POSITIVE SOLUTIONS OF FOURTH-ORDER SUPERLINEAR SINGULAR BOUNDARY VALUE PROBLEMS

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This paper investigates fourth-order superlinear singular two-point boundary value problems and obtains necessary and sufficient conditions for existence of $C^2$ or $C^3$ positive solutions on the closed interval.

1. INTRODUCTION

In this paper, we are concerned with the fourth-order singular two-point boundary value problem

\[ \begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \]

where $f \in C((0,1) \times [0, +\infty), [0, +\infty))$ and is quasi-homogeneous with respect to the second variable, namely, there are constants $\lambda, \mu, N, M$ with $1 < \lambda \leq \mu < \infty$ and $0 < N \leq 1 \leq M$ such that for all $0 < t < 1, u \geq 0$,

\[ \begin{align*} 
(2) & \quad c^\mu f(t, u) \leq f(t, cu) \leq c^\lambda f(t, u), \quad \text{if } 0 < c \leq N, \\
(3) & \quad c^\lambda f(t, u) \leq f(t, cu) \leq c^\mu f(t, u), \quad \text{if } c \geq M. 
\end{align*} \]

A typical quasi-homogeneous function is $f = f_1(t)u^{\lambda_1} + \cdots + f_m(t)u^{\lambda_m}$, where $\lambda \leq \lambda_i \leq \mu, i = 1, \ldots, m$.

Singular or nonsingular fourth-order boundary value problems have been extensively studied by many authors (see [1, 2, 3, 4, 5, 6, 7] for nonsingular cases and [8, 9] for singular cases). In [3, 4, 5] the right hand side function in the equation of (1) has separated variables, namely, $f(t, u) = \lambda a(t)g(u)$, and in [1, 6, 7, 8] the function $f$ involves the second derivative $u''$. O’Regan considered the singular case where $f(t, u, u'')$ is singular at $u = 0$ or $u'' = 0$, while in [9] singularity occurs at $t = 0$ or $t = 1$. Using a modified upper and lower solution method, Chen and Zhang [10] established necessary and sufficient conditions for existence of positive solutions to second-order sublinear...
boundary value problems on a half-line. Using a similar method, Wei [9] obtained necessary and sufficient conditions for existence of positive solutions to the fourth-order problem (1) in the sublinear case. The results in [9, 10] involve integrability conditions in terms of the function $f$ and the Green's function. To this connection, however, the upper and lower solution method can hardly be used to treat the superlinear case.

In this paper, based on a careful analysis of the Green's function, we shall apply a fixed point theorem in cones to the superlinear problem (1) and obtain necessary and sufficient conditions for existence of a positive solution with different smoothness on the closed interval.

2. MAIN RESULTS

Our main results are the two following theorems.

**Theorem 1.** The boundary value problem (1) has a positive solution $u \in C^2[0,1] \cap C^4(0,1)$, if and only if,

$$
\int_0^1 t(1-t)f(t,(1-t)t)\,dt < \infty.
$$

**Theorem 2.** The boundary value problem (1) has a positive solution $u \in C^3[0,1] \cap C^4(0,1)$, if and only if,

$$
\int_0^1 f(t,(1-t)t)\,dt < \infty.
$$

We note that (5) implies (4). To prove Theorems 1 and 2, we shall prepare some lemmas. First, we state a fixed point theorem in a cone as follows:

**Lemma 1.** ([11, Theorem 2.3.4].) Let $E$ be a Banach space and $P$ a cone in $E$. Suppose that $\Omega_1$ and $\Omega_2$ are two bounded open subsets of $E$ with $\Theta \subset \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. If $T : P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to P$ is a completely continuous operator satisfying

$$
\|Tx\| \leq \|x\| \text{ for } x \in P \cap \partial \Omega_1 \text{ and } \|Tx\| \geq \|x\| \text{ for } x \in P \cap \partial \Omega_2,
$$

then $T$ has a fixed point in $P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

Let $E = \{u \in C^2[0,1] : u(0) = u(1) = 0, u''(0) = u''(1) = 0\}$. Define the norm $\|u\|$ for every $u \in E$ by $\|u\| = |u|_0 + |u''|_0$, where $| \cdot |_0$ is the usual sup-norm for continuous functions over $[0,1]$. It is seen that $E$ equipped with the norm $\| \cdot \|$ is a Banach space.

Let $G(t,s)$ be the Green's function of the second-order boundary value problem

$$
\begin{cases}
-u''(t) = 0, \\
u(0) = u(1) = 0,
\end{cases}
$$

Our main results are the two following theorems.
that is,
\[ G(t, s) = \begin{cases} 
    s(1-t), & 0 \leq s \leq t \leq 1, \\
    t(1-s), & 0 \leq t \leq s \leq 1.
\end{cases} \]

Let
\[ h(t, s) = \int_0^1 G(t, \tau) G(\tau, s) d\tau. \]

Then \( h(t, s) \) is the Green's function of the homogeneous fourth-order boundary value problem corresponding to (1). It is easily seen that
\[ G(t, s) \leq G(s, s), \quad 0 \leq t, s \leq 1 \]
and for \( 1/4 \leq t \leq 3/4 \),
\[ G(t, s) \geq \frac{1}{4} G(s, s), \quad 0 \leq s \leq 1. \]

Denote
\[ P = \{ u \in E \mid u(t) \geq 0, u''(t) \leq 0, 0 \leq t \leq 1; \]
\[ u(t) \geq \frac{1}{4} |u|_0, -u''(t) \geq \frac{1}{4} |u''|_0, \frac{1}{4} \leq t \leq \frac{3}{4} \}. \]

It can be easily seen that \( P \) is a cone in \( E \).

Next, we define an operator \( T : P \rightarrow E \) by
\[ (Tu)(t) = \int_0^1 h(t, s) f(s, u(s)) ds, \quad u \in P. \]

We observe that a fixed point of \( T \) in \( E \) is indeed a positive solution of the boundary value problem (1).

Using the Green's function, for every \( u \in P \), we shall have an estimate for \( u(t) \) in terms of the magnitude of its second derivative, namely, for \( t \in [0, 1] \),
\[ u(t) = \int_0^1 G(t, s) (-u''(s)) ds \leq \left( \int_0^t s(1-t) ds + \int_t^1 t(1-s) ds \right) |u''|_0 \]
\[ = \frac{1}{2} t(1-t) |u''|_0. \]

Let \( u \in P \) and let \( c \) be a positive number such that \( c \geq M \) and \( |u''|_0/(2c) \leq N \). From (9), \( u(s)/(cs(1-s)) \leq |u''|_0/(2c) \leq M \). Then, from (2) and (3),
\[ |Tu(t)| \leq \int_0^1 G(s, s) c^\mu f(s, u(s)/c) ds \]
\[ = c^\mu \int_0^1 G(s, s) f\left(s, \frac{u(s)}{cs(1-s)}s(1-s)\right) ds \]
\[ \leq c^\mu \left(\frac{|u''|_0}{2c}\right) \int_0^1 s(1-s)f(s, s(1-s)) ds. \]

Hence, \( T \) is well defined on \( P \) provided that (4) or (5) holds.
**Lemma 2.** If (4) holds, then $T(P) \subset P$.

**Proof:** Let $u \in P$. Obviously, $Tu(t) \geq 0$ and $-(Tu)''(t) \geq 0$. For $1/4 \leq t \leq 3/4$, we claim that

$$Tu(t) \geq \frac{1}{4}|Tu|_0.$$  

Indeed, from (4), by Fubini's theorem, (8) can be rewritten as

$$
(Tu)(t) = \int_0^1 G(t, \tau) \int_0^1 G(\tau, s)f(s, u(s)) \, ds \, d\tau.
$$

It follows from (6) that

$$
|Tu|_0 \leq \int_0^1 G(\tau, \tau) \int_0^1 G(\tau, s)f(s, u(s)) \, ds \, d\tau.
$$

On the other hand, for $1/4 \leq t \leq 3/4$, (7) together with (11) gives

$$
(Tu)(t) \geq \frac{1}{4} \int_0^1 G(\tau, \tau) \int_0^1 G(\tau, s)f(s, u(s)) \, ds \, d\tau \geq \frac{1}{4}|Tu|_0.
$$

Next, we claim that $-(Tu)''(t) \geq (1/4)|Tu)''|_0$ for $t \in [1/4, 3/4]$. In fact, from

$$
-(Tu)''(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds,
$$

it follows from (6) and (7) that

$$
|Tu)''|_0 \leq \int_0^1 G(s, s)f(s, u(s)) \, ds
$$

and, for $1/4 \leq t \leq 3/4$,

$$
-(Tu)''(t) \geq \frac{1}{4} \int_0^1 G(s, s)f(s, u(s)) \, ds \geq \frac{1}{4}|Tu)''|_0.
$$

We now conclude that $T : P \rightarrow P$ from (12) and (13) and complete the proof. 

**Lemma 3.** If (4) holds, then $T$ is a completely continuous operator on $P$.

**Proof:** If $u_n \in P$ and $u_n \rightarrow u_0$ in $E$ as $n \rightarrow \infty$, then we have that $u_0 \in P$ by the definition of the cone $P$ and that $\{ \|u_n\| \}$ is bounded, say, $\|u_n\| \leq C_0$, $n \geq 1$. As a result, from (9), we have

$$
u_n(t) \leq \frac{C_0}{2}t(1-t).$$
Let \( c \) be a positive number such that \( c \geq M \) and \( C_0/2c \leq N \). From (2) and (3),
\[
|(Tu_n)(t)| \leq \int_0^1 s(1-s)f(s, u_n(s)) \, ds \\
\leq \int_0^1 s(1-s)c^{\mu}f(s, u_n(s)/c) \, ds \\
\leq \int_0^1 s(1-s)c^{\mu}\left(\frac{u_n(s)}{s(1-s)c}\right)^{\lambda}f(s, s(1-s)) \, ds \\
\leq c^{\mu-\lambda}\left(\frac{C_0}{2}\right)^{\lambda} \int_0^1 s(1-s)f(s, s(1-s)) \, ds.
\]

Now, from (4), an application of Lebesgue’s dominant convergence theorem gives the continuity of \( T \) on \( P \).

To prove \( T \) is a compact operator, we shall show that for every bounded sequence \( \{u_n\} \) in \( P \), the sequence \( \{Tu_n\} \subset P \) has a convergent subsequence in \( E \). Since \( \{Tu_n\} \) is bounded in \( E \), \( \{|(Tu_n)''|o\} \) is bounded and hence \( \{Tu_n(t)\} \) is equicontinuous. By Ascoli—Arzela’s lemma, it suffices to show that \( \{(Tu_n)''(t)\} \) is equicontinuous. Let \( C_0 \) be a positive number such that \( \|u_n\| \leq C_0 \), \( n = 1, 2, \ldots \). Then (14) holds from (9). Again, choose a \( c \geq \max\{M, C_0/(2N)\} \). Then
\[
(Tu)'''(t) = \int_0^t s f(s, u(s)) \, ds - \int_t^1 (1-s)f(s, u(s)) \, ds \\
\leq \int_0^t s f(s, u(s)) \, ds + \int_t^1 (1-s)f(s, u(s)) \, ds \\
\leq C_1\left(\int_0^t s f(s, s(1-s)) \, ds + \int_t^1 (1-s)f(s, s(1-s)) \, ds\right) \\
=: F(t),
\]
where \( C_1 = c^{\mu-\lambda}(C_0/2)^{\lambda} \). Since, in view of (4),
\[
\int_0^1 F(t) \, dt = C_1 \int_0^1 \int_0^t s f(s, u(s)) \, ds \, dt + C_1 \int_0^1 \int_t^1 (1-s)f(s, u(s)) \, ds \, dt \\
= 2C_1 \int_0^1 s(1-s)f(s, s(1-s)) \, ds < \infty,
\]
we have the equicontinuity of the sequence \( \{(Tu_n)'''(t)\} \) from the uniform continuity of the convergent integral of \( F(t) \) with respect to the Lebesgue measure over \([0,1]\).

Therefore, \( T \) is a compact operator on \( P \) and the proof of Lemma 3 is complete. \( \square \)

We are now in a position to prove our main results.
PROOF OF THEOREM 1: Necessity. Let \( u \in C^2[0,1] \cap C^4(0,1) \) be a positive solution of (1). Obviously, \( u''(t) \leq 0 \) for \( 0 \leq t \leq 1 \) and hence \( u(t) \) is concave. It follows from \( u(0) = u(1) = 0 \) that \( u'(0) > 0 \) and \( u'(1) < 0 \). Consequently, there must be a positive number \( k \) such that \( u(t) \geq kt(1-t) \). Let \( c \geq \max \{ M, 1/(kN) \} \). Then, for \( 0 < t < 1 \), \( t(1-t)/(cu(t)) < N \), and we get

\[
 f(t, t(1-t)) \leq c^{\mu} f(t, t(1-t)/u(t)) \leq c^{\mu-\lambda} k^{-\lambda} f(t, u(t)) = c^{\mu-\lambda} k^{-\lambda} u^{(4)}(t). 
\] (15)

Since \( u''(0) = u''(1) = 0 \), there is a \( t_0 \in (0,1) \) such that \( u'''(t_0) = 0 \). Then

\[
u''(t_0) = \int_0^{t_0} u'''(s) ds = -\int_0^{t_0} \int_0^s u^{(4)}(\tau) d\tau ds = -\int_0^{t_0} \tau u^{(4)}(\tau) d\tau. 
\] (16)

On the other hand,

\[
u''(t_0) = -\int_0^1 u'''(s) ds = -\int_0^1 \int_0^s u^{(4)}(\tau) d\tau ds.
\] (17)

Therefore,

\[
\int_0^1 t(1-t)u^{(4)}(t) dt = \left( \int_0^{t_0} + \int_{t_0}^1 \right) t(1-t)u^{(4)}(t) dt \
\leq \int_0^{t_0} t u^{(4)}(t) dt + \int_{t_0}^1 (1-t)u^{(4)}(t) dt \\
= 2(-u''(t_0)) < \infty.
\] (18)

We now obtain (4) from (15) and (18), and complete the proof of the necessity.

Sufficiency. Let \( \Omega_1 = \{ u \in E \mid \|u\| < r \} \), where

\[
r \leq \min \left\{ 2N, 2 \left( \int_0^1 s(1-s)f(s, s(1-s)) ds \right)^{1/(1-\lambda)} \right\}. 
\] (19)

Let \( u \in \partial \Omega_1 \cap P \). Then \( \|u\| = |u|_0 + |u''|_0 = r \), and \( |u|_0 \leq r \), \( |u''|_0 \leq r \). It follows from (9) that

\[
u(t) \leq \frac{1}{2} t(1-t)|u''|_0 \leq \frac{r}{2} t(1-t) \leq N t(1-t). 
\] (20)
In view of (2), (3), and (20), we have

\[ T u(t) = \int_0^1 h(t, s) f(s, u(s)) \, ds \]

\[ \leq \int_0^1 h(t, s) \left( \frac{u(s)}{s(1-s)} \right)^\lambda f(s, s(1-s)) \, ds \]

\[ \leq 2^{-\lambda} r^\lambda \int_0^1 s(1-s) f(s, s(1-s)) \, ds \]

and

\[ |T u|_0 \leq 2^{-\lambda} r^\lambda \int_0^1 s(1-s) f(s, s(1-s)) \, ds, \quad u \in \partial \Omega_1 \cap P. \]  

On the other hand,

\[ -(Tu)''(t) = \int_0^1 G(t, s) f(s, u(s)) \, ds \]

\[ \leq \int_0^1 G(t, s) \left( \frac{u(s)}{s(1-s)} \right)^\lambda f(s, s(1-s)) \, ds \]

\[ \leq 2^{-\lambda} r^\lambda \int_0^1 s(1-s) f(s, s(1-s)) \, ds, \]

and so

\[ \|(Tu)''\|_0 \leq 2^{-\lambda} r^\lambda \int_0^1 s(1-s) f(s, s(1-s)) \, ds. \]

Thus, from (21), (22), and (19),

\[ \|Tu\| = |Tu|_0 + |(Tu)''|_0 \leq 2^{1-\lambda} r^\lambda \int_0^1 s(1-s) f(s, s(1-s)) \, ds \]

\[ \leq r = \|u\|, \quad u \in \partial \Omega_1 \cap P. \]

Next, set \( \Omega_2 = \{ u \in E \mid \|u\| < R \} \), where

\[ R = \max \left\{ 288 M, 2^{(\theta+1)/(\lambda-1)} \left( \int_1^{3/4} s(1-s) f(s, s(1-s)) \, ds \right)^{1/(1-\lambda)} \right\}. \]

Let \( u \in \partial \Omega_2 \cap P \). Then \( \|u\| = |u|_0 + |u''|_0 = R, \ |u|_0 \leq R, \ |u''|_0 \leq R \). From (9), we have

\[ |u|_0 \leq \frac{1}{8} |u''|_0, \quad |u''|_0 \geq \frac{8}{9} R. \]
Also, by the definition of the cone $P$, we have that for $1/4 \leq t \leq 3/4$,

$$u(t) = \int_0^1 G(t, s)(-u''(s)) \, ds \geq \int_{1/4}^{3/4} G(t, s)(-u''(s)) \, ds \geq \frac{1}{4^2} |u''|_0 \int_{1/4}^{3/4} G(s, s) \, ds \geq \frac{1}{2^8} |u''|_0,$$

and hence,

$$|u|_0 \geq \frac{1}{2^8} |u''|_0.$$  \hfill (26)

Since $u \in P$, from (26), we have

$$\frac{u(s)}{s(1-s)} \geq 4u(s) \geq |u|_0 \geq \frac{1}{2^8} |u''|_0,$$  \hfill (27)

and so, from (24) and (25), for $1/4 \leq s \leq 3/4$,

$$\frac{u(s)}{s(1-s)} \geq \frac{8}{2^8} R \geq M.$$  \hfill (28)

For $1/4 \leq t \leq 3/4$, from (27) and (28), we have

$$Tu(t) = \int_0^1 G(t, \tau) \int_0^1 G(\tau, s)f(s, u(s)) \, ds \, d\tau \geq \int_{1/4}^{3/4} G(t, \tau) \int_{1/4}^{3/4} G(\tau, s)f(s, u(s)) \, ds \, d\tau \geq \frac{1}{4^2} \int_{1/4}^{3/4} (1-\tau) \, d\tau \int_{1/4}^{3/4} s(1-s) \left( \frac{u(s)}{s(1-s)} \right)^{\lambda} f(s, s(1-s)) \, ds \geq \frac{1}{2^8} |u|_0^\lambda \int_{1/4}^{3/4} s(1-s) f(s, s(1-s)) \, ds.$$  \hfill (29)

On the other hand, from (27),

$$-(Tu)''(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds \geq \int_{1/4}^{3/4} G(t, s)\left( \frac{u(s)}{s(1-s)} \right)^{\lambda} f(s, s(1-s)) \, ds \geq 2^{-8\lambda-2} |u''|_0^\lambda \int_{1/4}^{3/4} s(1-s) f(s, s(1-s)) \, ds,$$
and hence,

\[(Tu)''_0 \geq 2^{-8\lambda-2}|u''|_0^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.\]

Now, from (29), (30), and the fact that $a^\lambda + b^\lambda \geq 2^{1-\lambda}(a + b)^\lambda$ for $\lambda \geq 1$ and $a, b > 0$, we arrive at

\[
\|Tu\| \geq (2^{-8}|u|^\lambda_0 + 2^{-8\lambda-2}|u''|_0^\lambda) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\]

\[
\geq 2^{-8\lambda-2}(|u|^\lambda_0 + |u''|^\lambda_0) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\]

\[
\geq 2^{-9\lambda-1}(|u|^\lambda_0 + |u''|^\lambda_0) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.
\]

Consequently, by the definition of $R$, we have

\[
\|Tu\| = |Tu|_0 + |(Tu)''_0|_0 \geq 2^{-9\lambda-1}R^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\]

\[
\geq R = \|u\|, \quad u \in \partial \Omega_2 \cap P.
\]

Finally, from (23) and (31), by Lemma 1, the operator $T$ has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive $C^2[0,1]$ solution to the boundary value problem (1).

The proof of Theorem 1 is complete.

PROOF OF THEOREM 2: We prove the sufficiency first. Since (5) implies (4), Theorem 1 provides a $C^2[0,1]$ solution $u \in P$. From (9), $u(t) \leq (1/2)t(1-t)|u''|_0$.

To prove that $u \in C^3[0,1]$, choose a positive number $c \geq \max\{M, |u''|_0/(2N)\}$. Then, from (2) and (3), we have

\[
\int_0^1 |u^{(4)}(s)| \, ds = \int_0^1 f(s, u(s)) \, ds \leq c^\lambda \int_0^1 f\left(s, \frac{u(s)}{c}\right) \, ds
\]

\[
\leq c^\lambda \left(|u''|_0^\lambda / 2c\right) \int_0^1 f\left(s, s(1-s)\right) \, ds.
\]

Now, $u^{(4)}(t)$ is absolute integrable over $[0,1]$ from (5), and hence, $u \in C^3[0,1]$.

To prove the necessity, let there be a positive solution $u \in C^3[0,1]$ of (1). The same reasoning at the beginning of the proof of Theorem 1 asserts that $u(t) \geq k_1t(1-t)$ for
all \( t \in [0,1] \) and for some constant \( k_1 > 0 \). Let \( c \geq \max\{M, 1/(k_1 N)\} \). Then, from \( (2) \) and \( (3) \),
\[
f(t, t(1-t)) \leq c\alpha f\left(t, \frac{(t(1-t)u(t))/(cu(t))}{cu(t)}\right) \leq c^{\mu-\lambda} k_1^{-\lambda} f(t, u(t)),
\]
and hence,
\[
\int_0^1 f(t, t(1-t)) \, dt \leq c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 f(t, u(t)) \, dt = c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 u^{(4)}(t) \, dt
\]
\[
= c^{\mu-\lambda} k_1^{-\lambda} [u''(1) - u''(0)] < +\infty.
\]
Thus, \( (5) \) holds and the proof of Theorem 2 is complete. 

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