ISOMETRIC REPRESENTATION OF $M(G)$ ON $B(H)$

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In a recent paper, E. Størmer, among other things, proves the existence of an isometric isomorphism from the measure algebra $M(G)$ of a locally compact abelian group G into $BB(L^2(G))$, ([**6**], Proposition 4.6). Here we give another proof for this result which works for non-commutative G as well as commutative G. We also prove that the algebra $L^1(G, \lambda)$, with λ the left (or right) Haar measure, is not isometrically isomorphic with an algebra of operators on a Hilbert space. The proofs of these two results are taken from the author's Ph.D. thesis [**4**], submitted to the University of Edinburgh before Størmer's paper. The author wishes to thank Dr. A. M. Sinclair for his help and encouragement.

We adopt the notation of [**5**], the exception being that for every $a \in G$ and $f \in L^1(G, \lambda)$ we let $L_a f$ be the function in $L^1(G, \lambda)$ defined by $(L_a f)(x) = f(ax)$ for every $x \in G$.

First we need the following two lemmas.

Lemma 1. *For $n \geq 2$ let $F_1, F_2, \ldots, F_n$ be pairwise disjoint compact subsets of G. Then there is an open neighbourhood A of e such that for $x \in F_i$, $y \in F_j$, $(i \neq j)$ the sets xA and yA are disjoint $(i, j = 1, 2, \ldots, n)$.*

Proof. For each $i \neq j$ the set $B_{i,j} = \{x^{-1}y : x \in F_i, y \in F_j\}$ is compact and disjoint from e. Since there are only finitely many such sets, there is an open neighbourhood U of e such that $U \cap (\bigcup B_{i,j}) = \varnothing$. The required set is any open neighbourhood A of e such that $AA^{-1} \subset U$.

In the lemma to follow, λ is the left Haar measure on G, and $H = L^2(G, \lambda)$.

Lemma 2. *Let μ be a positive measure in $M(G)$. Then the map ψ from $L^1(G, \mu)$ into $BB(H)$ defined by*

$$\langle \psi(f)Tg, h \rangle = \int_G f(t) \langle L_{t^{-1}} T L_t g, h \rangle \, d\mu(t), \tag{1}$$

$(f \in L^1(G, \mu)$, $T \in B(H)$, $g, h \in H)$, is an isometric isomorphism.

Proof. The continuity of translations ([**5**], Theorem 20.4) implies that $\langle L_{t^{-1}} T L_t g, h \rangle$ is a continous function of t. Moreover, the boundedness of $\langle L_{t^{-1}} T L_t g, h \rangle$ implies that for every $T \in B(H)$ and $f \in L^1(G, \mu)$ the integral on the right side of (1) exists and defines a bounded sesquilinear form on H. Since for every $t \in G$, L_t and $L_{t^{-1}}$ are isometries we have

$$\left| \int_G f(t) \langle L_{t^{-1}} T L_t g, h \rangle \, d\mu(t) \right| \leq \|f\| \, \|T\| \, \|g\|_2 \, \|h\|_2. \tag{2}$$

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Thus,

$$\|\psi(f)T\| \leq \|T\| \, \|f\|. \tag{3}$$

Therefore, ψ is norm decreasing. To prove ψ is an isometry, we proceed as follows. Let

$$f = \sum_{k=1}^{n} c_k \chi_{F_k},$$

where F_k $(k=1,2,\dots,n)$ are pairwise disjoint compact sets, and χ_{F_k} is the characteristic function of F_k $(k=1,2,\dots,n)$. We have

$$\|f\| = \sum_{k=1}^{n} |c_k| \, \mu(F_k).$$

Let $c_k = |c_k| e^{i\theta_k}$ be the polar form of the complex number $c_k (k=1,2,\dots,n)$. For the compact sets $F_1, F_2, \dots, F_n$, we choose the open set A as in lemma 1 with $\lambda(A) < \infty$. Then, $g = \chi_A \in L^2(G, \lambda)$ and $g \neq 0$.

Let M be the linear span of the set $\{L_t g : t \in \bigcup_{i=1}^{n} F_i\}$. If $t \in F_i$ and $s \in F_j$ $(i \neq j)$, then the two sets tA and sA are disjoint. Thus, the functions $L_t g = \chi_{tA}$ and $L_s g = \chi_{sA}$ are orthogonal. We define the operator S on M as follows. If $h = \sum_{p,q} \lambda_{p,q} L_{t_{p,q}} g$, with $t_{p,q} \in F_p$ $(p=1,2,\dots,n)$, then

$$Sh = \sum_{p,q} e^{-i\theta_p} \lambda_{p,q} L_{t_{p,q}} g.$$

Obviously, S is a linear isometry. We extend S to the closure $\bar{M}$ of M by continuity and we let $T = \bar{S} \oplus 1$ act on $\bar{M} \oplus (\bar{M})^{\perp} = H$. Then, T is an isometry, and we have

$$\begin{aligned} \langle \psi(f)Tg, g \rangle &= \int_G f(t) \langle L_{t^{-1}} T L_t g, g \rangle \, d\mu(t) \\ &= \sum_{k=1}^{n} c_k \int_{F_k} \langle L_{t^{-1}} T L_t g, g \rangle \, d\mu(t) \\ &= \sum_{k=1}^{n} c_k \int_{F_k} \langle e^{-i\theta_k} L_{t^{-1}} L_t g, g \rangle \, d\mu(t) \\ &= \sum_{k=1}^{n} c_k e^{-i\theta_k} \mu(F_k) \|g\|^2 \\ &= \sum_{k=1}^{n} |c_k| \, \mu(F_k) \|g\|^2 = \|f\| \, \|g\|^2. \end{aligned} \tag{4}$$

Thus, $\|\psi(f)\| = \|f\|$.

For a general simple function

$$f=\sum_{k=1}^{n} c_k\chi_{F_k}\in L^1(G,\mu),$$

where the F_k are pairwise disjoint sets, we can, by regularity of the measure μ, find compact sets $F'_k\subset F_k$ such that $\mu(F_k)-\mu(F'_k)$ is arbitrarily small. If $f'=\sum c_k\chi_{F'_k}$, then by the above paragraph $\|\psi(f')\|=\|f'\|$, and the continuity of ψ implies $\|\psi(f)\|=\|f\|$.

Finally, since simple functions are dense in $L^1(G,\mu)$, the continuity of ψ implies that $\|\psi(f)\|=\|f\|$ for every $f\in L^1(G,\mu)$.

THEOREM 1. *If $H=L^2(G,\lambda)$, then there exists an isometric isomorphism from the algebra $M(G)$ into $BB(H)$.*

Proof. We define the map θ from $M(G)$ into $BB(H)$ by

$$\langle\theta(\mu)Tg,h\rangle=\int_G\langle L_{t^{-1}}TL_tg,h\rangle\,d\mu(t)\qquad(\mu\in M(G),\,T\in B(H),\,g,h\in H).$$

Obviously, θ is linear. By the Radon–Nikodym theorem there is a Borel measurable function k with $k(x)=1,(x\in G)$, and $d\mu=k\,d|\mu|$. Thus,

$$\langle\theta(\mu)Tg,h\rangle=\int_G k(t)\langle L_{t^{-1}}TL_tg,h\rangle\,d\,|\mu|\,(t).$$

Let ψ be the mapping of $L^1(G,|\mu|)$ into $BB(H)$ as in Lemma 2. Then,

$$\|\theta(\mu)\|=\|\psi(k)\|=\|k\|=\int_G|k(x)|\,d\,|\mu|\,(x)=\|\mu\|.$$

Thus, θ is isometric. Given $\mu,\nu\in M(G)$, we have

$$\begin{aligned}\langle\theta(\mu)\theta(\nu)Tg,h\rangle&=\int_G\langle L_{t^{-1}}\theta(\nu)TL_tg,h\rangle\,d\mu(t)\\&=\int_G\langle\theta(\nu)TL_tg,L_th\rangle\,d\mu(t)=\int_G\int_G\langle L_{t^{-1}}L_{s^{-1}}TL_sL_tg,h\rangle\,d\nu(s)\,d\mu(t)\\&=\int_G\int_G\langle L_{(ts)^{-1}}TL_{ts}g,h\rangle\,d\nu(s)\,d\mu(t)=\int_G\langle L_{x^{-1}}TL_xg,h\rangle\,d(\mu*\nu)(x)\\&=\langle\theta(\mu*\nu)Tg,h\rangle\qquad(g,h\in H,\,T\in B(H)).\end{aligned}$$

Thus, $\theta(\mu*\nu)=\theta(\mu)\theta(\nu)$ and θ is an isometric isomorphism from $M(G)$ into $BB(H)$.

In order to prove that $M(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space, it is sufficient to prove the following result.

THEOREM 2. *If G has at least two elements, then the algebra $L^1(G,\lambda)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space.*

Proof. Suppose that θ is an isometric isomorphism from $L^1(G,\lambda)$ into $B(H)$. Let K be the closed linear span of the set $\{\theta(f)x : f \in L^1(G,\lambda), x \in H\}$. Then, since $L^1(G,\lambda)$ has a bounded approximate identity of norm one, and θ is an isometry, $\psi(f) = \theta(f) \mid K, (f \in L^1(G,\lambda))$, is an isometric isomorphism from $L^1(G,\lambda)$ into $B(K)$. Thus, without loss of generality, we can assume that the closed linear span of the set $\{\theta(f)x : f \in L^1(G,\lambda), x \in H\}$ is equal to H. From this and $\|\theta\| = 1$ it follows that θ is a $*$-representation of $L^1(G,\lambda)$ on H, ([**1**] Exercise 69.30), and thus, $L^1(G,\lambda)$ is isometrically isomorphic with a C*-algebra. Since the double centralizer of a C*-algebra is a C*-algebra ([**3**], Theorem 2.11), and the double centralizer of $L^1(G,\lambda)$ is $M(G)$ [**7**] this would imply that $M(G)$ is isometrically isomorphic to a C*-algebra.

But it can easily be verified that the set of Hermitian elements ([**2**], Definition 1, p. 46) of $M(G)$ is equal to $\{\lambda\delta_e : \lambda \in \mathbb{R}\}$. Since the set of self-adjoint elements of a unital C*-algebra is equal to the set of Hermitian elements, as defined in Numerical Range theory ([**2**], Example 3, p. 47), we would have

$$M(G) = \{\lambda\delta_e : \lambda \in \mathbb{R}\} + i\{\lambda\delta_e : \lambda \in \mathbb{R}\},$$

a contradiction.

It should be noted that in the case of infinite-dimensional $L^1(G,\lambda)$ a much stronger statement is possible [**8**, Corollary]: $L^1(G,\lambda)$ is not topologically isomorphic to any quotient of a subalgebra of a C*-algebra by a closed ideal.

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