This paper investigates fourth-order superlinear singular two-point boundary value problems and obtains necessary and sufficient conditions for existence of $C^2$ or $C^3$ positive solutions on the closed interval.

1. INTRODUCTION

In this paper, we are concerned with the fourth-order singular two-point boundary value problem

\[
\begin{aligned}
&u^{(4)}(t) = f(t, u(t)), & 0 < t < 1, \\
u(0) = u(1) = u''(0) = u''(1) = 0,
\end{aligned}
\]

where $f \in C((0, 1) \times [0, +\infty), [0, +\infty))$ and is quasi-homogeneous with respect to the second variable, namely, there are constants $\lambda$, $\mu$, $N$, $M$ with $1 < \lambda \leq \mu < \infty$ and $0 < N \leq 1 \leq M$ such that for all $0 < t < 1$, $u \geq 0$,

\[
\begin{aligned}
&c^\mu f(t, u) \leq f(t, cu) \leq c^\lambda f(t, u), \quad \text{if } 0 < c \leq N, \\
&c^\lambda f(t, u) \leq f(t, cu) \leq c^\mu f(t, u), \quad \text{if } c \geq M.
\end{aligned}
\]

A typical quasi-homogeneous function is $f = f_1(t)u^{\lambda_1} + \cdots + f_m(t)u^{\lambda_m}$, where $\lambda \leq \lambda_i \leq \mu$, $i = 1, \ldots, m$.

Singular or nonsingular fourth-order boundary value problems have been extensively studied by many authors (see [1, 2, 3, 4, 5, 6, 7] for nonsingular cases and [8, 9] for singular cases). In [3, 4, 5] the right hand side function in the equation of (1) has separated variables, namely, $f(t, u) = \lambda a(t)g(u)$, and in [1, 6, 7, 8] the function $f$ involves the second derivative $u''$. O'Regan considered the singular case where $f(t, u, u'')$ is singular at $u = 0$ or $u'' = 0$, while in [9] singularity occurs at $t = 0$ or $t = 1$. Using a modified upper and lower solution method, Chen and Zhang [10] established necessary and sufficient conditions for existence of positive solutions to second-order sublinear...

Received 8th January, 2002
This research was supported by the Chinese NSF under Grant 10071043

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/02 $A2.00+0.00.$
boundary value problems on a half-line. Using a similar method, Wei [9] obtained necessary and sufficient conditions for existence of positive solutions to the fourth-order problem (1) in the sublinear case. The results in [9, 10] involve integrability conditions in terms of the function $f$ and the Green's function. To this connection, however, the upper and lower solution method can hardly be used to treat the superlinear case.

In this paper, based on a careful analysis of the Green's function, we shall apply a fixed point theorem in cones to the superlinear problem (1) and obtain necessary and sufficient conditions for existence of a positive solution with different smoothness on the closed interval.

2. MAIN RESULTS

Our main results are the two following theorems.

**THEOREM 1.** The boundary value problem (1) has a positive solution $u \in C^2[0,1] \cap C^4(0,1)$, if and only if,

$$\int_0^1 t(1-t)f(t,t(1-t))\,dt < \infty.$$  

**THEOREM 2.** The boundary value problem (1) has a positive solution $u \in C^3[0,1] \cap C^4(0,1)$, if and only if,

$$\int_0^1 f(t,t(1-t))\,dt < \infty.$$  

We note that (5) implies (4). To prove Theorems 1 and 2, we shall prepare some lemmas. First, we state a fixed point theorem in a cone as follows:

**LEMMA 1.** ([11, Theorem 2.3.4].) Let $E$ be a Banach space and $P$ a cone in $E$. Suppose that $\Omega_1$ and $\Omega_2$ are two bounded open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$ is a completely continuous operator satisfying

$$\|Tx\| \leq \|x\| \text{ for } x \in P \cap \partial \Omega_1 \text{ and } \|Tx\| \geq \|x\| \text{ for } x \in P \cap \partial \Omega_2,$$

then $T$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $E = \{u \in C^2[0,1] : u(0) = u(1) = 0, u''(0) = u''(1) = 0\}$. Define the norm $\|u\|$ for every $u \in E$ by $\|u\| = |u|_0 + |u''|_0$, where $|\cdot|_0$ is the usual sup-norm for continuous functions over $[0,1]$. It is seen that $E$ equipped with the norm $\|\cdot\|$ is a Banach space.

Let $G(t, s)$ be the Green's function of the second-order boundary value problem

$$\begin{cases} -u''(t) = 0, \\
  u(0) = u(1) = 0,
\end{cases}$$
that is,
\[ G(t,s) = \begin{cases} 
  s(1-t), & 0 \leq s \leq t \leq 1, \\
  t(1-s), & 0 \leq t \leq s \leq 1.
\end{cases} \]

Let
\[ h(t, s) = \int_0^1 G(t, \tau)G(\tau, s)d\tau. \]

Then \( h(t, s) \) is the Green's function of the homogeneous fourth-order boundary value problem corresponding to (1). It is easily seen that
\[ (7) \quad G(t,s) > \frac{1}{4}G(s,s), \quad 0 \leq s \leq 1 \]
and for \( 1/4 \leq t \leq 3/4 \),
\[ G(t,s) \geq \frac{1}{4}G(s,s), \quad 0 \leq s \leq 1. \]

Denote
\[ P = \left\{ u \in E \mid u(t) \geq 0, \ u''(t) \leq 0, \ 0 \leq t \leq 1; \ u(t) \geq \frac{1}{4}|u|_0, \ -u''(t) \geq \frac{1}{4}|u''|_0, \ \frac{1}{4} \leq t \leq \frac{3}{4} \right\}. \]

It can be easily seen that \( P \) is a cone in \( E \).

Next, we define an operator \( T : P \to E \) by
\[ (8) \quad (Tu)(t) = \int_0^1 h(t, s)f(s, u(s))ds, \quad u \in P. \]

We observe that a fixed point of \( T \) in \( E \) is indeed a positive solution of the boundary value problem (1).

Using the Green's function, for every \( u \in P \), we shall have an estimate for \( u(t) \) in terms of the magnitude of its second derivative, namely, for \( t \in [0, 1] \),
\[ u(t) = \int_0^1 G(t,s)(-u''(s)) \, ds \leq \left( \int_0^t s(1-t) \, ds + \int_t^1 t(1-s) \, ds \right)|u''|_0 \]
\[ = \frac{1}{2} t(1-t)|u''|_0. \]

Let \( u \in P \) and let \( c \) be a positive number such that \( c \geq M \) and \( |u''|_0/(2c) \leq N \). From (9), \( u(s)/(cs(1-s)) \leq |u''|_0/(2c) \leq M \). Then, from (2) and (3),
\[ |Tu(t)| \leq \int_0^1 G(s,s)c^\alpha f(s, u(s)/c) \, ds \]
\[ = c^\alpha \int_0^1 G(s,s)f\left(s, \frac{u(s)}{cs(1-s)}s(1-s) \right) \, ds \]
\[ \leq c^\alpha \left( \frac{|u''|_0}{2c} \right)^{\lambda} \int_0^1 s(1-s)f(s, s(1-s)) \, ds. \]

Hence, \( T \) is well defined on \( P \) provided that (4) or (5) holds.
LEMMA 2. If (4) holds, then $T(P) \subset P$.

PROOF: Let $u \in P$. Obviously, $Tu(t) \geq 0$ and $-(Tu)'(t) \geq 0$. For $1/4 \leq t \leq 3/4$, we claim that

$$ Tu(t) \geq \frac{1}{4}|Tu|_0. $$

Indeed, from (4), by Fubini's theorem, (8) can be rewritten as

$$ (Tu)(t) = \int_0^1 G(t, \tau) \int_0^1 G(\tau, s) f(s, u(s)) \, ds \, d\tau. $$

It follows from (6) that

$$ |Tu|_0 \leq \int_0^1 G(\tau, \tau) \int_0^1 G(\tau, s) f(s, u(s)) \, ds \, d\tau. $$

On the other hand, for $1/4 \leq t \leq 3/4$, (7) together with (11) gives

$$ (Tu)(t) > \frac{1}{4} \int_0^1 G(t, \tau) \int_0^1 G(\tau, s) f(s, u(s)) \, ds \, d\tau \geq \frac{1}{4}|Tu|_0. $$

Next, we claim that $-(Tu)''(t) \geq (1/4)|(Tu)''|_0$ for $t \in [1/4, 3/4]$. In fact, from

$$ -(Tu)''(t) = \int_0^1 G(t, s) f(s, u(s)) \, ds, $$

it follows from (6) and (7) that

$$ |(Tu)''|_0 \leq \int_0^1 G(s, s) f(s, u(s)) \, ds $$

and, for $1/4 \leq t \leq 3/4$,

$$ -(Tu)''(t) \geq \frac{1}{4} \int_0^1 G(s, s) f(s, u(s)) \, ds \geq \frac{1}{4}|(Tu)''|_0. $$

We now conclude that $T : P \to P$ from (12) and (13) and complete the proof.

LEMMA 3. If (4) holds, then $T$ is a completely continuous operator on $P$.

PROOF: If $u_n \in P$ and $u_n \to u_0$ in $E$ as $n \to \infty$, then we have that $u_0 \in P$ by the definition of the cone $P$ and that $\{\|u_n\|\}$ is bounded, say, $\|u_n\| \leq C_0$, $n \geq 1$. As a result, from (9), we have

$$ u_n(t) \leq \frac{C_0}{2} t(1 - t). $$
Let \( c \) be a positive number such that \( c \geq M \) and \( C_0/2c \leq N \). From (2) and (3),
\[
|(Tu_n)(t)| \leq \int_0^1 s(1-s)f(s,u_n(s))\,ds \\
\leq \int_0^1 s(1-s)c^\mu f(s,u_n(s)/c)\,ds \\
\leq \int_0^1 s(1-s)c^\mu \left( \frac{u_n(s)}{s(1-s)c} \right)^\lambda f(s,s(1-s))\,ds \\
\leq c^{\mu-\lambda} \left( \frac{C_0}{2} \right)^\lambda \int_0^1 s(1-s)f(s,s(1-s))\,ds.
\]

Now, from (4), an application of Lebesgue's dominant convergence theorem gives the continuity of \( T \) on \( P \).

To prove \( T \) is a compact operator, we shall show that for every bounded sequence \( \{u_n\} \) in \( P \), the sequence \( \{Tu_n\} \subset P \) has a convergent subsequence in \( E \). Since \( \{Tu_n\} \) is bounded in \( E \), \( \{(Tu_n)''\}_n \) is bounded and hence \( \{Tu_n(t)\} \) is equicontinuous. By Ascoli—Arzela's lemma, it suffices to show that \( \{(Tu_n)''(t)\} \) is equicontinuous. Let \( C_0 \) be a positive number such that \( \|u_n\| \leq C_0 \), \( n = 1, 2, \ldots \). Then (14) holds from (9).

Again, choose a \( c \geq \max\{M,C_0/(2N)\} \). Then
\[
(Tu)''(t) = \int_0^t s f(s,u(s))\,ds - \int_t^1 (1-s)f(s,u(s))\,ds \\
\leq \int_0^t s f(s,u(s))\,ds + \int_t^1 (1-s)f(s,u(s))\,ds \\
\leq C_1 \left( \int_0^t s f(s,s(1-s))\,ds + \int_t^1 (1-s)f(s,s(1-s))\,ds \right) \\
= : F(t),
\]
where \( C_1 = c^{\mu-\lambda}(C_0/2)^\lambda \). Since, in view of (4),
\[
\int_0^1 F(t)\,dt = C_1 \int_0^1 \int_0^t s f(s,u(s))\,ds\,dt + C_1 \int_0^1 \int_t^1 (1-s)f(s,u(s))\,ds\,dt \\
= 2C_1 \int_0^1 s(1-s)f(s,s(1-s))\,ds < \infty,
\]
we have the equicontinuity of the sequence \( \{(Tu_n)''(t)\} \) from the uniform continuity of the convergent integral of \( F(t) \) with respect to the Lebesgue measure over \([0,1]\).

Therefore, \( T \) is a compact operator on \( P \) and the proof of Lemma 3 is complete. \( \square \)

We are now in a position to prove our main results.
PROOF OF THEOREM 1: Necessity. Let \( u \in C^2[0,1] \cap C^4(0,1) \) be a positive solution of (1). Obviously, \( u''(t) \leq 0 \) for \( 0 \leq t \leq 1 \) and hence \( u(t) \) is concave. It follows from \( u(0) = u(1) = 0 \) that \( u'(0) > 0 \) and \( u'(1) < 0 \). Consequently, there must be a positive number \( k \) such that \( u(t) \geq kt(1-t) \). Let \( c \geq \max\{M,1/(kN)\} \). Then, for \( 0 < t < 1 \), \( (1-t)/(cu(t)) < N \), and we get

\[
\begin{align*}
\frac{f(t, t(1-t))}{c} \leq c^\mu f\left( t, t(1-t)/cu(t) \right) \\
\leq c^{\mu-\lambda} k^{-\lambda} f\left( t, u(t) \right) = c^{\mu-\lambda} k^{-\lambda} u^{(4)}(t).
\end{align*}
\]

(15)

Since \( u''(0) = u''(1) = 0 \), there is a \( t_0 \in (0,1) \) such that \( u'''(t_0) = 0 \). Then

\[
\begin{align*}
u''(t_0) = \int_0^{t_0} u'''(s) \, ds = -\int_0^{t_0} \int_{t_0}^s u^{(4)}(\tau) \, d\tau \, ds = -\int_0^{t_0} \tau u^{(4)}(\tau) \, d\tau.
\end{align*}
\]

(16)

On the other hand,

\[
\begin{align*}v''(t_0) = -\int_{t_0}^1 u'''(s) \, ds = -\int_{t_0}^1 \int_{t_0}^s u^{(4)}(\tau) \, d\tau \, ds \\
= -\int_{t_0}^1 (1-\tau)u^{(4)}(\tau) \, d\tau.
\end{align*}
\]

(17)

Therefore,

\[
\begin{align*}
\int_0^1 t(1-t)u^{(4)}(t) \, dt = \left( \int_0^{t_0} + \int_{t_0}^1 \right) t(1-t)u^{(4)}(t) \, dt \\
\leq \int_0^{t_0} tu^{(4)}(t) \, dt + \int_{t_0}^1 (1-t)u^{(4)}(t) \, dt \\
= 2(-u''(t_0)) < \infty.
\end{align*}
\]

(18)

We now obtain (4) from (15) and (18), and complete the proof of the necessity.

Sufficiency. Let \( \Omega_1 = \{ u \in E \mid ||u|| < r \} \), where

\[
r \leq \min\left\{ 2N, 2\left( \int_0^{1} s(1-s)f(s,s(1-s)) \, ds \right)^{1/(1-\lambda)} \right\}.
\]

(19)

Let \( u \in \partial\Omega_1 \cap P \). Then \( ||u|| = |u|_0 + |u''|_0 = r \), and \( |u|_0 \leq r, |u''|_0 \leq r \). It follows from (9) that

\[
u(t) \leq \frac{1}{2} t(1-t)|u''|_0 \leq \frac{r}{2} t(1-t) \leq Nt(1-t).
\]

(20)
In view of (2), (3), and (20), we have

\[ Tu(t) = \int_0^1 h(t, s) f(s, u(s)) \, ds \]

\[ \leq \int_0^1 h(t, s) \left( \frac{u(s)}{s(1-s)} \right)^\lambda f(s, s(1-s)) \, ds \]

\[ \leq 2^{-\lambda r^A} \int_0^1 s(1-s) f(s, s(1-s)) \, ds \]

and

\[ |Tu|_0 \leq 2^{-\lambda r^A} \int_0^1 s(1-s) f(s, s(1-s)) \, ds, \quad u \in \partial \Omega_1 \cap P. \]

On the other hand,

\[ -(Tu)''(t) = \int_0^1 G(t, s) f(s, u(s)) \, ds \]

\[ \leq \int_0^1 G(t, s) \left( \frac{u(s)}{s(1-s)} \right)^\lambda f(s, s(1-s)) \, ds \]

\[ \leq 2^{-\lambda r^A} \int_0^1 s(1-s) f(s, s(1-s)) \, ds, \]

and so

\[ |(Tu)''|_0 \leq 2^{-\lambda r^A} \int_0^1 s(1-s) f(s, s(1-s)) \, ds. \]

Thus, from (21), (22), and (19),

\[ ||Tu|| = |Tu|_0 + |(Tu)''|_0 \leq 2^{1-\lambda r^A} \int_0^1 s(1-s) f(s, s(1-s)) \, ds \]

\[ \leq r = ||u||, \quad u \in \partial \Omega_1 \cap P. \]

Next, set \( \Omega_2 = \{ u \in E \mid ||u|| < R \} \), where

\[ R = \max \left\{ 288M, 2^{(\theta+1)/(\lambda-1)} \left( \int_{1/4}^{3/4} s(1-s) f(s, s(1-s)) \, ds \right)^{1/(1-\lambda)} \right\}. \]

Let \( u \in \partial \Omega_2 \cap P \). Then \( ||u|| = |u|_0 + |u''|_0 = R, \quad |u|_0 \leq R, \quad |u''|_0 \leq R \). From (9), we have

\[ |u|_0 \leq \frac{1}{8} |u''|_0, \quad |u''|_0 \geq \frac{8}{9} R. \]
Also, by the definition of the cone $P$, we have that for $1/4 \leq t \leq 3/4$,

$$u(t) = \int_0^1 G(t, s)(-u''(s)) \, ds \geq \int_{1/4}^{3/4} G(t, s)(-u''(s)) \, ds$$

$$\geq \frac{1}{4^2} |u''|_0 \int_{1/4}^{3/4} G(s, s) \, ds \geq \frac{1}{2^8} |u''|_0,$$

and hence,

(26) \quad |u|_0 \geq \frac{1}{2^8} |u''|_0.

Since $u \in P$, from (26), we have

(27) \quad \frac{u(s)}{s(1-s)} \geq 4u(s) \geq |u|_0 \geq \frac{1}{2^8} |u''|_0,

and so, from (24) and (25), for $1/4 \leq s \leq 3/4$,

(28) \quad \frac{u(s)}{s(1-s)} \geq \frac{8}{2^8} R \geq M.

For $1/4 \leq t \leq 3/4$, from (27) and (28), we have

$$Tu(t) = \int_0^1 G(t, \tau) \int_0^1 G(\tau, s)f(s, u(s)) \, ds \, d\tau$$

$$\geq \int_{1/4}^{3/4} G(t, \tau) \int_{1/4}^{3/4} G(\tau, s)f(s, u(s)) \, ds \, d\tau$$

$$\geq \frac{1}{4^2} \int_{1/4}^{3/4} \tau(1-\tau) \, d\tau \int_{1/4}^{3/4} s(1-s) \left( \frac{u(s)}{s(1-s)} \right)^{\lambda} f(s, s(1-s)) \, ds$$

(29) \quad \geq \frac{1}{2^8} |u|_0^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.

On the other hand, from (27),

$$-(Tu)'''(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds$$

$$\geq \int_{1/4}^{3/4} G(t, s) \left( \frac{u(s)}{s(1-s)} \right)^{\lambda} f(s, s(1-s)) \, ds$$

$$\geq 2^{-8\lambda-2} |u''|_0^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.$$
and hence,

\[(30) \quad \| (Tu)'' \|_0 \geq 2^{-8\lambda - 2}|u''|_0^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.\]

Now, from (29), (30), and the fact that \(a^\lambda + b^\lambda \geq 2^{1-\lambda}(a+b)^\lambda\) for \(\lambda \geq 1\) and \(a, b > 0\), we arrive at

\[
\|Tu\| \geq (2^{-8}|u|_0^\lambda + 2^{-8\lambda - 2}|u''|_0^\lambda) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\geq 2^{-8\lambda - 2}(|u|_0^\lambda + |u''|_0^\lambda) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\geq 2^{-9\lambda - 1}(|u|_0 + |u''|_0) \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds.
\]

Consequently, by the definition of \(R\), we have

\[
\|Tu\| = \|Tu\|_0 + \| (Tu)'' \|_0 \geq 2^{-9\lambda - 1}R^\lambda \int_{1/4}^{3/4} s(1-s)f(s, s(1-s)) \, ds
\geq R = \|u\|, \quad u \in \partial \Omega_2 \cap P.
\]

Finally, from (23) and (31), by Lemma 1, the operator \(T\) has at least one fixed point \(u \in P \cap (\Omega_2 \setminus \Omega_1)\) which is a positive \(C^2[0,1]\) solution to the boundary value problem (1).

The proof of Theorem 1 is complete.

**Proof of Theorem 2:** We prove the sufficiency first. Since (5) implies (4), Theorem 1 provides a \(C^2[0,1]\) solution \(u \in P\). From (9), \(u(t) \leq (1/2)t(1-t)|u''|_0\).

To prove that \(u \in C^3[0,1]\), choose a positive number \(c \geq \max\{M, |u''|_0/(2N)\}\). Then, from (2) and (3), we have

\[
\int_0^1 |u^{(4)}(s)| \, ds = \int_0^1 f(s, u(s)) \, ds \leq c^\alpha \int_0^1 f(s, u(s)/c) \, ds
\leq c^\alpha \left(\frac{|u''|_0}{2c}\right)^\lambda \int_0^1 f(s, s(1-s)) \, ds.
\]

Now, \(u^{(4)}(t)\) is absolute integrable over \([0,1]\) from (5), and hence, \(u \in C^3[0,1]\).

To prove the necessity, let there be a positive solution \(u \in C^3[0,1]\) of (1). The same reasoning at the beginning of the proof of Theorem 1 asserts that \(u(t) \geq k_1t(1-t)\) for
all $t \in [0,1]$ and for some constant $k_1 > 0$. Let $c \geq \max\{M, 1/(k_1 N)\}$. Then, from (2) and (3),

$$f(t, t(1-t)) \leq c^\mu f\left(t, \frac{t(1-t)u(t)}{cu(t)}\right) \leq c^\mu k_1^{-\lambda} f(t, u(t)),$$

and hence,

$$\int_0^1 f(t, t(1-t)) dt \leq c^\mu k_1^{-\lambda} \int_0^1 f(t, u(t)) dt = c^\mu k_1^{-\lambda} \int_0^1 u^{(4)}(t) dt = c^\mu k_1^{-\lambda} \left[u''(1) - u''(0)\right] < +\infty.$$

Thus, (5) holds and the proof of Theorem 2 is complete. \[\square\]

REFERENCES


