \geq c := \frac{a}{(1 + a(a-1)x_1^2)^2} > 0 \tag{5.9}$$

and the eigenvalues $\kappa_1, \ldots, \kappa_n$ of the shape operator A_H are given by

$$\kappa_1 = \ldots = \kappa_{n-1} = \frac{a + (n-1)(1 + a(a-1)x_1^2)}{n(1 + a(a-1)x_1^2)^2},$$
$$\kappa_n = \frac{a(a + (n-1)(1 + a(a-1)x_1^2))}{n(1 + a(a-1)x_1^2)^3}. \tag{5.10}$$

From (5.9) and (5.10) it follows that $A_H > \left(\frac{n-1}{n}\right) cI_n$ and

$$\kappa_n - \frac{n-1}{n} c = \frac{a^2}{n(1 + a(a-1)x_1^2)^3} \to 0$$

as $a \to 0$.

For an n-dimensional submanifold M^n in $\mathbb{E}^m$, let $\mathbb{E}^{n+1}$ be the linear subspace of dimension $n+1$ spanned by the tangent space at a point $p \in M$ and the mean curvature vector $H(p)$ at p. Geometrically, the shape operator A_{n+1} of M^n in E^m at p is the shape operator of the orthogonal projection of M^n into $\mathbb{E}^{n+1}$. Moreover, it is known that if the shape operator of a hypersurface in $\mathbb{E}^{n+1}$ is definite at a point p, then it is strictly convex at p. For this reason a submanifold M^n in $\mathbb{E}^m$ is said to be *H-strictly convex* if the shape operator A_H is positive-definite at each point in M^n.

Theorem 4 implies immediately the following.

COROLLARY 3. *Let M^n be a submanifold of a Euclidean m-space $\mathbb{E}^m$. If M^n has positive sectional curvatures, then M^n is H-strictly convex.*

REMARK 5.2. If the condition $c \equiv \inf K \neq \bar{c}$ at p in Theorem 4 is replaced by $\inf K = \bar{c}$ at p, then (5.1) will be replaced by $A_H \geq 0$ at p.

REFERENCES

1. B. Y. Chen, *Geometry of Submanifolds*, (Marcel Dekker, New York, 1973).

2. B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Archiv der Math.* **60** (1993), 568–578.

3. B. Y. Chen, A Riemannian invariant for submanifolds in space forms and its applications, *Geometry and Topology of Submanifolds* (F. Dillen and L. Verstraelen, eds.), vol. VI, 1994, pp. 58–81.

4. B. Y. Chen, A Riemannian invariant and its applications to submanifold theory, *Results in Math.* **27** (1995), 17–26.

5. J. Erbacher, Reduction of the codimension of an isometric immersion, *J. Differential Geometry* **5** (1971), 333–340.

6. P. F. Leung, On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold, *Proc. Edinburgh Math. Soc.* **28** (1985), 305–311.

MICHIGAN STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
EAST LANSING
MICHIGAN 48824-1027 U.S.A.
E-mail: bychen@math.msu.edu