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NONOSCILLATION OF SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Some sufficient conditions are given for all solutions of the nonlinear differential equation $y''(x) + p(x)f(y) = 0$ to be nonoscillatory, where p is positive and

$$0 < \inf_{y\neq 0} \frac{f(y)}{y^\gamma} \leq \sup_{y\neq 0} \frac{f(y)}{y^\gamma} < \infty$$

for a quotient γ of odd positive integers, $\gamma > 1$.

1. **Introduction.** The main purpose of this paper is to present some criteria for all solutions of the following second order nonlinear differential equation

$$y''(x) + p(x)f(y) = 0 \tag{1.1}$$

to be nonoscillatory, where $p: R^+ \longrightarrow R^+$, $R^+ = [0, \infty)$, is positive and continuous, $f: R \longrightarrow R$ is continuous and

$$0 < \inf_{y\in R\setminus\{0\}} \frac{f(y)}{y^\gamma} \leq \sup_{y\in R\setminus\{0\}} \frac{f(y)}{y^\gamma} < \infty \tag{1.2}$$

for a quotient γ of odd positive integers, $\gamma > 1$.

Equation (1.1) includes the so-called (generalized) Emden-Fowler equation

$$\frac{d}{dt}\Big[p(t)\frac{du}{dt}\Big] + q(t)u^\gamma = 0, \quad t \geq 0 \tag{1.3}$$

under the standard Liouville transformation

$$s = \int_0^t \frac{d\tau}{p(\tau)} \quad \text{and} \quad y(s) = u(t) \quad \text{if} \quad \int_0^\infty \frac{dt}{p(t)} = \infty$$

or the alternative transformation

$$s = \Big[\int_t^\infty \frac{d\tau}{p(\tau)}\Big]^{-1}, \quad y(s) = su(t) \quad \text{if} \quad \int_0^\infty \frac{dt}{p(t)} < \infty$$

where $p: R^+ \longrightarrow R^+ \setminus \{0\}$ and $q: R^+ \longrightarrow R^+$ are continuous (*cf.* [20]). This equation arises in various applications such as gas dynamics, fluid dynamics, relativistic mechanics,

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nuclear physics and chemically reacting systems, and its positive solutions give rise to ground states for certain semilinear elliptic problems (*cf.* [14], [15]).

A great deal of research has been devoted to the study of oscillation and nonoscillation of solutions and various criteria have been established to guarantee the oscillation of all solutions or the existence of nonoscillatory solutions. For details, we refer to [10], [15], [20] and the references therein. However, as Fowler [13] indicated, the Emden-Fowler equation may at the same time possess oscillatory and nonoscillatory solutions. Therefore it is an interesting but difficult problem to find general conditions to guarantee all solutions are nonoscillatory. In [16], Nehari proved that all solutions to the equation

$$y''(x) + p(x)y^{\gamma} = 0 \tag{1.4}$$

are nonoscillatory if $p(x)(x\ln x)^{\frac{\gamma+3}{2}}$ is nonincreasing for sufficiently large x. This result was subsequently improved by Chiou [5], Nehari [17] and Kaper-Kwong [15]. In [15], it was shown that all solutions to equation (1.4) are nonoscillatory if there exists a $\beta > 0$ such that $p(x)x^{\frac{\gamma+3}{2}}(\ln x)^{\beta}$ is nonincreasing for all sufficiently large x. However, the techniques employed involve clever but complicated use of differential and integral inequalities and identities.

In [8]–[12], the first author of this paper and his collaborators established some nonoscillation criteria by using a suitable change of variables and the energy function method. For example, in [12] it was shown that all solutions to equation (1.4) are nonoscillatory if $p(x)x^{\frac{\gamma+2}{3}}(\ln x)^{\beta}$ is nonincreasing and $p(x)x^{\frac{\gamma+2}{3}}(\ln x)^{\eta}$ is bounded for some pair (β, η) with $\beta > 0$ and $\frac{2\beta}{\gamma+3} + \frac{2\eta}{\gamma-1} \geq 1$.

It is the purpose of this paper to continue the above geometrically enlightening method for equation (1.1) by employing a more general transformation. In Section 2, we will introduce our transformation and energy functions, and prove our general results which involve some auxillary functions. It turns out that nonoscillation criteria depend on the choice of this function. In Section 3, we will illustrate how to select this function to produce a series of nonoscillation criteria. It is shown that all solutions of equation (1.4) are nonoscillatory if there are pairs (β_i, η_i) with $\beta_1 > 0$, $\beta_i \geq 0$, $\frac{2\beta_1}{\gamma+3} + \frac{2\eta_1}{\gamma-1} \geq 1$. $\frac{\beta_i}{\gamma+3} + \frac{\eta_i}{\gamma-1} \geq 0$ for $i = 2, 3, \ldots, N$, such that $p(x)x^{\frac{\gamma+3}{2}}\prod_{i=1}^{N}[\ln^i(x)]^{\beta_i}$ is nonincreasing and $p(x)x^{\frac{\gamma+3}{2}}\prod_{i=1}^{N}[\ln^i(x)]^{\eta_i}$ is bounded, where $\ln^1 x = \ln x, \ln^i(x) = \ln\big(\ln^{i-1}(x)\big)$ for $1 < i \leq N$.

2. **Transformations and energy functions.** Making the following change of variables

$$x = e^t,\ y = g(t)e^{\alpha t}u, \quad 0 < \alpha \leq \frac{1}{2} \tag{2.1}$$

where $g\colon R \longrightarrow R$ is an undetermined twice continuously differentiable function satisfying $g(t) > 0$ for sufficiently large t, we can rewrite equation (1.1) as

$$\big(\rho(t)u'\big)' + a(t,u)u = 0 \tag{2.2}$$

where

(2.3) $$\rho(t) = g^2(t)e^{(2\alpha-1)t}$$

(2.4) $$a(t,u) = \rho(t)\Big[\sigma(t)f\big(g(t)e^{\alpha t}u\big)/g^{\gamma}(t)e^{\alpha\gamma t}u^{\gamma} - \lambda(t)\Big]$$

(2.5) $$\sigma(t) = p(e^t)g^{\gamma-1}(t)e^{[\alpha(\gamma-1)+2]t}$$

(2.6) $$\lambda(t) = -\left[\frac{g''(t)}{g(t)} + \frac{(2\alpha-1)g'(t)}{g(t)} + \alpha(\alpha-1)\right].$$

It is easy to prove the following facts

(2.7) $$a(t,u) > 0 \quad \text{implies} \quad |u| > k_1\alpha(t),$$

(2.8) $$a(t,u) > 0 \quad \text{if} \quad |u| > k_2\alpha(t),$$

(2.9) $$\int_0^u a(t,s)s\,ds > 0 \quad \text{implies} \quad |u| > k_1\left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}}\alpha(t),$$

(2.10) $$\int_0^u a(t,s)s\,ds > 0 \quad \text{if} \quad |u| > k_2\left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}}\alpha(t),$$

where

(2.11) $$k_1 = \left[\frac{1}{c_2}\right]^{\frac{1}{\gamma-1}}, \quad k_2 = \left[\frac{1}{c_1}\right]^{\frac{1}{\gamma-1}}$$

(2.12) $$c_2 = \sup_{y\in R\setminus\{0\}} \frac{f(y)}{y^{\gamma}}, \quad c_1 = \inf_{y\in R\setminus\{0\}} \frac{f(y)}{y^{\gamma}}$$

and

(2. 13) $$\alpha(t) = \left[\frac{\lambda(t)}{\sigma(t)}\right]^{\frac{1}{\gamma-1}}.$$

Our general nonoscillation criterion is as follows

THEOREM 2.1. *Suppose*

(i) $k_1\left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}} > k_2$,

(ii) $\lim_{t\to\infty}\rho(t) = \infty$, $\lim_{t\to\infty}\lambda(t) = \alpha(1-\alpha)$,

(iii) $\frac{d}{dt}\left[\frac{f(t)}{t^{\gamma}}\right] \le 0$, $\frac{d}{dt}[g(t)e^{\alpha t}] \ge 0$,

(iv) $\frac{d}{dt}[\rho^2(t)\sigma(t)] \le 0$, $\frac{d}{dt}[\rho^2(t)\lambda(t)] \ge 0$,

(v) $\liminf_{t\to\infty}\int_t^{t+\tau} d[\rho^2(s)\lambda(s)\alpha^2(s)] > 0$

with $\tau e^{\tau} = \delta$ *and* $\delta > 0$ *satisfying*

$$0 < \delta < \frac{1}{k_2\sqrt{\alpha(1-\alpha)\Delta}}\left[k_1\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} - k_2\right] \quad \textit{and} \quad \Delta = \frac{\gamma-1}{\gamma+1}.$$

Then all nontrivial solutions of equation (2.2) are nonoscillatory.

PROOF. By way of contradiction, we assume that there exists a nontrivial oscillatory solution $u(t)$ of equation (2.2). We construct the following energy function

(2. 14)
$$\begin{aligned} E(t) &= [\rho(t)u'(t)]^2 + 2\rho(t)\int_0^{u(t)} a(t,s)s\,ds \\ &= [\rho(t)u'(t)]^2 + 2\rho^2(t)\left[\sigma(t)\int_0^{u(t)}\frac{f\big(g(t)e^{\alpha t}s\big)}{g^{\gamma}(t)e^{\alpha\gamma t}}\,ds - \frac{1}{2}\lambda(t)u^2(t)\right]. \end{aligned}$$

With some manipulation, we get

$$(2.15)\qquad E'(t) = 2[\rho^2(t)\sigma(t)]' \int_0^{u(t)} \frac{f\big(g(t)e^{\alpha t}s\big)}{g^\gamma(t)e^{\alpha\gamma t}}\,ds - [\rho^2(t)\lambda(t)]'u^2(t) + 2\rho^2(t)\sigma(t)\int_0^{u(t)} \frac{\partial}{\partial t}\left[\frac{f\big(g(t)e^{\alpha t}s\big)}{g^\gamma(t)e^{\alpha\gamma t}}\right] ds.$$

Condition (iv) guarantees that the first two terms of $E'(t)$ are non-positive. From condition (iii), we have

$$\begin{aligned} 2\rho^2(t)\sigma(t)\int_0^{u(t)} \frac{\partial}{\partial t}\left[\frac{f\big(g(t)e^{\alpha t}s\big)}{g^\gamma(t)e^{\alpha\gamma t}}\right] ds &= 2\rho^2(t)\sigma(t)\int_0^{u(t)} \frac{f'(ws)w^\gamma s - \gamma w^{\gamma-1}f(ws)}{w^{2\gamma}}w'\,ds \\ &= 2\rho^2(t)\sigma(t)\int_0^{u(t)} \left[\frac{dw}{dt}\right]\frac{d}{dws}\left[\ln\frac{f(ws)}{(ws)^\gamma}\right]\cdot\frac{f(ws)ws}{w^{\gamma+1}}\,ds \\ &\leq 0 \end{aligned}$$

where $w = g(t)e^{\alpha t}$. Therefore, $E'(t) \leq 0$ for all $t \geq t_0$. Since $u(t)$ is nontrivial, $E(t) = [\rho(t)u'(t)]^2 > 0$ whenever $u(t) = 0$. It then follows from the oscillating property of u and the monotonicity of $E(t)$ that $E(t) > 0$ for all $t \geq t_0$.

Whenever $u'(t) = 0$, $E(t) = 2\rho(t)\int_0^{u(t)} a(t,s)s\,ds > 0$. From (2.9) and condition (i), we get

$$(2.16)\qquad |u(t)| > k_1\left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}}\alpha(t) \geq k_2\alpha(t).$$

Therefore, $|u(\theta)| > k_2\alpha(\theta)$ for all θ in a neighborhood of t.

Now suppose t_n and t_{n+1} are successive zeros of u. We can find $\{r_n\}$, $\{\tau_n\}$ and $\{s_n\}$, satisfying $t_n < r_n < \tau_n < s_n < t_{n+1}$ with $|u(r_n)| = k_2\alpha(r_n)$, $|u(s_n)| = k_2\alpha(s_n)$, $|u| > k_2\alpha$ on (r_n, s_n) and $u'(\tau_n) = 0$. Hence from (2.16), we get

$$(2.17)\qquad |u(\tau_n)| > k_1\left[\frac{\gamma+1}{2}\right]^{\frac{1}{\gamma-1}}\alpha(\tau_n).$$

We assume $u(t) > 0$ on (t_n, t_{n+1}) (in case $u(t) < 0$, the argument is the same). Since $|u(t)| > k_2\alpha(t)$ on (r_n, s_n) we have $a(t,u) > 0$ from (2.8) for $t \in (r_n, s_n)$. Then, equation (2.2) implies $\rho(t)u'(t)$ is nonincreasing on (r_n, s_n). Consequently, with (2.17) in mind, we have

$$(2.18)\qquad \begin{aligned} (\tau_n - r_n)u'(r_n) &\geq \int_{r_n}^{\tau_n}\left[\frac{g(t)e^{\alpha t}}{g(r_n)e^{\alpha r_n}}\right]^2 \frac{e^{-t}}{e^{-r_n}}u'(t)\,dt \\ &\geq \int_{r_n}^{\tau_n}\frac{e^{-t}}{e^{-r_n}}u'(t)\,dt \\ &\geq e^{-(\tau_n - r_n)}\int_{r_n}^{\tau_n}u'(t)\,dt \\ &= e^{-(\tau_n - r_n)}[u(\tau_n) - u(r_n)] \\ &= e^{-(\tau_n - r_n)}[u(\tau_n) - k_2\alpha(r_n)] \\ &> e^{-(\tau_n - r_n)}\left[k_1\left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}\alpha(\tau_n) - k_2\alpha(r_n)\right]. \end{aligned}$$

Since $\alpha(t) = \left[\frac{\rho^2(t)\lambda(t)}{\rho^2(t)\sigma(t)}\right]^{\frac{1}{\gamma-1}}$, the condition (iv) implies $\alpha(t)$ is nondecreasing. We have, by (2.18),

$$(2.19) \qquad (\tau_n - r_n)u'(r_n) > e^{-(\tau_n - r_n)}\Big[k_1\Big(\frac{\gamma+1}{2}\Big)^{\frac{1}{\gamma-1}} - k_2\Big]\alpha(r_n).$$

To estimate $u'(r_n)$, we observe that $E(r_n) \leq E_0 = E(t_0)$ for $t_n \geq t_0$, and

$$(2.20) \qquad [\rho(r_n)u'(r_n)]^2 + \int_0^{k_2\alpha(r_n)} 2\rho^2(r_n)[\sigma(r_n)c_1 s^\gamma - \lambda(r_n)s]\,ds \leq E(r_n) \leq E_0.$$

Note that

$$\begin{aligned}
2\rho^2(r_n)&\int_0^{k_2\alpha(r_n)}[c_1\sigma(r_n)s^\gamma - \lambda(r_n)s]\,ds \\
&= \frac{2c_1\rho^2(r_n)\sigma(r_n)k_2^{\gamma+1}}{\gamma+1}\alpha^{\gamma+1}(r_n) - \rho^2(r_n)\lambda(r_n)k_2^2\alpha^2(r_n) \\
&= \frac{2c_1k_2^{\gamma+1}}{\gamma+1}\rho^2(r_n)\sigma(r_n)\Big[\frac{\lambda(r_n)}{\sigma(r_n)}\Big]^{\frac{\gamma+1}{\gamma-1}} - \rho^2(r_n)\lambda(r_n)k_2^2\Big[\frac{\lambda(r_n)}{\sigma(r_n)}\Big]^{\frac{2}{\gamma-1}} \\
&= k_2^2\rho^2(r_n)\lambda(r_n)\Big[\frac{\lambda(r_n)}{\sigma(r_n)}\Big]^{\frac{2}{\gamma-1}}\Big[\frac{2c_1k_2^{\gamma-1}}{\gamma+1} - 1\Big] \\
&= -\Delta k_2^2\rho^2(r_n)\lambda(r_n)\alpha^2(r_n).
\end{aligned}$$

Therefore, from equation (2.20), we deduce

$$[\rho(r_n)u'(r_n)]^2 \leq \Delta k_2^2\rho^2(r_n)\lambda(r_n)\alpha^2(r_n) + E_0$$

which implies

$$(2.21) \qquad u'(r_n) \leq [\rho^{-2}(r_n)E_0 + \Delta k_2^2\lambda(r_n)\alpha^2(r_n)]^{\frac{1}{2}}.$$

It is easy to see that, from the condition (v), $\lim_{t\to\infty}\rho^2(t)\lambda(t)\alpha^2(t) = \infty$. If we choose $c > \sqrt{\Delta\alpha(1-\alpha)}$, then (2.21) gives

$$(2.22) \qquad u'(r_n) \leq ck_2\alpha(r_n) \quad \text{for sufficiently large } n.$$

Combining (2.19) and (2.22), we see that

$$(\tau_n - r_n) > \frac{1}{ck_2}e^{-(\tau_n - r_n)}\Big[k_1\Big(\frac{\gamma+1}{2}\Big)^{\frac{1}{\gamma-1}} - k_2\Big]$$

or,

$$(\tau_n - r_n)e^{(\tau_n - r_n)} > \frac{1}{ck_2}\Big[k_1\Big(\frac{\gamma+1}{2}\Big)^{\frac{1}{\gamma-1}} - k_2\Big] = \delta,$$

from which it follows

$$(2.23) \qquad s_n - r_n > \tau.$$

On the other hand, by (2.15) we get

$$E'(t) \le \frac{2[\rho^2(t)\sigma(t)]'}{\gamma+1} c_1 u^{\gamma+1}(t) - [\rho^2(t)\lambda(t)]' u^2(t).$$

and on (r_n, s_n), we have $|u| > k_2\alpha$. Then

$$\begin{aligned} E'(t) &\le \frac{2c_1}{\gamma+1}[\rho^2(t)\sigma(t)]' k_2^{\gamma+1}\alpha^{\gamma+1}(t) - [\rho^2(t)\lambda(t)]' k_2^2\alpha^2(t) \\ &= \frac{2}{\gamma+1} G'\Big(\frac{F}{G}\Big)^{\frac{\gamma+1}{\gamma-1}} - F'\Big(\frac{F}{G}\Big)^{\frac{2}{\gamma-1}} \qquad (2.24) \\ &= -\frac{\gamma-1}{\gamma+1}\frac{d}{dt}\Big[F\Big(\frac{F}{G}\Big)^{\frac{2}{\gamma-1}}\Big] \end{aligned}$$

where $F(t) = \rho^2(t)\lambda(t)$ and $G(t) = c_1\rho^2(t)\sigma(t)$. Integrating (2.24) and using (2.23), we have

$$\begin{aligned} E(\infty) - E(t_0) &= \int_{t_0}^{\infty} E'(t)\,dt \\ &\le \sum_{n=1}^{\infty} \int_{s_n}^{r_n} E'(t)\,dt \\ &\le -\frac{\gamma-1}{\gamma+1}\sum_{n=1}^{\infty}\int_{s_n}^{r_n} \frac{d}{dt}\Big[F\Big(\frac{F}{G}\Big)^{\frac{2}{\gamma-1}}\Big]\,dt \\ &\le -\frac{\gamma-1}{\gamma+1}\sum_{n=1}^{\infty}\int_{s_n}^{r_n} d[k_2^2\rho^2(t)\lambda(t)\alpha^2(t)] \\ &= -\frac{k_2^2(\gamma-1)}{\gamma+1}\sum_{n=1}^{\infty}\int_{s_n}^{r_n} d[\rho^2(t)\lambda(t)\alpha^2(t)] \\ &= -\infty \end{aligned}$$

which contradicts $E(t) > 0$. This completes the proof.

3. **Nonoscillation criteria.** We are now in the position to state our main nonoscillation criterion

THEOREM 3.1. *Suppose that* $k_1\big[\frac{1+\gamma}{2}\big]^{\frac{1}{\gamma-1}} > k_2$ *and for some* α *with* $0 < \alpha \le \frac{1}{2}$, *we have*

(i) $d\big[\frac{f(x)}{x^\gamma}\big] \le 0$, $d[g(\ln x)x^\alpha] \ge 0$;

(ii) $d[g^{\gamma+3}(\ln x)x^{\alpha(\gamma+3)}p(x)] \le 0$, $d[g^4(\ln x)x^{2[2\alpha-1]}\lambda(\ln x)] \ge 0$,

(iii) $\lim_{x\to\infty} g^2(\ln x)x^{2\alpha-1} = \infty$ *and* $\lim_{x\to\infty}\lambda(\ln x) = \alpha(1-\alpha)$,

(iv) $\liminf_{x\to\infty}\int_x^{kx} d\Big[g^2(\ln s)\big/\big(p(s)s^{(\gamma-1)(1-\alpha)+2}\big)^{\frac{2}{\gamma-1}}\Big] > 0$

where $k = e^\tau$, g, λ *and* τ *are defined in Theorem 2.1. Then all solutions of (1.1) are nonoscillatory.*

PROOF. It suffices to verify condition (iv) of Theorem 2.1. Since $\lim_{t\to\infty}\lambda(t) = \alpha(1-\alpha)$, we can delete the term $\lambda(s)$ in the condition (iv) of Theorem 2.1. Then

$$\liminf_{t\to\infty}\int_\tau^{t+\tau} d[\rho^2(s)\lambda(s)\alpha^2(s)] > 0$$

is equivalent to

$$\liminf_{x\to\infty}\int_x^{\kappa x} d\Big[g^2(\ln s)\big/\Big(\rho(s)s^{(\gamma-1)(1-\alpha)+2}\Big)^{\frac{2}{\gamma-1}}\Big]\,ds>0$$

with $x=e^t$. This completes the proof.

Taking $g(t)=t^\mu$ for $\mu\geq 0$ in the above theorem, we get the following

COROLLARY 3.1. *Suppose that $k_1\big[\frac{\gamma+1}{2}\big]^{\frac{1}{\gamma-1}}>k_2$ and for some β,α with $\beta\geq 0$ and $0<\alpha\leq\frac{1}{2}$, we have*
(i) $d\big[\frac{f(x)}{x^\gamma}\big]\leq 0$, $d[p(x)x^{\alpha(\gamma+3)}(\ln x)^\beta]\leq 0$,
(ii) $\liminf_{x\to\infty}\int_x^{\kappa x} d\Big[(\ln s)^{\frac{2\beta}{\gamma+3}}\big/\Big(p(s)s^{\alpha(\gamma-1)+2}\Big)^{\frac{2}{\gamma-1}}\Big]>0$
where $\kappa=e^\tau$ and τ is defined in Theorem 2.1. Then solutions of equation (1.1) are nonoscillatory.

This corollary implies the main result of [12] if we select $\alpha=\frac{1}{2}$, $c_1=c_2=k_1=k_2=1$ and $f(x)=x^\gamma$.

The following result indicates our nonoscillation criteria depend on the choice of the auxiliary function g. For simplicity, we let $f(x)=x^\gamma$.

COROLLARY 3.2. *If $\beta_1>0$, $\beta_2\geq 0$ and $\eta_1,\eta_2\in R$ are given so that*

$$\frac{2\beta_1}{\gamma+3}+\frac{2\eta_1}{\gamma-1}\geq 1$$

and

$$\frac{2\beta_2}{\gamma+3}+\frac{2\eta_2}{\gamma-1}\geq 0,$$

then all solutions of equation (1.4) are nonoscillatory in the case where
(i) $p(x)x^{\frac{\gamma+3}{2}}(\ln x)^{\beta_1}(\ln\ln x)^{\beta_2}$ *is nonincreasing,*
(ii) $p(x)x^{\frac{\gamma+3}{2}}(\ln x)^{\eta_1}(\ln\ln x)^{\eta_2}$ *is bounded.*

PROOF. Let $g(t)=t^\mu\ln^\nu t$, $\beta_1=\mu(\gamma+3)>0$, $\beta_2=\nu(\gamma+3)\geq 0$ and $\alpha=\frac{1}{2}$. Then

$$\begin{aligned}
&d\Big[g^2(\ln s)\big/\Big(\rho(s)s^{(\gamma-1)(1-\alpha)+2}\Big)^{\frac{2}{\gamma-1}}\Big]\\
&\quad=d\Big[g^2(\ln s)\big/\Big(\rho(s)^{\frac{2}{\gamma-1}}s^{\frac{4}{\gamma-1}+1}\Big)\Big]\\
&\quad=d\Big[\Big(p(s)s^{\frac{\gamma+3}{2}}\ln^{\beta_1}s(\ln\ln s)^{\beta_2}\Big)^{-\frac{2}{\gamma-1}}(\ln s)^{\frac{2\beta_1}{\gamma+3}+\frac{2\beta_1}{\gamma-1}}(\ln\ln s)^{\frac{2\beta_2}{\gamma+3}+\frac{2\beta_2}{\gamma-1}}\Big]\\
&\quad\geq[p(s)s^{\frac{\gamma+3}{2}}\ln^{\eta_1}s(\ln\ln s)^{\eta_2}]^{-\frac{2}{\gamma-1}}(\ln s)^{\frac{2(\eta_1-\beta_1)}{\gamma-1}}(\ln\ln s)^{\frac{2(\eta_2-\beta_2)}{\gamma-1}}\\
&\qquad\times d[(\ln s)^{\frac{2\beta_1}{\gamma+3}+\frac{2\beta_1}{\gamma-1}}(\ln\ln s)^{\frac{2\beta_2}{\gamma+3}+\frac{2\beta_2}{\gamma-1}}]\\
&\quad\geq M^{-\frac{2}{\gamma-1}}\Big(\frac{2\beta_1}{\gamma+3}+\frac{2\beta_1}{\gamma-1}\Big)\cdot\frac{1}{s}(\ln s)^{\frac{2\beta_1}{\gamma+3}+\frac{2\eta_1}{\gamma-1}-1}(\ln\ln s)^{\frac{2\beta_2}{\gamma+3}+\frac{2\eta_2}{\gamma-1}}\,ds\\
&\quad\geq c\,d(\ln s)
\end{aligned}$$

for some constant c, where by (ii) we assume

$$|p(s)s^{\frac{\gamma+3}{2}}(\ln s)^{\eta_1}(\ln\ln s)^{\eta_2}| \leq M$$

for a constant $M > 0$. Then it follows that

$$\int_x^{\kappa x} d\Big[g^2(\ln s)/\Big(p(s)^{\frac{2}{\gamma-1}} s^{\frac{4}{\gamma-1}+1}\Big)\Big] \geq c\ln\kappa > 0.$$

Therefore, condition (iv) of Theorem 3.1 is satisfied. The proof is then completed.

By selecting more delicate auxiliary functions g, we can establish more general nonoscillation criteria. For example, letting $g(t) = t^{\mu_1}\prod_{i=2}^n\big(\ln^{i-1}(t)\big)^{\mu_i}$, where $\ln^1(t) = \ln t$, $\ln^2(t) = \ln(\ln t)$ and $\ln^i(t) = \ln\big(\ln^{i-1}(t)\big)$, we get

COROLLARY 3.3. *If there exist n pairs (β_i, η_i) with $\beta_1 > 0$, $\beta_i \geq 0$ for $2 \leq i \leq n$ such that*

(i) $\frac{2\beta_1}{\gamma+3} + \frac{2\eta_1}{\gamma-1} \geq 1$ *and* $\frac{2\beta_i}{\gamma+3} + \frac{2\eta_i}{\gamma-1} \geq 0$ *for* $2 \leq i \leq n$.

(ii) $p(x)x^{\frac{\gamma+3}{2}}\prod_{i=1}^n\big(\ln^i(x)\big)^{\beta_i}$ *is nonincreasing,*

(iii) $p(x)x^{\frac{\gamma+3}{2}}\prod_{i=1}^n\big(\ln^i(x)\big)^{\eta_i}$ *is bounded.*

Then all solutions of equation (1.4) are nonoscillatory.

The proof is similar to that for Corollary 3.2, and therefore is omitted.

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