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ON COMPETITION BETWEEN MODES AT THE ONSET OF BÉNARD-MARANGONI CONVECTION IN A LAYER OF FLUID

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Abstract

In this paper we use classical linear stability theory to analyse the onset of steady and oscillatory Bénard-Marangoni convection in a horizontal layer of fluid in the more physically-relevant case when both the non-dimensional Rayleigh and Marangoni numbers are linearly dependent. We present examples of situations in which there is competition between modes at the onset of convection when the layer is heated from below.

1. Introduction

The onset and subsequent evolution of convection in a fluid is a problem of great importance to many industrial applications, such as crystal growth (see, for example, Hurle [6], Ostrach [10] and Schwabe [15]) and welding of steels (Mills and Keene [8]). Convection is also important in many other contexts, for example, in geophysics, where convection in the oceanic and continental surfaces and the atmosphere can occur due to non-uniform heating from the sun. Indeed, the atmospheric structure of planets and the Earth's magnetic field are determined largely by convective instabilities (Zierep and Oertel [18]). We note that the problem of convection in a fluid has been of considerable interest to applied mathematicians for over a hundred years. Convective instabilities driven by either buoyancy (Bénard) or thermocapillary (Marangoni) effects have been the subject of a great deal of theoretical and experimental investigation since the pioneering theoretical works of Rayleigh [14] and Pearson [11] respectively. However, a great deal less work has been done on the more general case in which both mechanisms act simultaneously, a situation usually termed Bénard-Marangoni convection (and hereafter abbreviated to B-M convection), which is the subject of the present work.

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The first study of B-M convection in a planar horizontal fluid layer with a non-deformable free surface was performed by Nield [9], who showed that for steady convection the two destabilising mechanisms reinforce one another. Subsequently Takashima [16] showed numerically that oscillatory convection cannot occur if the free surface is non-deformable. Davis and Homsy [2] extended Nield's work [9] on steady convection to include a deformable free surface and found that weak surface deformation stabilises Bénard-dominated convection and destabilises Marangoni-dominated convection.

In all the early work described above the effects of buoyancy and thermocapillary, represented by the non-dimensional Rayleigh number R and Marangoni number M respectively, were taken to be *independent*. However, in a typical physical experiment the control parameter is the temperature difference across the layer (which appears linearly in both R and M), and so recent work has focused on the more physically-relevant case in which R and M are *linearly dependent*. Benguria and Depassier [1] numerically investigated the onset of B-M convection in a planar layer heated from below. They identified the existence of an oscillatory branch to the marginal stability curve for the onset of convection when the free surface is sufficiently deformable and the Marangoni effects are sufficiently strong, although in all the examples they described the onset of convection was steady. Recently Hashim and Wilson [5] extended the work of Benguria and Depassier [1], and in particular gave examples of situations in which the onset of convection was oscillatory. Pérez-García and Carneiro [12] also considered the same problem numerically and found examples of situations in which there is competition between a steady and an oscillatory mode and between two oscillatory modes at the onset of convection when the layer is heated from above.

An exciting recent development is the growth of interest in understanding how the *nonlinear* competition between different modes can lead to pattern formation near or at the onset of convection. For example, Johnson and Narayanan [7] studied experimentally surface-tension-gradient-driven convection in a layer of silicone oil with an upper air gap, which was placed between two plates, near a "codimension-two" point at which two steady modes coexist. Johnson and Narayanan [7] observed a dynamic switching between two different flow patterns near this point. VanHook *et al.* [17] investigated experimentally the formation of surface-tension-gradient-driven convection patterns in a thin layer of silicone oil bounded below by a heated rigid plane boundary and above by an air layer. Specifically VanHook *et al.* [17] observed that both the long- and short-wavelength (hexagonal) modes can coexist for a range of liquid depths. VanHook *et al.* [17] observed that the presence of the hexagons suppresses the long-wavelength mode, while the presence of the long-wavelength mode may induce the formation of hexagons. Motivated in part by these results, Golovin *et al.* [3] recently studied theoretically the nonlinear evolution and secondary instabilities of Marangoni convection in a two-layer liquid-gas system with a deformable liquid-gas interface, bounded from below and from above by rigid plates. Golovin *et al.* [3] derived a system of amplitude equations describing the evolution of a short-wave mode and its interaction with the long-wave mode. The equations they obtained are valid when both instability modes coexist. This work shows that the nonlinear interaction between these two types of Marangoni convection can yield new secondary instabilities of the basic hexagonal convection pattern and result in complex non-stationary behaviour.

In this paper we use classical linear stability theory to investigate the competition between modes at the onset of B-M convection in a horizontal planar layer of fluid heated from below in the most physically-relevant case when R and M are linearly dependent. In particular, we find for the first time a situation in which there exists a competition between two steady modes and one oscillatory mode at the onset of convection. These results extend some of the numerical results of Pérez-García and Carneiro [12]. The structure of the paper is as follows. In Sections 2 and 3 we briefly formulate and solve the appropriate linear stability problem. In Section 4 we present the results of numerical calculations which show examples of situations in which there are competitions between different modes at the onset of convection. Finally, in Section 5 we summarise the work.

2. Problem formulation

We wish to examine the stability of a horizontal layer of quiescent fluid of infinite extent and thickness d which is bounded below by a rigid solid planar boundary maintained at a constant temperature T_1 and above by a free surface initially at temperature T_2 and subject to a uniform vertical temperature gradient. The fluid is Newtonian and incompressible with density $\rho = \rho_0[1-\alpha(T-T_2)]$, where the constant ρ_0 is the value ρ at $T = T_2$ and $\alpha > 0$ is the coefficient of thermal volume expansion. The free surface is in contact with a passive gas which is at constant pressure and constant temperature T_∞ and has surface tension given by the simple linear law $\tau = \tau_0 - \gamma(T-T_2)$, where the constant τ_0 is the value of τ at $T = T_2$ and the constant γ is positive for most common fluids. At the free surface the temperature obeys Newton's law of cooling and so T_1, T_2 and T_∞ are related by $\Delta T = T_1 - T_2 = (T_2 - T_\infty)hd/k$ where h is the heat transfer coefficient between the free surface and the passive gas above and k is the thermal conductivity of the fluid. We choose Cartesian axes with the x- and y-axes in the plane of the rigid lower boundary and the z-axis vertically upwards and non-dimensionalise the governing equations and boundary conditions using d, κ/d, d^2/κ and $(T_1 - T_2)/d$ as appropriate scales for distance, velocity, time and temperature gradient respectively. Using the Boussinesq approximation the linearised Navier-Stokes and heat equations and boundary conditions governing the

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onset of B-M convection are given by (see Hashim [4])

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{s}{P_r}\right) w - a^2 R T = 0, \tag{1}$$

$$(D^2 - a^2 - s)T + w = 0, \tag{2}$$

subject to

$$sf - w = 0, \tag{3}$$

$$\left(D^2 - 3a^2 - \frac{s}{P_r}\right) Dw - a^2\left(P_r G + \frac{a^2}{C_r}\right) f = 0, \tag{4}$$

$$(D^2 + a^2)w + a^2 M(T - f) = 0, \tag{5}$$

$$DT + B_i(T - f) = 0, \tag{6}$$

evaluated on the undisturbed position of the upper free surface $z = 1$, and

$$w = 0, \tag{7}$$

$$Dw = 0, \tag{8}$$

$$T = 0, \tag{9}$$

evaluated on the lower rigid boundary $z = 0$, where the operator $D = \mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to the vertical coordinate z. The variables $w = w(z)$, $T = T(z)$ and f denote the vertical variation of the z-velocity and temperature and the magnitude of the free surface deflection of the linear perturbation to the basic state with total wave number a in the horizontal x-y plane and complex growth rate s. The non-dimensional groups appearing in the problem are the Rayleigh number, $R = g\alpha\Delta T d^3/\nu\kappa$, the Marangoni number, $M = \gamma\Delta T d/\rho_0\kappa\nu$, the capillary number, $C_r = \rho_0\nu\kappa/\tau_0 d$, the Galileo number, $G = gd^3/\nu^2$, the Prandtl number, $P_r = \nu/\kappa$, and the Biot number, $B_i = hd/k$, where, in addition to the parameters defined above, g denotes acceleration due to gravity, ν the kinematic viscosity and κ the thermal diffusivity of the fluid. Note that T and f can be calculated directly from (1) and boundary condition (4) respectively. Eliminating T between (1) and (2) yields a single linear sixth-order ordinary differential equation for w,

$$\left[(D^2 - a^2)(D^2 - a^2 - s)\left(D^2 - a^2 - \frac{s}{P_r}\right) + a^2 R\right] w = 0. \tag{10}$$

The Rayleigh number R and the Marangoni number M are related by $M = \Gamma R$ where $\Gamma = \gamma/\rho_0 g\alpha d^2$. We shall investigate the most physically-relevant case in which Γ is held constant and so R and M are linearly dependent. Note that in the limit $C_r \to 0$ the free surface is weakly deformable, while in the limit $C_r \to \infty$ it is strongly deformable. In the special case $C_r = \infty$, $P_r = 1$ and $B_i = 0$ we recover the problem treated numerically by Benguria and Depassier [1], while if we write $G = B_o/P_rC_r$, where $B_o = \rho_0 g d^2/\tau_0$ is the Bond number, then we recover the problem treated numerically by Pérez-García and Carneiro [12] in the special case $B_o = 0.1$, $P_r = 1$ and $B_i = 0$.

3. Solution of the linearised problem

Equations (1) and (2) together with the boundary conditions (3)–(9) constitute a linear eigenvalue problem for the unknown temporal exponent s. The general solution of (10) is

$$w(z) = \sum_{i=1}^{6} A_i e^{\xi_i z}, \tag{11}$$

where $\xi_1, \ldots, \xi_6$ are the six distinct roots of the sixth-order algebraic equation

$$(\xi^2 - a^2)(\xi^2 - a^2 - s)\left(\xi^2 - a^2 - \frac{s}{P_r}\right) + a^2 R = 0, \tag{12}$$

and A_i ($i = 1, \ldots, 6$) are arbitrary constants. Imposing boundary conditions (3), (5) and (6)–(9), where expressions for T and f are obtained from (1) and (4) respectively, yields a linear system of the form $\mathrm{M}\mathbf{A} = 0$, where $\mathbf{A} = [A_1, \ldots, A_6]^T$. In general, the 6×6 coefficient matrix M (whose entries depend on a, R, s, C_r, G, Γ, P_r and B_i) is complex and may be rather complicated, and so, in general, it has to be calculated either numerically or symbolically using a symbolic algebra package. In this paper we use a FORTRAN 77 program employing the Numerical Algorithms Group (NAG) routine F03ADF and running on a SUN SPARCstation 1+ to evaluate the determinant of M using LU factorisation with partial pivoting. A modification of the Powell [13] hybrid algorithm, which is a combination of Newton's method and the method of steepest descent, implemented using NAG routine C05NBF is then used to find the eigenvalues of M by solving the two non-linear equations obtained from the real and imaginary parts of the determinant of M.

The marginal stability curves in the (a, R) plane on which $\mathrm{Re}(s) = 0$ separate regions of unstable modes with $\mathrm{Re}(s) > 0$ from those of stable modes with $\mathrm{Re}(s) < 0$. The critical Rayleigh number for the onset of convection is the global minimum of R over $a \geq 0$.

4. Numerical results

In this section we shall present numerical results which show the existence of competition between modes at the onset of convection in the case when $\Gamma > 0$.

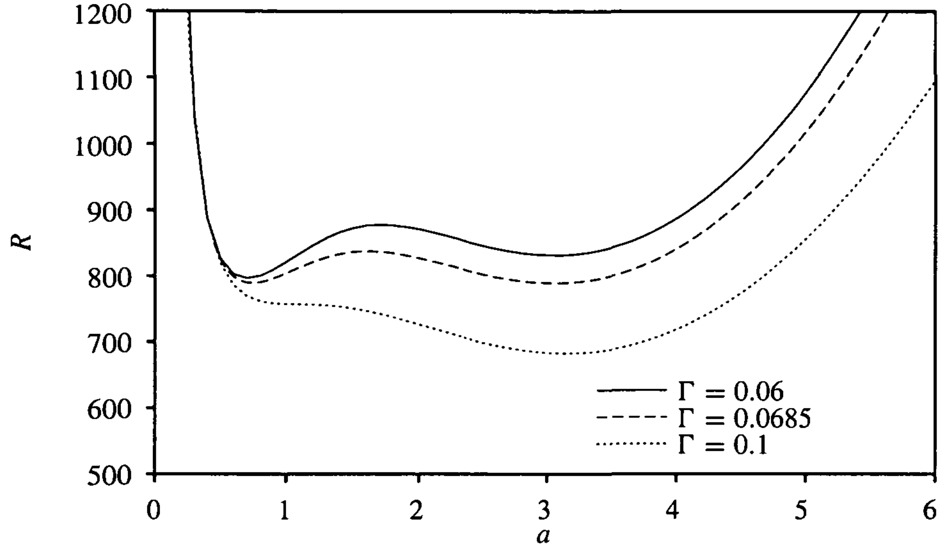

FIGURE 1. Numerically-calculated oscillatory marginal stability curves showing a competition between two oscillatory modes for $\Gamma = 0.0685$ in the case $C_r = \infty$, $G = 250$, $P_r = 1$ and $B_i = 0$.

Figures 1 and 2 illustrate two interesting possibilities for competition between different modes at the onset of convection. Figure 1 shows part of the marginal curves for the case $C_r = \infty$, $G = 250$, $P_r = 1$ and $B_i = 0$ when oscillatory convection is the preferred mode of instability and the oscillatory marginal curves have two local minima. The two minima are approximately equal when $\Gamma = 0.0685$. Similar competition between two oscillatory modes was previously found by Pérez-García and Carneiro [12] when $\Gamma < 0$ (corresponding to situations in which surface tension increases with temperature). Pérez-García and Carneiro [12] found similar phenomena when $\Gamma_c \simeq -10.79048$, using their own numerical method in which an approximate solution for $w(z)$ is used, for the parameter values $C_r = 10^{-3}$, $P_r = 1$, $G = 100$ and $B_i = 0$. In this case we recomputed the value and found $\Gamma_c \simeq -10.879$ using the present numerical method in which an exact analytical form of solution for $w(z)$ is used.

Figure 2 illustrates an alternative possibility which has not been identified before, namely the competition between three different modes. The figure shows part of the marginal stability curves for the case $C_r = 2.73 \times 10^{-3}$, $G = 36.63$, $\Gamma = 0.049$, $P_r = 1$ and $B_i = 0$ in which two steady (long- and short-wave) modes and an oscillatory (short-wave) mode all occur simultaneously at the onset of convection. We note that in the case shown the critical wavenumber on the oscillatory branch is smaller than the critical wavenumber for the short-wave steady mode. The critical Rayleigh number for the long-wave steady mode is in complete agreement with the

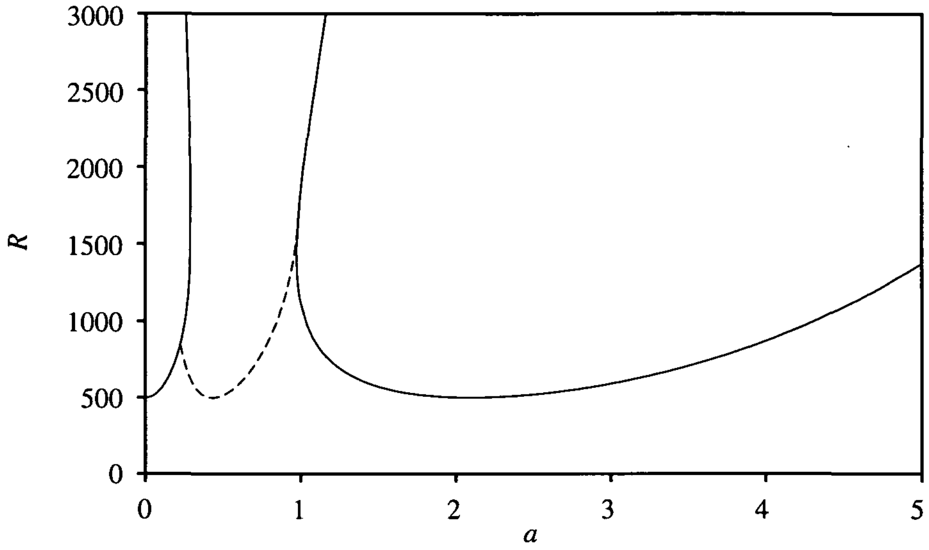

FIGURE 2. Numerically-calculated marginal stability curves (steady—solid line, oscillatory—dashed line) in the case $C_r = 2.73 \times 10^{-3}$, $G = 36.63$, $\Gamma = 0.0490$, $P_r = 1$ and $B_i = 0$.

analytical expression obtained by Hashim and Wilson [5] in the limit $a \to 0$.

In Figure 3 we have plotted numerically-calculated values of Γ_c (at which a steady mode and an oscillatory mode compete) as a function of C_r and extend the results of Pérez-García and Carneiro [12] in the case $P_r = 1$, $G = 0.1/C_r$ and $B_i = 0$. The regions below ($\Gamma < \Gamma_c$) and above ($\Gamma > \Gamma_c$) the curve represent situations in which convection first sets in as oscillatory and steady motions respectively. We found that $\Gamma_c \simeq 0$ when $C_r \simeq 1.58 \times 10^{-3}$ in the case $P_r = 1$, $G \simeq 63.29$ and $B_i = 0$. In this case Pérez-García and Carneiro [12] obtained $C_r = 1.7 \times 10^{-3}$ using their numerical method. Figure 3 shows in particular that oscillatory motions can be the preferred mode of instability for small enough $\Gamma_c > 0$ (that is, $R > 0$) and large enough C_r and thus extends the results of Pérez-García and Carneiro [12] who showed only that convection first sets in as oscillatory motions for $\Gamma_c < 0$.

5. Conclusions

In this paper we used classical linear stability theory to investigate the competition between modes at the onset of B-M convection in a horizontal planar layer of fluid heated from below in the most physically-relevant case when R and M are linearly dependent. The linear analysis presented in this paper revealed a number of situations with competition between different kinds of modes which could be studied in a similar

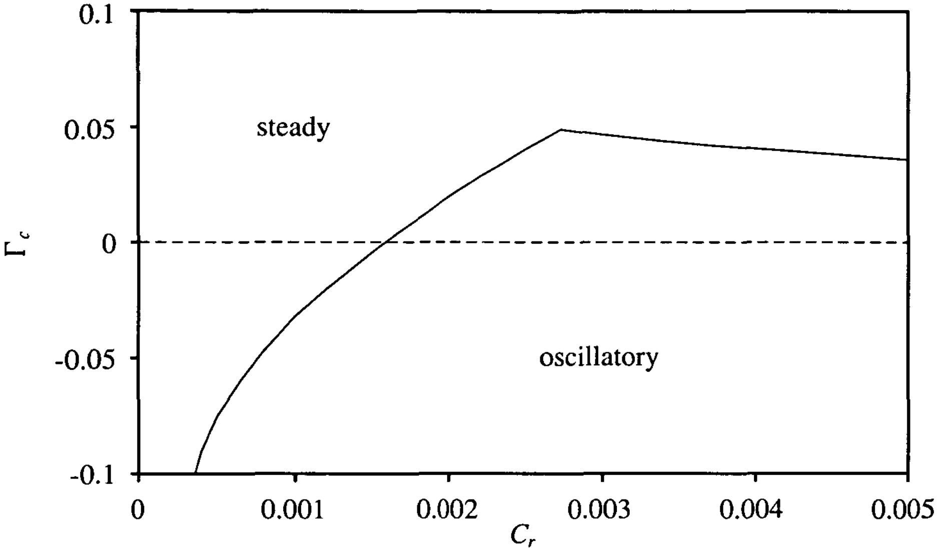

FIGURE 3. Numerically-calculated values of Γ_c at which both steady and oscillatory modes compete plotted as a function of C_r in the case $P_r = 1$, $G = 0.1/C_r$ and $B_i = 0$. Note that $\Gamma_c \simeq 0$ when $C_r \simeq 1.58 \times 10^{-3}$.

manner to that used by Golovin *et al.* [3]. In particular, we obtained for the first time a situation in which two steady modes and an oscillatory mode compete at the onset of convection. It would be very interesting to attempt an experimental verification of the novel features revealed by analysis in this paper.

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