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COMMUTATIVE DISTRIBUTIVE LAWS

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1. Introduction

Kock (1970) defined the notion of a commutative monad in a symmetric monoidal closed category $\mathscr{V}$ and in Kock (1971) showed that the algebras for such a monad had a canonical structure as a closed category and that the monad had a canonical closed structure. In this paper we are concerned with the relationship between distributive laws and commutivity. In particular, the following question arises: given a distributive law between two monads on $\mathscr{V}$ when is the composite monad commutative? To answer this question we define commutative distributive laws and show that if the composite is commutative then the distributive law must be commutative. We also show that if $\mathscr{S}$ and $\mathscr{T}$ are commutative monads in $\mathscr{V}$ with a commutative distributive law between them then the composite is commutative. So we get that if $\mathscr{S}$ and $\mathscr{T}$ are commutative then the composite is commutative if and only if the distributive law is commutative. In addition we show that if the monads and the distributive law are commutative then the lifting of the monad $\mathscr{S}$ to the category of $\mathscr{T}$-algebras has a canonical structure as a closed monad (closed relative to the canonical closed category structure on the $\mathscr{T}$-algebras).

2. Commutative distributive laws

We assume throughout that $\mathscr{V}$ is a symmetric monoidal closed category with equalizers (although some of the results below do not need the equalizer hypothesis). We also assume that the reader is familiar with $\mathscr{V}$-category theory and the theory of $\mathscr{V}$-monads (see Bunge (1969), Dubuc (1970), or Kock (1970).

Recall that the natural isomorphism $p: \mathscr{V}(A \otimes B, C) \to \mathscr{V}(A, \mathscr{V}(B, C))$ gives rise to the adjunction $- \otimes B \vdash \mathscr{V}(B, -)$, $B \in \mathscr{V}$. Following Kock (1970) we denote the front adjunction by $f: 1 \to \mathscr{V}(B, - \otimes B)$ and the back adjunction by $\mathrm{ev}: \mathscr{V}(B, -) \otimes B \to 1$.

Let T be a $\mathscr{V}$-functor on $\mathscr{V}$. Kock (1970) defines the natural transformation t'' of bifunctors $t''_{A,B}: A \otimes TB \to T(A \otimes B)$ by $t''_{A,B} = ev_{TB,T(A\otimes B)} \cdot ((f_{A,B} \cdot T) \otimes 1_{TB})$. He then defines $t'_{A,B}: TA \otimes B \to T(A \otimes B)$ by $t'_{AB} = T(c) \cdot t''_{B,A} \cdot c$ where c is the symmetry. If T is the $\mathscr{V}$-functor part of a $\mathscr{V}$-monad $\mathscr{T} = (T, \eta, \mu)$, he then shows that if we put $\psi_{A,B}$ equal to the composite $\mu_{A\otimes B} \cdot T(t''_{A,B}) \cdot t'_{A,TB}$ and $\psi^\circ = \eta_I$ then (T, ψ, ψ°) becomes a monoidal functor ((Kock 1970) Theorem 2.1, page 6). Also if $\tilde{\psi}_{A,B} = \mu_{A\otimes B} \cdot T(t'_{A,B}) \cdot t''_{TA,B}$ and $\tilde{\psi}^\circ = \eta_I$ then $(T, \tilde{\psi}, \tilde{\psi}^\circ)$ is a monoidal functor. If $\psi = \tilde{\psi}$ then the monad $\mathscr{T}$ is called commutative.

If $\mathscr{T} = (T, \eta, \mu)$ and $\mathscr{S} = (S, \eta', \mu')$ are two $\mathscr{V}$-monads on a $\mathscr{V}$-category $\mathscr{A}$ a $\mathscr{V}$-distributive law from $\mathscr{T}$ to $\mathscr{S}$ is a $\mathscr{V}$-natural transformation $\lambda: TS \to ST$ such that (1) $\lambda \cdot T\eta' = \eta' T$; (2) $\lambda \cdot \eta S = S\eta$; (3) $\lambda \cdot \mu S = S\mu \cdot \lambda T \cdot T\lambda$; and (4) $\lambda \cdot T\mu' = \mu' T \cdot S\lambda \cdot \lambda S$. We record here a $\mathscr{V}$-version of a result of Beck [1]. His proof generalizes easily to the $\mathscr{V}$-case.

PROPOSITION. *Let $\mathscr{S} = (S, \eta', \mu')$ and $\mathscr{T} = (T, \eta, \mu)$ be $\mathscr{V}$-monads on a $\mathscr{V}$-category $\mathscr{A}$. Then the following are equivalent.*

(1) *There exists a $\mathscr{V}$-distributive law $\lambda: TS \to ST$.*

(2) *There exists a $\mathscr{V}$-monad $\tilde{\mathscr{S}} = (\tilde{\mathscr{S}}, \tilde{\eta}, \tilde{\mu})$ in $\mathscr{A}^T$ which lifts $\mathscr{S}$ (i.e., $SU^T = U^T\tilde{S}$; $\eta' U^T = U^T\tilde{\eta}$; $\mu' U^T = U^T\tilde{\mu}$).*

(3) *There exists a $\mathscr{V}$-multiplication $m: STST \to ST$ such that (a) $(\mathscr{S}\mathscr{T})_m = (ST, \eta'\eta, m)$ is a $\mathscr{V}$-monad in $\mathscr{A}$; (b) the $\mathscr{V}$-natural transformation $S\eta: S \to ST$ and $\eta' T: T \to ST$ are $\mathscr{V}$-monad maps; and (c) the middle unitary law $m \cdot STS\eta \cdot ST\eta' = ST$ holds.*

We assume from now on that $\mathscr{S}$ and $\mathscr{T}$ are $\mathscr{V}$-monads on $\mathscr{V}$.

DEFINITION 2.1. A $\mathscr{V}$-distributive law $\lambda: TS \to ST$ is called commutative if the following diagram commutes for all A, B in $\mathscr{V}$.

$$\begin{tikzcd}
TA \otimes SB \arrow[r, "s''"] \arrow[d, "t'"'] & S(TA \otimes B) \arrow[dd, "S(t')"] \\
T(A \otimes SB) \arrow[d, "T(s'')"'] & \\
TS(A \otimes B) \arrow[r, "\lambda"] & ST(A \otimes B)
\end{tikzcd}$$

LEMMA 2.2. *$\lambda: TS \to ST$ is a commutative distributive law if and only if $\lambda \cdot T(s') \cdot t'' = S(t'') \cdot s'$.*

PROOF ($\Rightarrow$)

$$\begin{aligned}
\lambda \cdot T(s') \cdot t'' &= \lambda \cdot TS(c) \cdot T(s'') \cdot T(c) \cdot t'' \\
&= \lambda \cdot TS(c) \cdot T(s'') \cdot t' \cdot c \\
&= ST(c) \cdot \lambda \cdot T(s'') \cdot t' \cdot c \\
&= ST(c) \cdot S(t') \cdot s'' \cdot c \\
&= S(t'') \cdot S(c) \cdot s'' \cdot c \\
&= S(t'') \cdot s'.
\end{aligned}$$

The converse is clear.

Proposition 2.3. *If λ is a commutative distributive law then $\psi_{ST} = S\psi_T \cdot \psi_S$ and $\tilde{\psi}_{ST} = S\tilde{\psi}_T \cdot \tilde{\psi}_S$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
STA \otimes STB \arrow[r, "s''"] & S(STA \otimes TB) \arrow[r, "S(s')"] \arrow[d, "S(t'')"] & S^2(TA \otimes TB) \arrow[r, "\mu'"] \arrow[dd, "S^2(t'')"] & S(TA \otimes TB) \arrow[dd, "S(t'')"] \\
 & ST(STA \otimes B) \arrow[d, "ST(s')"] \arrow[r, phantom, "1."] & & \\
 & STS(TA \otimes B) \arrow[r, "s\lambda"] \arrow[d, "STS(t')"] & S^2T(TA \otimes B) \arrow[r, "\mu'T"] \arrow[d, "S^2T(t')"] & ST(TA \otimes B) \arrow[d, "ST(t')"] \\
 & STST(A \otimes B) \arrow[r, "S\lambda T"] & S^2T^2(A \otimes B) \arrow[r, "\mu'T"] & ST^2(A \otimes B) \arrow[d, "S\mu"] \\
 & & & ST(A \otimes B)
\end{tikzcd}$$

This commutes since 1. commutes by commutativity of λ and the rest commutes by the naturality of the maps involved. But the clockwise direction about the diagram is $S\psi_T \cdot \psi_S$ and the counterclockwise direction is ψ_{ST}. So $\psi_{ST} = S\psi_T \cdot \psi_S$. Similarly $\tilde{\psi}_{ST} = S\tilde{\psi}_T \cdot \tilde{\psi}_S$.

Corollary 2.4. *If $\mathscr{S}$ and $\mathscr{T}$ are commutative $\mathscr{V}$-monads and λ is a commutative distributive law then $\mathscr{S}\mathscr{T}$ is a commutative $\mathscr{V}$-monad.*

Proposition 2.5. *If $\mathscr{S}\mathscr{T}$ is a commutative monad then λ is a commutative distributive law.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
SA \otimes TB \arrow[rr, "SA \otimes \eta' TB", "1"'] \arrow[d, "t''"'] & & SA \otimes STB \arrow[rr, "S\eta A \otimes STB", "4"'] \arrow[d, "(st)''"] & & STA \otimes STB \arrow[d, "(st)''"] \\
T(SA \otimes B) \arrow[rr, "\eta' T"] \arrow[ddd, "T(s')"'] & & ST(SA \otimes B) \arrow[rr, "ST(S\eta A \otimes STB)"] \arrow[d, "ST(s')"] & & ST(STA \otimes B) \arrow[ddd, "ST(st')"] \\
 & 2 & STS(A \otimes B) \arrow[d, "S\lambda"] \arrow[ddrr] & 5 & \\
 & 3 & S^2T(A \otimes B) \arrow[d, "\mu' T"] & 6 & \\
TS(A \otimes B) \arrow[rr, "\lambda"] \arrow[uurr] & & ST(A \otimes B) & & STST(A \otimes B) \arrow[ll, "\mu^{ST}"] \\
 & \eta' TS & STS\eta & &
\end{tikzcd}$$

1 commutes by Lemma 1.1 of Kock (1970); 2 commutes by naturality of η'; 3 commutes since $\eta'ST \cdot \lambda = S\lambda \cdot \eta'TS$ and $\mu'T \cdot \eta'ST = ST$. 4 commutes by naturality of $(st)''$; 5 commutes by Lemma 1.1 of Kock (1970); 6 commutes since

$$\mu^{ST} \cdot STS\eta = \mu'T \cdot S^2\mu \cdot S\lambda T \cdot ST\eta = \mu'T \cdot S^2\mu \cdot S^2T\eta \cdot S\lambda$$

$$= \mu'T.\, S\lambda \cdot \mathrm{So}\lambda \cdot T(s') \cdot t'' = \mu^{ST} \cdot ST(st') \cdot (st)'' \cdot S\eta A \otimes STB \cdot SA \otimes \eta'TB$$

Now consider the following diagram:

$$\begin{tikzcd}
SA \otimes TB \arrow[r, "S\eta \otimes TB"] \arrow[d, "s'"'] \arrow[dr, phantom, "1"] & STA \otimes TB \arrow[r, "STA \otimes \eta'T"] \arrow[d, "(st)'"] \arrow[dr, phantom, "2"] & STA \otimes STB \arrow[d, "(st)'"] \\
S(A \otimes TB) \arrow[r, "S\eta"] \arrow[dd, "S(t'')"'] \arrow[ddr, phantom, "3"] & ST(A \otimes TB) \arrow[r, "ST(A \otimes \eta'T)"] \arrow[dd, "ST(t'')"] \arrow[ddr, phantom, "4"] & ST(A \otimes STB) \arrow[d, "ST(st'')"] \\
 & & STST(A \otimes B) \arrow[d, "S\lambda T"] \\
ST(A \otimes B) & ST^2(A \otimes B) \arrow[l, "S\mu"] & S^2T^2(A \otimes B) \arrow[l, "\mu'T^2"]
\end{tikzcd}$$

1 commutes by Lemma 1.1 of Kock (1970); 2 commutes by naturality of t'; 3 commutes since $S\mu \cdot ST(t'') \cdot S\eta = S\mu \cdot S\eta T \cdot (t'') = S(t)$. 4 commutes since

$$\mu'T^2 \cdot S\lambda T \cdot ST(st'') \cdot ST(A \otimes \eta'T) = \mu'T^2 \cdot S\lambda T \cdot ST\eta'T \cdot ST(t'')$$

$$= \mu'T^2 \cdot S\eta'T^2 \cdot ST(t'') = ST(t'').$$

So $S(t'') \cdot s' = S\mu \cdot \mu'T^2 \cdot S\lambda T \cdot ST(st'') \cdot (st)' \cdot STA \otimes \eta'TB \cdot S\eta_A \otimes TB$.

Now since $\mathscr{S}\mathscr{T}$ is commutative we have

$$\begin{aligned}
\lambda \cdot T(s') \cdot t'' &= \mu^{ST} \cdot ST(st') \cdot (st)'' \cdot S\eta A \otimes STB \cdot SA \otimes \eta'TB \\
&= \mu^{ST} \cdot ST(st'') \cdot st' \cdot S\eta A \otimes STB \cdot SA \otimes \eta'TB \\
&= S(t'') \cdot s'.
\end{aligned}$$

Hence λ is commutative.

THEOREM 2.6. *If $\mathscr{S}$ and $\mathscr{T}$ are commutative $\mathscr{V}$-monads in $\mathscr{V}$ then $\mathscr{S}\mathscr{T}$ is commutative if and only if λ is a commutative distributive law.*

3. The closed lifting

The main purpose of this section is to show that the adjunction $F \vdash U$: $\mathscr{V}^T \leftrightarrows \mathscr{V}^{ST}$ which generates $\tilde{\mathscr{S}}$ in $\mathscr{V}^T$ is closed, i.e., F and U are closed functors and ε: $FU \to 1$ and η: $1 \to UF = \tilde{S}$ are closed natural transformations if $\mathscr{S}$, $\mathscr{T}$ and λ are commutative.

We recall some notation and definitions from Kock (1971). In that paper Kock showed that the algebras for a commutative monad $\mathscr{T} = (T, \eta, \mu)$ had a canonical structure as a closed monad. He defines the map $\theta^T_{AB}: T\mathscr{V}(A, B) \to \mathscr{V}(A, T$ (called λ in Kock (1971). This map turns out to be the map corresponding to

$$T\mathscr{V}(A,B) \otimes A \xrightarrow{t'} T(\mathscr{V}(A,B) \otimes A) \xrightarrow{T(ev)} TB$$

under the adjunction $- \otimes A \vdash \mathscr{V}(A, -)$. Using this he constructs the closed structure on $\mathscr{V}^T$ as follows:

To give the internal hom functor of $\mathscr{V}^T$, let $\bar{A} = (A, a)$ and $\bar{B} = (B, b)$ be objects of $\mathscr{V}^T$. Then an object $(\mathscr{V}^T(\bar{A}, \bar{B}), \langle a, b \rangle)$ is constructed as follows. Let the following diagram be a chosen equalizer in $\mathscr{V}$

$$\begin{tikzcd}
\mathscr{V}^T(\bar{A}, \bar{B}) \arrow[r, "U^T"] & \mathscr{V}(A,B) \arrow[r, "T"] \arrow[rr, bend right, "\mathscr{V}(a,B)"'] & \mathscr{V}(TA,TB) \arrow[r, "\mathscr{V}(TA,b)"] & \mathscr{V}(TA,B).
\end{tikzcd}$$

Define the structure $\langle a, b \rangle$ by commutativity of the diagram

$$\begin{tikzcd}
T(\mathscr{V}^T(\bar{A}, \bar{B})) \arrow[r, "T(U^T)"] \arrow[dd, "\langle a,b \rangle"'] & T(\mathscr{V}(A,B)) \arrow[d, "\theta^T"] \\
 & \mathscr{V}(A,TB) \arrow[d, "\mathscr{V}(A,b)"] \\
\mathscr{V}^T(A,B) \arrow[r, "U^T"] & \mathscr{V}(A,B)
\end{tikzcd}$$

If β: $\bar{B} \to \bar{B}'$ and α: $\bar{A}' \to \bar{A}$ are morphisms in $\mathscr{V}^T$, we define $\mathscr{V}^T(\alpha, \beta)$ by commutativity of the diagram

$$\begin{tikzcd}
\mathscr{V}^T(\bar{A}, \bar{B}) \arrow[r, "U^T"] \arrow[d, "\mathscr{V}^T(\alpha,\beta)"'] & \mathscr{V}(A,B) \arrow[d, "\mathscr{V}(\alpha,\beta)"] \\
\mathscr{V}^T(\bar{A}', \bar{B}') \arrow[r, "U^T"] & \mathscr{V}(A',B')
\end{tikzcd}$$

For base object $\bar{I}$ in $\mathscr{V}^T$ take $\bar{I} = (T1, \mu_1)$. The isomorphism $\bar{\imath}_{\bar{A}}$ is constructed by commutivity of

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "\bar{i}_A"'] & \mathscr{V}(I, A) \arrow[r, "T"] & \mathscr{V}(T1, TA) \arrow[d, "\mathscr{V}(1, a)"] \\
\mathscr{V}^T(\bar{I}, \bar{A}) \arrow[rr, "U^T"] & & \mathscr{V}(TI, A)
\end{tikzcd}$$

The morphism $\bar{j}$ is constructed by commutivity of

$$\begin{tikzcd}
T1 \arrow[r, "T(j_A)"] \arrow[d, "\bar{j}_A"'] & T\mathscr{V}(A, A) \arrow[r, "\theta^T"] & \mathscr{V}(A, TA) \arrow[d, "\mathscr{V}(1, a)"] \\
\mathscr{V}^T(\bar{A}, \bar{A}) \arrow[rr, "U^T"] & & \mathscr{V}(A, A)
\end{tikzcd}$$

Finally $\bar{L}^{\bar{A}}_{BC}$ is constructed by commutivity of

$$\begin{tikzcd}
\mathscr{V}^T(\bar{B}, \bar{C}) \arrow[r, "U^T"] \arrow[d, "\bar{L}^{\bar{A}}_{BC}"'] \arrow[dr, "L'"] & \mathscr{V}(B, C) \arrow[r, "L^A_{BC}"] & \mathscr{V}(\mathscr{V}(A, B), \mathscr{V}(A, \hat{C})) \arrow[d, "\mathscr{V}(U^T, 1)"] \\
\mathscr{V}^T(\mathscr{V}^T(\bar{A}, \bar{B}), \mathscr{V}^T(\bar{A}, \bar{C})) \arrow[r, "U^T"] & \mathscr{V}(\mathscr{V}^T(\bar{A}, \bar{B}), \mathscr{V}^T(\bar{A}, \bar{C})) \arrow[r, "\mathscr{V}(1, U^T)"] & \mathscr{V}(\mathscr{V}^T(\bar{A}, \bar{B}), \mathscr{V}(A, C)
\end{tikzcd}$$

We also follow Kock in the following convention. When we want to show that two expressions are equal we shall write down a string of equations giving the result; the equality signs carry explanations: a θ on top of the equality sign means: "by naturality of θ." A 'T' means "by naturality of T," a 'd' means "by properties of distributive laws," a 'D' means "by definition," and a 'U^T' means "by the equalizing property of U^T." A number (3.2) refers to that lemma and a * means "will be explained after the equation."

Proposition 3.1. *If $\phi: \mathscr{T} \to \mathscr{S}$ is a $\mathscr{V}$-monad map between commutative monads then $\mathscr{V}^\phi: \mathscr{V}^S \to \mathscr{V}^T$ is a closed functor.*

Proof. Define $\hat{V}^\phi: V^\phi(\mathscr{V}^S(\bar{A}, \bar{B})) \to \mathscr{V}^T(V^\phi(\bar{A}), V^\phi(\bar{B}))$ by $U^T \cdot \hat{V}^\phi = U^S$. This is well-defined since

$$\mathscr{V}(a \cdot \phi, B) \cdot U^S = \mathscr{V}(\phi, B) \cdot \mathscr{V}(a, B) \cdot U^S \overset{U^S}{=} \mathscr{V}(\phi, B) \cdot \mathscr{V}(SA, b) \cdot S \cdot U^S$$
$$\overset{V}{=} \mathscr{V}(TA, b) \cdot \mathscr{V}(\phi, SB) \cdot S \cdot U^S \overset{\phi}{=} \mathscr{V}(TA, b) \cdot \mathscr{V}(TA, \phi) \cdot T \cdot U^S.$$

That $\hat{V}^\phi$ is an T-algebra map follows from:

$$U^T \cdot \hat{V}^\phi \cdot \langle a, b\rangle^S \cdot \phi \overset{D}{=} U^S \cdot \langle a, b\rangle^S \cdot \phi \overset{D}{=} \mathscr{V}(A, b) \cdot \theta^S \cdot \phi \cdot TU^S$$
$$\overset{D}{=} \mathscr{V}(A, b) \cdot \theta^S \cdot \phi \cdot TU^T \cdot T\hat{V}^\phi \overset{*}{=} \mathscr{V}(A, b) \cdot \mathscr{V}(A, \phi) \cdot \theta^T \cdot TU^T \cdot T\hat{V}^\phi$$
$$\overset{D}{=} U^T \cdot \langle V^\phi \bar{A}, V^\phi \bar{B})^T \cdot T\hat{V}^\phi.$$

Here * follows from 1.4 of Kock (1971).

Now define $V_0^\phi\colon (T1,\mu_1) \to V^\phi(S1,\mu_1)$ by $V_0^\phi = \phi_1$. This is clearly a T-algebra map.

Axiom CF1 says that the diagram

$$\begin{tikzcd}
V^\phi(S1,\mu_1) \arrow[r, "V^\phi(\bar{j})"] & V^\phi(\mathscr{V}^S(\bar{A},\bar{A})) \arrow[d, "\hat{V}^\phi"] \\
(T1,\mu_1) \arrow[u, "V_0^\phi"] \arrow[r, "\bar{j}"] & \mathscr{V}^T(V^\phi(\bar{A}), V^\phi(\bar{A}))
\end{tikzcd}$$

should commute. Now $U^T \cdot \hat{V}^\phi \cdot V^\phi(\bar{j}) \cdot V_0^\phi \overset{D}{=} U^S \cdot \bar{j} \cdot \phi_1 = \mathscr{V}(1,a) \cdot \theta^S \cdot Sj \cdot \phi_1 \overset{\phi}{=} \mathscr{V}(1,a) \cdot \theta^S \cdot \phi \cdot T(j) \overset{D}{=} \mathscr{V}(1,a) \cdot \mathscr{V}(A,\phi) \cdot \theta^T \cdot T(j) \overset{*}{=} U^T \cdot \bar{j}$. Here * follows from 1.4 of Kock (1971). So we get CF1.

Axiom CF2 says that the diagram

$$\begin{tikzcd}
V^\phi(\mathscr{V}^S((S_1,\mu),\bar{A})) \arrow[r, "\hat{V}^\phi"] & \mathscr{V}^T(V^\phi(S1,\mu,)\, V^\phi\bar{A}) \arrow[d, "{V^T(V_0^\phi, V\phi\bar{A})}"] \\
V\phi(A,a) \arrow[u, "V^\phi(\bar{\imath})"'] \arrow[r, "\bar{\imath}"] & \mathscr{V}^T((T1,\mu), V\phi(A,a))
\end{tikzcd}$$

should commute. Now $U^T \cdot \mathscr{V}^T(V_0^\phi, V^\phi(\bar{A})) \cdot \hat{V}^\phi \cdot V^\phi(\bar{\imath}) \overset{D}{=} \mathscr{V}(\phi 1, A) \cdot U^S \cdot \bar{\imath} \overset{D}{=} \mathscr{V}(\phi 1, A) \cdot \mathscr{V}(S1, a) \cdot S \cdot i \overset{V}{=} \mathscr{V}(T1, a) \cdot V(\phi 1, SA) \cdot S \cdot i \overset{\phi}{=} \mathscr{V}(T1,a) \cdot$ $= \mathscr{V}(\phi 1, A) \cdot \mathscr{V}(S1,a) \cdot S \cdot i = \mathscr{V}(T1,a) \cdot V(\phi 1, SA) \cdot S \cdot i \overset{\phi}{=} \mathscr{V}(T1,a) \cdot \mathscr{V}(T1,\phi A) \cdot T \cdot i \overset{D}{=} U^T \cdot \bar{\imath}$. So we get CF2.

Axiom CF3 says that the diagram

$$\begin{tikzcd}
V\phi(\mathscr{V}^S(\bar{B},\bar{C})) \arrow[r, "V\phi(\tilde{L})"] \arrow[d, "\hat{V}\phi"'] & V\phi(\mathscr{V}^S(\mathscr{V}^S(\bar{A},\bar{B}),\mathscr{V}^S(\bar{A},\bar{C})) \arrow[r, "\hat{V}\phi"] & \mathscr{V}^T(V^\phi\mathscr{V}^S(\bar{A},\bar{B}), V^\phi\mathscr{V}^S \arrow[d, "\mathscr{V}^T("] \\
\mathscr{V}^T(V^\phi\bar{B}, V^\phi(\bar{C})) \arrow[r, "\bar{L}"] & \mathscr{V}^T(\mathscr{V}^T(V^\phi\bar{A}, V^\phi\bar{B}), \mathscr{V}^T(V^\phi\bar{A}, V^\phi\bar{B})) \arrow[r, "{\mathscr{V}^T(\hat{V}^\phi, 1)}"] & \mathscr{V}^T(V^\phi\mathscr{V}^S(\bar{A},\bar{B}), \mathscr{V}^T(V^\phi,
\end{tikzcd}$$

should commute. Now $\mathscr{V}(1,U^T) \cdot U^T \cdot \mathscr{V}^T(\hat{V}^\phi, 1) \cdot \bar{L} \cdot \hat{V}^\phi \overset{D}{=} \mathscr{V}(\hat{V}^\phi, V(A,C)) \cdot \mathscr{V}(1,U^T) \cdot U^T \cdot \bar{L} \cdot \hat{V}^\phi \overset{D}{=} \mathscr{V}(\hat{V}^\phi, V(A,C)) \cdot \mathscr{V}(U^T, V(A,C)) \cdot L^A \cdot U^T \cdot \hat{V}^\phi \overset{D}{=} \mathscr{V}(U^S, \mathscr{V}(A,C)) \cdot L^A \cdot U^S \overset{D}{=} \mathscr{V}(1,U^S) \cdot U^S \cdot V^\phi(\tilde{L}) \overset{D}{=} \mathscr{V}(1,U^S) \cdot U^T \cdot \hat{V}^\phi \cdot V^\phi(\tilde{L}) \overset{D}{=} U^T \cdot \mathscr{V}^T(1, \hat{V}^\phi) \cdot \hat{V}^\phi \cdot V^\phi(\tilde{L})$. So we get CF3.

Hence V^φ is a closed functor.

LEMMA 3.2. *If λ is commutative then (a) $\mathscr{V}(TA,\lambda)\cdot T\cdot\theta^S_{A,B}=\theta^S_{TA,TB}\cdot S(T)$ and (b) $\mathscr{V}(SA,\lambda)\cdot\theta^T_{SA,SB}\cdot T(S)=S\cdot\theta^T_{A,B}$.*

PROOF. (a) Under the adjunction between $-\otimes TA$ and $\mathscr{V}(TA,-)$ the left hand side of the equation correspond to the counterlockwise direction around the diagram below:

$$\begin{tikzcd}
S\mathscr{V}(A,B)\otimes TA \arrow[rrr, "s"] \arrow[dr, "t''"] \arrow[dd, "\theta\otimes 1"'] & & & S(\mathscr{V}(A,B)\otimes TA) \arrow[dd, "S(t'')"] \\
 & T(S\mathscr{V}(A,B)\otimes A) \arrow[d, "T(\theta\otimes 1)"'] \arrow[dr, "T(s')"] & & \\
\mathscr{V}(A,SB)\otimes TA \arrow[r, "t''"] \arrow[d, "T\otimes 1"'] & T(\mathscr{V}(A,SB)\otimes A) \arrow[d, "T(ev)"'] & TS(\mathscr{V}(A,B)\otimes A) \arrow[r, "\lambda"] \arrow[dl, "TS(ev)"] & ST(\mathscr{V}(A,B)\otimes A) \arrow[d, "ST(ev)"] \\
\mathscr{V}(TA,TSB)\otimes TA \arrow[r, "ev"] & TSB \arrow[rr, "\lambda"] & & STB
\arrow[from=1-1, to=3-2, phantom, "1"]
\arrow[from=3-1, to=4-2, phantom, "2"]
\arrow[from=3-2, to=3-3, phantom, "3"]
\arrow[from=1-4, to=3-3, phantom, "4"]
\arrow[from=3-3, to=4-4, phantom, "5"]
\end{tikzcd}$$

1 and 5 commute by naturality; 2 commutes since both legs correspond to T under adjunction (see Kock (1970); 3 commutes by Lemma 1.2 of Kock (1971); and 4 commutes by commutativity of λ.

$$\begin{aligned}
\text{Now } ST(ev)\cdot S(t')\cdot s' &= S(ev)\cdot S(T\otimes 1)\cdot s' && \text{(see 2 above)}\\
&= S(ev)\cdot s'\cdot S(T_{AB})\otimes TA && \text{(naturality)}\\
&= ev\cdot\theta\otimes 1\cdot S(T_{AB})\otimes TA && \text{(Lemma 1.2 of Kock (1971)).}
\end{aligned}$$

Hence $\lambda\cdot ev\cdot T_{ASB}\otimes 1\cdot\theta\otimes 1=ev\cdot\theta\otimes 1\cdot S(T_{AB})\otimes 1$. So by the adjunction we get the result.

(b) By the adjunction of $-\otimes SA$ with $\mathscr{V}(SA,-)$ we get that the left hand side of the equation corresponds to $ev\cdot\mathscr{V}(SA,\lambda_B)\otimes 1\cdot\theta^T\otimes 1\cdot T(S_{A,B})\otimes 1$ $=\lambda_B\cdot T(ev)\cdot T(S_{A,B}\otimes 1)\cdot t'$. Now consider the following diagram.

$$\begin{tikzcd}
T\mathscr{V}(A,B)\otimes SA \arrow[rr, "t'"] \arrow[dr, "s''"] \arrow[ddd, "\theta^T\otimes 1"'] & & T(\mathscr{V}(A,B)\otimes SA) \arrow[r, "T(S\otimes 1)"] \arrow[d, "T(s'')"'] & T(\mathscr{V}(SA,SB)\otimes SA) \arrow[d, "T(ev)"] \\
 & S(T\mathscr{V}(A,B)\otimes A) \arrow[d, "S(\theta^T\otimes 1)"'] \arrow[dr, "s(t')"] & TS(\mathscr{V}(A,B)\otimes A) \arrow[r, "TS(ev)"] \arrow[d, "\lambda"] & TSB \arrow[d, "\lambda"] \\
 & S(\mathscr{V}(A,TB)\otimes A) \arrow[rr, bend right=15, "S(ev)"'] & ST(\mathscr{V}(A,B)\otimes A) \arrow[r, "ST(ev)"] & STB \\
\mathscr{V}(A,TB)\otimes SA) \arrow[rr, "S\otimes 1"] & & \mathscr{V}(SA,STB)\otimes SA \arrow[ur, "ev"'] &
\arrow[from=4-1, to=3-2, phantom, "s''"]
\arrow[from=1-1, to=3-2, phantom, "1"]
\arrow[from=1-3, to=2-2, phantom, "2"]
\arrow[from=1-3, to=2-4, phantom, "3"]
\arrow[from=2-3, to=3-4, phantom, "4"]
\arrow[from=3-2, to=3-3, phantom, "5"]
\arrow[from=4-1, to=3-3, phantom, "6"]
\end{tikzcd}$$

1 and 4 commute by naturality; 2 commutes since λ is commutative; 6 commutes since both legs corresponds to S under the adjunction; 3 is just T applied to 6; finally 5 commutes since it is S applied to a commutative diagram (the diagram commutes by Lemma 1.2 of Kock (1971). So we get the desired result.

THEOREM 3.3. *Suppose* $\mathscr{S}$, $\mathscr{T}$ *and* λ *are all commutative, then* $F \vdash U$: $\mathscr{V}^T \to \mathscr{V}^{ST}$ *is closed.*

PROOF. Let $\bar{A} = (A, a)$ $\bar{B} = (B, b)$ be $\mathscr{T}$-algebras. Then $F(A, a) = (SA, Sa \cdot \mu'_{TA} \cdot S\lambda)$. We must show that there exists a natural transformation $\hat{F}{:}F\mathscr{V}^T(\bar{A}, \bar{B}) \to \mathscr{V}^{ST}(F\bar{A}, F\bar{B})$ and a map $F_0{:}\,(ST1, \mu_1^{ST}) \to F(T1, \mu_1)$ which satisfy CF1, CF2, CF3 of Eilenberg and Kelly (1966). To define F it suffices to find a map $\hat{F}{:}\, S\mathscr{V}^T(\bar{A}, \bar{B}) \to \mathscr{V}^{ST}(F\bar{A}, F\bar{B})$ which is an $\mathscr{S}\mathscr{T}$-algebra map. But to define a map into $\mathscr{V}^{ST}(F\bar{A}, F\bar{B})$ it suffices to define a map into $\mathscr{V}(SA, SB)$ which equalize the following diagram.

$$\begin{tikzcd}
\mathscr{V}(SA, SB) \arrow[r, "ST"] \arrow[rr, bend right, "\mathscr{V}(Sa \cdot \mu'_{TA} \cdot S\lambda, -)"'] & \mathscr{V}(STSA, STSB) \arrow[r, "\mathscr{V}(-, Sb \cdot \mu'_{TB} \cdot S\lambda)"] & \mathscr{V}(STSA, SB)
\end{tikzcd}$$

Now define $\hat{F} = \mathscr{V}(-, \mu') \cdot S \cdot \theta^S \cdot SU^T$. To show that $\hat{F}$ equalizes the above diagram consider diagram 1. Each subdiagram commutes either by Lemma 3.2, $\mathscr{V}$-naturality, or the definition of the maps involved. Using a similar diagram one can show that $V(Sa \cdot \mu'_{TA} \cdot S\lambda, -) \cdot \hat{F} = \mathscr{V}(-, \mu') \cdot S \cdot \mathscr{V}(\lambda, -) \cdot \mathscr{V}(-, \mu') \cdot S \cdot \theta \cdot S \cdot \mathscr{V}(a, B)$. Hence there exists a unique $\hat{F}{:}\; S\mathscr{V}^T(\bar{A}, \bar{B}) \to \mathscr{V}^{ST}(F\bar{A}, F\bar{B})$ such that $U^{ST} \cdot \hat{F} = \mathscr{V}(-, \mu') \cdot S \cdot \theta^S \cdot SU^T$.

To show that $\hat{F}$ is an $\mathscr{S}\mathscr{T}$-algebra map it suffices to show that $U^{ST} \cdot \hat{F} \cdot S(\langle a, b\rangle^T) \cdot \mu_T \cdot S\lambda = U^{ST} \cdot \langle F(a), F(b)\rangle^{ST} \cdot ST(\hat{F})$. Consider diagram 2.

1 and 2 commute by Lemmas 1.4 and 1.5 of Kock (1971); 3 commutes by Lemma 3.2 above; 4 commutes since $\mathscr{S}$ is commutative monad (Kock (1971)); 5 commutes since λ is a distributive law; 6 and 7 commute by properties of monads; everything else commutes by naturality. So diagram 2 commutes. Hence $\hat{F}$ is a ST-algebra map.

It is clear that $\hat{F}$ is natural. Now define $F_0 = id$. To verify CF1, CF2, CF3 we proceed as follows: For $CF1$ we need to show that $\bar{\bar{j}} = \hat{F} \cdot F(\bar{j})$. To do this it suffices to show that $U^{ST} \cdot \bar{\bar{j}} = U^{ST} \cdot \hat{F} \cdot F(\bar{j})$. But

$$U^{ST}\bar{\bar{j}} \overset{D}{=} \mathscr{V}(-, Sa \cdot \mu'_{TA} \cdot S\lambda) \cdot \theta^{ST} \cdot ST(j) \text{ and } U^{ST} \cdot \hat{F} \cdot F(\bar{j})$$

$$\overset{D}{=} \mathscr{V}(-, \mu') \cdot S \cdot \theta^S \cdot SU^T \cdot F(\bar{j}) \overset{D}{=} \mathscr{V}(-, \mu') \cdot S \cdot \theta^S \cdot S\mathscr{V}(-, a) \cdot S\theta^T \cdot ST(j)$$

$$\overset{*}{=} \mathscr{V}(SA, \mu') \cdot \theta^S \cdot S(S) \cdot S\mathscr{V}(A, a) \cdot S\theta^T \cdot STj \overset{S}{=} \mathscr{V}(SA, \mu') \cdot \theta^S$$

$$\cdot S\mathscr{V}(SA, Sa) \cdot S(S) \cdot S\theta^T \cdot STj \overset{3.2}{=} \mathscr{V}(SA, \mu') \cdot \theta^S \cdot S\mathscr{V}(SA, Sa)$$

$$\cdot S\hat{\mathscr{V}}(SA,\lambda)\cdot S\theta^{T}\cdot ST(S)\cdot STj \overset{**}{=} \hat{\mathscr{V}}(SA,Sa)\cdot\hat{\mathscr{V}}(SA,\mu'_{TA})$$

$$\cdot\hat{\mathscr{V}}(SA,S\lambda)\cdot\theta^{S}\cdot S\theta^{T}\cdot STj \overset{D}{=} U^{ST}\bar{\bar{j}}.$$

Here * follows from the commutativity of $\mathscr{S}$ and ** follows easily from Lemma 1.4 of Kock (1971) and naturality.

CF2 says that the diagram

$$\begin{tikzcd}
F(\hat{\mathscr{V}}^{T}(\bar{I},\bar{A})) \arrow[r, "\hat{F}"] & \hat{\mathscr{V}}^{ST}(F\bar{I},F\bar{A}) \\
F(A,a) \arrow[u, "F\bar{i}"] \arrow[ur, "\bar{\bar{i}}"'] &
\end{tikzcd}$$

should commute. We have

$$U^{ST}\cdot\hat{F}\cdot F\bar{i} \overset{D}{=} \hat{\mathscr{V}}(ST1,\mu'_{A})\cdot S\cdot\theta^{S}\cdot S\hat{\mathscr{V}}(T1,a)\cdot S(T)\cdot Si_{A} \overset{\theta}{=} \hat{\mathscr{V}}(ST1,\mu'_{A})$$

$$\cdot S\cdot\hat{\mathscr{V}}(T1,Sa)\cdot\theta^{S}\cdot S(T)\cdot Si_{A} \overset{S,\mathscr{V}}{=} \hat{\mathscr{V}}(ST1,Sa)\cdot\hat{\mathscr{V}}(ST1,\mu'_{TA})$$

$$\cdot S\cdot\theta^{S}\cdot S(T)\cdot S(i) \overset{3.2}{=} \hat{\mathscr{V}}(ST1,Sa)\cdot\hat{\mathscr{V}}(ST1,\mu'_{TA})\cdot S$$

$$\cdot\hat{\mathscr{V}}(T1,\lambda_{A})\cdot T\cdot\theta^{S}\cdot Si = \hat{\mathscr{V}}(ST1,\ Sa\cdot\mu'_{TA})$$

$$\cdot\hat{\mathscr{V}}(T1,S\lambda_{A})\cdot S\cdot T\cdot i_{SA} \overset{D}{=} U^{ST}\cdot\bar{\bar{i}}.$$

So we have *CF*2.

CF3 says that the diagram:

$$\begin{tikzcd}
(\bar{B},\bar{C}) \arrow[r, "F(\bar{L})"] \arrow[d] & F\hat{\mathscr{V}}^{T}(\hat{\mathscr{V}}^{T}(\bar{A},\bar{B}),\hat{\mathscr{V}}^{T}(\bar{A},\bar{C})) \arrow[r, "\hat{F}"] & \hat{\mathscr{V}}^{ST}(F\hat{\mathscr{V}}^{T}(\bar{A},\bar{B}),F\hat{\mathscr{V}}^{T}(\bar{A},\bar{C})) \arrow[d, "\hat{\mathscr{V}}^{ST}(1,\hat{F})"] \\
\bar{B},F\bar{C}) \arrow[r, "\bar{\bar{L}}"] & \hat{\mathscr{V}}^{ST}(\hat{\mathscr{V}}^{ST}(F\bar{A},F\bar{B}),\hat{\mathscr{V}}^{ST}(F\bar{A},F\bar{C})) \arrow[r, "\hat{\mathscr{V}}^{ST}(\hat{F},1)"] & \hat{\mathscr{V}}^{ST}(F\hat{\mathscr{V}}^{T}(\bar{A},\bar{B}),\hat{\mathscr{V}}^{ST}(F\bar{A},F\bar{C}))
\end{tikzcd}$$

commutes. We have

$$\hat{\mathscr{V}}(-,U^{ST})\cdot U^{ST}\cdot\hat{\mathscr{V}}^{ST}(\hat{F},1)\cdot\bar{\bar{L}}\cdot\hat{F} \overset{D}{=} \hat{\mathscr{V}}(\hat{F},1)\cdot\hat{\mathscr{V}}(1,U^{ST})\cdot L'\cdot\hat{F}$$

$$\overset{D}{=} \hat{\mathscr{V}}(\hat{F},1)\cdot\hat{\mathscr{V}}(U^{ST},1)\cdot L^{SA}\cdot U^{ST}\cdot\hat{F} \overset{D,\mathscr{V}}{=} \hat{\mathscr{V}}(SU^{ST},1)\cdot\hat{\mathscr{V}}(\hat{S},1)\cdot L^{SA}\cdot\hat{S}\cdot SU^{ST}$$

$$\overset{*}{=} \hat{\mathscr{V}}(S(U^{ST}),1)\cdot\hat{\mathscr{V}}(1,\hat{S})\cdot\hat{S}\cdot SL\cdot SU^{ST} \overset{D,\mathscr{V}}{=} \hat{\mathscr{V}}(1,\hat{S})\cdot\hat{\mathscr{V}}(1,\mu)\cdot S\cdot\theta^{S}$$

$$\cdot S\hat{\mathscr{V}}(-,U^{ST})\cdot S(U^{ST})\cdot S(\bar{\bar{L}}) \overset{D}{=} \hat{\mathscr{V}}(1,U^{ST})\cdot U^{ST}\cdot\hat{\mathscr{V}}^{ST}(1,\hat{F})\cdot\hat{F}\cdot F(\bar{L})$$

where equation * follows from Lemma 3.2 of Kock (1971) and $S = V(1,\mu')\cdot S\cdot\theta^{S}$.

That U is a closed functor follows from Proposition 3.1 since it arises from he monad map $\eta'T: T\to ST$.

To show $\tilde{\eta}\colon 1 \to UF$ is a closed natural transformation we must verify CN1 and CN2 of Eilenberg and Kelly (1966). CN1 says that $\tilde{\eta}_1 = (UF)^\circ$. But $(UF)^\circ = U(F^\circ) \cdot U^\circ \overset{D}{=} U(id) \cdot \eta'_{T1} = \tilde{\eta}_1$.

CN2 says that the following diagram should commute

$$\begin{tikzcd}
\mathscr{V}^T(\bar{A}, \bar{B}) \arrow[rr, "\mathscr{V}^T(1, \tilde{\eta})"] \arrow[d, "\tilde{\eta}"'] & & \mathscr{V}^T(\bar{A}, UF\bar{B}) \\
UF\mathscr{V}^T(\bar{A}, \bar{B}) \arrow[r, "U(\hat{F})"] & U\mathscr{V}^{ST}(F\bar{A}, F\bar{B}) \arrow[r, "\hat{U}"] & \mathscr{V}^T(UF\bar{A}, UF\bar{B}) \arrow[u, "\mathscr{V}^T(\tilde{\eta}, 1)"']
\end{tikzcd}$$

But this follows easily from the fact that $\eta'\colon 1 \to S$ is a closed natural transformation (see Kock 1971).

To show that $\varepsilon\colon FU \to 1$ is a closed natural transformation we first of all verify CN1 which says $\varepsilon_1 \cdot (FU)^\circ = (ST1, \mu_1)$. But $\varepsilon_1 \cdot (FU)^\circ \overset{D}{=} \varepsilon_1 \cdot F(U^\circ) \cdot F^\circ \overset{D}{=} \mu^{ST} \cdot S\eta ST \cdot \eta' ST = \mu' T \cdot S^2\mu \cdot S\lambda T \cdot S\eta ST \cdot \eta' ST \overset{\eta}{=} \mu' T \cdot S^2\mu \cdot S^2\eta T \cdot \eta' ST \overset{T}{=} \mu' T \cdot \eta' ST \overset{T}{=} (ST1, \mu_1)$. So CN1 holds.

CN2 says that

$$\begin{tikzcd}
FU\mathscr{V}^{ST}(\bar{A}, \bar{B}) \arrow[r, "F(\hat{U})"] \arrow[d, "\varepsilon"'] & F\mathscr{V}^T(U\bar{A}, U\bar{B}) \arrow[r, "\hat{F}"] & \mathscr{V}^{ST}(FU\bar{A}, FU\bar{B}) \arrow[d, "\mathscr{V}^{ST}(1, \varepsilon)"] \\
\mathscr{V}^{ST}(\bar{A}, \bar{B}) \arrow[rr, "\mathscr{V}^{ST}(\varepsilon, 1)"'] & & \mathscr{V}^{ST}(FU\bar{A}, \bar{B})
\end{tikzcd}$$

should commute. Now

$$\begin{aligned}
& U^{ST} \cdot \mathscr{V}^{ST}(1, \varepsilon)\hat{F} \cdot F(\hat{U}) \overset{D}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(SA, S\eta_B) \cdot \mathscr{V}(SA, \mu') \cdot S \cdot \theta^S \cdot S(U^{ST}) \\
& \quad \overset{\mu'}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(SA, \mu'_{TB}) \cdot \mathscr{V}(SA, S^2\eta) \cdot S \cdot \theta^S \cdot S(U^{ST}) \\
& \quad \overset{S}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(SA, \mu'_{TB}) \cdot S \cdot \mathscr{V}(A, S\eta_B) \cdot \theta^S \cdot S(U^{ST}) \\
& \quad \overset{*}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(SA, \mu'_{TB}) \cdot \mathscr{V}(SA, S^2\mu_B) \cdot \mathscr{V}(SA, S\lambda TB) \cdot \mathscr{V}(SA, S\eta STB) \\
& \quad \cdot S \cdot \mathscr{V}(A, S\eta_B) \cdot \theta^S \cdot S(U^{ST}) \overset{S\eta}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(SA, \mu^{ST}) \cdot \mathscr{V}(S\eta_A, STSTB) \\
& \quad \cdot ST \cdot \mathscr{V}(A, S\eta_B) \cdot \theta^S \cdot S(U^{ST}) \overset{\mathscr{V}}{=} \mathscr{V}(SA, b) \cdot \mathscr{V}(S\eta_A, STB) \cdot \mathscr{V}(STA, \mu^{ST}) \\
& \quad \cdot ST \cdot \mathscr{V}(A, S\eta_B) \cdot \theta^S \cdot S(U^{ST}) \overset{\mathscr{V}, \theta}{=} \mathscr{V}(S\eta_A, B) \cdot V(STA, b) \cdot \mathscr{V}(STA, \mu^{ST}) \\
& \quad \cdot ST \cdot \theta^{ST} \cdot S\eta_{\mathscr{V}(A,B)} \cdot S(U^{ST}) \overset{**}{=} \mathscr{V}(S\eta_A, B) \cdot \mathscr{V}(STA, b) \cdot \mathscr{V}(STA, ST(b)) \\
& \quad \cdot ST \cdot \theta^{ST} \cdot S\eta_{\mathscr{V}(A,B)} \cdot S(U^{ST}) \overset{ST}{=} \mathscr{V}(S\eta_A, B) \cdot \mathscr{V}(STA, b) \cdot ST \\
& \quad \cdot \mathscr{V}(A, b) \cdot \theta^{ST} \cdot S\eta_{\mathscr{V}(A,B)} \cdot S(U^{ST}) \overset{D}{=} U^{ST} \cdot \mathscr{V}^{ST}(\varepsilon, 1) \cdot \varepsilon.
\end{aligned}$$

Here * follows from the fact that $S^2\mu_B \cdot S\lambda TB \cdot S\eta STB = S^2\mu_B \cdot S^2\eta TB = 1$, and ** follows from the fact that b is an $\mathscr{S}\mathscr{T}$-algebra structure and so $b \cdot \mu^{ST} = b \cdot ST(b)$. So CN2 holds. Therefore ε is a closed natural transformation.

Corollary 3.4. *$\tilde{S}$, the lifted monad in $\mathscr{V}^T$, has a canonical closed monad structure if $\mathscr{S}$, $\mathscr{T}$ and λ are commutative.*

4. Two examples

1. A monoid in $\mathscr{V}$ is an object M of $\mathscr{V}$ together with maps $e: I \to M$ and $m: M \otimes M \to M$ satisfying (1) $m \cdot M \otimes m = m \cdot m \otimes M$ (2) $m \cdot M \otimes e = m \cdot e \otimes M = M$. If M is a monoid in $\mathscr{V}$ then we can form a $\mathscr{V}$-monad $\mathscr{M} = (-\otimes M, \eta, \mu)$ where η and μ are obvious. If $\mathscr{S} = (S, \eta', \mu')$ is any $\mathscr{V}$-monad in $\mathscr{V}$ then the map $s': SA \otimes M \to S(A \otimes M)$ form the components of a $\mathscr{V}$-distributive law. By Proposition 1.5 of Kock (1970) this distributive law is always commutative.

2. A comonoid in $\mathscr{V}$ is an object C of $\mathscr{V}$ together with the maps $\varepsilon: C \to I$ and $\delta: C \to C \otimes C$ such that $\delta \otimes C \cdot \delta = C \otimes \delta \cdot \delta$ and $\varepsilon \otimes C \cdot \delta = C \otimes \varepsilon \cdot \delta = C$. It is clear that the $\mathscr{V}$-functor $\mathscr{V}(C, -): \mathscr{V} \to \mathscr{V}$ can be given the structure of a $\mathscr{V}$-monad in an obvious way using ε and δ. Now if $T = (T, \eta, \mu)$ is any $\mathscr{V}$-monad in $\mathscr{V}$, then the map $\theta^T_{C,-}: T\mathscr{V}(C, \ \ ) \to \mathscr{V}(C, T(\ \ ))$ can be easily shown to be a $\mathscr{V}$-distributive law using Lemmas 1.4, 1.6 and 1.7 of Kock (1971). We claim that this $\mathscr{V}$-distributive law is always commutative.

Lemma 4.1 *$1 \otimes ev \cdot a = ev \cdot s' \otimes C$: $(A \otimes \mathscr{V}(C,B)) \otimes C \to A \otimes B$ where $s': A \otimes \mathscr{V}(C,B) \to \mathscr{V}(C, A \otimes B)$ is the canonical map.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
(C,B))\otimes C \arrow[rr, "(f\otimes 1)\otimes 1"] \arrow[dr, phantom, "1"] & & (\mathscr{V}(B,A\otimes B)\otimes\mathscr{V}(C,B))\otimes C \arrow[dl, "a"'] \arrow[dd, "4"', "M\otimes 1"] \arrow[dr, "(L\otimes 1)\otimes 1"] & \\
 & V(B,A\otimes B)\otimes(\mathscr{V}(C,B)\otimes C) \arrow[d, "I\otimes ev"', "3"] & & {[\mathscr{V}(\mathscr{V}(C,B),\mathscr{V}(C,A\otimes B))\otimes\mathscr{V}(C,B)]\otimes C} \arrow[dl, "ev\otimes 1"] \\
 & \mathscr{V}(B,A\otimes B)\otimes B \arrow[dr, "ev"] & \mathscr{V}(C,A\otimes B)\otimes C \arrow[d, "ev"] & \\
(C,B)\otimes C) \arrow[uur, "f\otimes(1\otimes 1)"] \arrow[rr, "1\otimes ev"', "2"] & & A\otimes B &
\end{tikzcd}$$

1 commutes by naturality; 2 commutes since $f \otimes 1 \cdot 1 \otimes ev = 1 \otimes ev \cdot f \otimes (1 \otimes 1)$ and $ev_{B,A \otimes B} \cdot f_A \otimes B = A \otimes B$; 3 commutes by Lemma 1.3 of Kock (1970) page 3; 4 commutes by definition of M (M is the composition of $\mathscr{V}$).

Proposition 4.2. *θ is a commutative distributive law.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
{[TA \otimes \mathscr{V}(C,B)] \otimes C} \arrow[r, "u"] \arrow[dd, "t' \otimes 1"'] & TA \otimes (\mathscr{V}(C,B) \otimes C) \arrow[rr, "TA \otimes ev"] \arrow[dr, "t'"] & & TA \otimes B \arrow[dddd, "t'"] \\
 & 1 & T(A \otimes (V(C,B) \otimes C) \arrow[dl, "T(a)"] \arrow[dddr, "T(1 \otimes ev)"'] & \\
{[T(A \otimes \mathscr{V}(C,B))] \otimes C} \arrow[r, "t'"] \arrow[dd, "T(s'') \otimes 1"'] & T((A \otimes \mathscr{V}(C,B)) \otimes C) \arrow[d, "T(s'' \otimes 1)"'] & & \\
4 & T(\mathscr{V}(C,A \otimes B) \otimes C) \arrow[drr, "T(ev)"] & 3 & \\
T\mathscr{V}(C,A \otimes B) \otimes C \arrow[ur, "t'"] \arrow[r, "\theta^1 \otimes 1"'] & \mathscr{V}(C,T(A \otimes B)) \otimes C \arrow[rr, "ev"'] & & T(A \otimes B)
\arrow[from=1-4, to=2-3, phantom, "2"]
\end{tikzcd}$$

1 commutes by Proposition 1.5 of Kock (1970); 2 and 4 commute by naturality of t'; 3 commutes by the lemma; and 5 commutes by Lemma 1.2 of Kock (1971).) Note that the counterclock way around the diagram corresponds to $\theta \cdot T(s\ ) \cdot t'$ under the usual adjunction.

Now consider the following diagram.

$$\begin{tikzcd}
{[TA \otimes \mathscr{V}(C,B)] \otimes C} \arrow[r, "s'' \otimes C"] \arrow[d, "a"'] \arrow[dr, phantom, "1"] & \mathscr{V}(C,TA \otimes B) \otimes C \arrow[r, "\mathscr{V}(C,t') \otimes 1"] \arrow[d, "ev"] \arrow[dr, phantom, "2"] & \mathscr{V}(C,T(A \otimes B) \arrow[d, "ev"] \\
TA \otimes (\mathscr{V}(C,B) \otimes C) \arrow[r, "1 \otimes ev"'] & TA \otimes B \arrow[r, "t'"'] & T(A \otimes B)
\end{tikzcd}$$

1 commutes by the lemma and 2 commutes by naturality of ev. Since the clockwise direction corresponds to $\mathscr{V}(C,t') \cdot s''$ we get that θ is commutative.

$$\begin{tikzcd}
S\mathscr{V}(A,B) \arrow[r, "S(T)"] \arrow[ddd, "\theta"'] & S\mathscr{V}(TA,TB) \arrow[rrrr, "S\mathscr{V}(TA,b)"] \arrow[d, "\theta"] & & & & S\mathscr{V}(TA,B) \arrow[d, "\theta"] \\
& \mathscr{V}(TA,STB) \arrow[r, "S"] \arrow[rrrr, bend left, "\mathscr{V}(-,Sb)"] & \mathscr{V}(STA,S^2TB) \arrow[drrr, "\mathscr{V}(-,S^2b)"] \arrow[dddrr, "\mathscr{V}(-,\mu')"] \arrow[dddr, "\mathscr{V}(\lambda,-)"] & & & \mathscr{V}(TA,SB) \arrow[d, "S"] \\
& & & & & \mathscr{V}(STA,S^2B) \arrow[ddd, "\mathscr{V}(-,\mu')"] \\
\mathscr{V}(A,SB) \arrow[r, "T"] \arrow[dd, "S"'] & \mathscr{V}(TA,TSB) \arrow[r, "S"] \arrow[uu, "\mathscr{V}(TA,\lambda)"'] & \mathscr{V}(STA,STSB) \arrow[uu, "\mathscr{V}(-,S\lambda)"'] \arrow[dd, "\mathscr{V}(\lambda,-)"'] & & & \\
& & & \mathscr{V}(TSA,S^2TB) \arrow[ddl, "\mathscr{V}(-,\mu'T)"'] & V(STA,STB) \arrow[ddll, "\mathscr{V}(\lambda,-)"] \arrow[dr, "\mathscr{V}(-,Sb)"] & \\
\mathscr{V}(SA,S^2B) \arrow[r, "T"] \arrow[d, ",\mu')"'] & \mathscr{V}(TSA,TS^2B) \arrow[r, "\mathscr{V}(-,\lambda S)"'] \arrow[d, "\mathscr{V}(-,T\mu')"'] & \mathscr{V}(TSA,STSB) \arrow[ur, "\mathscr{V}(-.S\lambda)"] & & & \mathscr{V}(STA,SB) \arrow[d, "V(\lambda,-)"] \\
\mathscr{V}(SA,SB) \arrow[r, "T"] & \mathscr{V}(TSA,TSB) \arrow[r, "\mathscr{V}(-,\lambda)"'] \arrow[dd, "S"'] & \mathscr{V}(TSA,STB) \arrow[rrr, "\mathscr{V}(-,Sb)"] \arrow[dd, "S"'] & & & \mathscr{V}(TSA,SB) \arrow[d, "S"] \\
& & & & & \mathscr{V}(STSA,S^2B) \arrow[d, "\mathscr{V}(-,\mu')"] \\
& \mathscr{V}(STSA,STSB) \arrow[r, "\mathscr{V}(-,S\lambda)"'] & \mathscr{V}(STSA,S^2TB) \arrow[rr, "\mathscr{V}(-,\mu')"'] & & \mathscr{V}(STSA,STB) \arrow[r, "\mathscr{V}(-,Sb)"'] \arrow[ur, "\mathscr{V}(-,S^2b)"] & \mathscr{V}(STSA,SB)
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
STS\mathscr{V}(A,B) \arrow[rr, "STA^S"] \arrow[ddd, "S\lambda"'] & & ST\mathscr{V}(A,SB) \arrow[r, "ST(S)"] \arrow[d, "S\theta^T"] & ST\mathscr{V}(SA,S^2B) \arrow[rr, "ST\mathscr{V}(-{,}\mu')"] \arrow[d, "S\theta^T"] & & ST\mathscr{V}(SA,SB) \arrow[d, "S\theta^T"] \\
 & & S\mathscr{V}(A,TSB) \arrow[dd, "S\mathscr{V}(-{,}\lambda)"'] \arrow[dr, "S(S)"'] \arrow[r, phantom, "3"] & S\mathscr{V}(SA,TS^2B) \arrow[rr, "S\mathscr{V}(-{,}T\mu')"] \arrow[d, "S\mathscr{V}(-{,}\lambda S)"] & & S\mathscr{V}(SA,TSB) \arrow[d, "S\mathscr{V}(-{,}\lambda)"] \\
 & 1 & & S\mathscr{V}(SA,STSB) \arrow[d, "S\mathscr{V}(-{,}S\lambda)"'] & 5 & S\mathscr{V}(SA,STB) \arrow[d, "\theta^S"] \\
S^2T\mathscr{V}(A,B) \arrow[r, "S^2\theta^T"] \arrow[d, "\mu'T"'] & S^2\mathscr{V}(A,TB) \arrow[r, "S\theta^S"] \arrow[ddl, "\mu'"] & S\mathscr{V}(A,STB) \arrow[r, "S(S)"] \arrow[d, "\theta^S"] & S\mathscr{V}(SA,S^2TB) \arrow[r, "\theta^S"'] \arrow[urr, "S\mathscr{V}(-{,}\mu'_T)"] \arrow[d, phantom, "4"] & \mathscr{V}(SA,S^3TB) \arrow[r, "\mathscr{V}(-{,}S\mu')"'] \arrow[d, "\mathscr{V}(-{,}\mu'_S)"] & \mathscr{V}(SA,S^2TB) \arrow[ddl, "\mathscr{V}(-{,}\mu')"] \\
ST\mathscr{V}(A,B) \arrow[d, "S\theta^T"'] & 2 & \mathscr{V}(A,S^2TB) \arrow[r, "S"] \arrow[d, "\mathscr{V}(-{,}\mu')"'] & \mathscr{V}(SA,S^3TB) \arrow[r, "\mathscr{V}(-{,}\mu'S)"'] \arrow[d, "\mathscr{V}(-{,}S\mu')"'] & \mathscr{V}(SA,S^2TB) \arrow[d, "\mathscr{V}(-{,}\mu')"'] & 6 \\
S\mathscr{V}(A,TB) \arrow[rr, "\theta^S"] \arrow[d, "S\mathscr{V}(-{,}b)"'] & & \mathscr{V}(A,STB) \arrow[r, "S"] \arrow[d, "\mathscr{V}(-{,}Sb)"'] & \mathscr{V}(SA,S^2TB) \arrow[r, "7", "\mathscr{V}(-{,}\mu')"'] \arrow[d, "\mathscr{V}(-{,}S^2b)"'] & \mathscr{V}(SA,STB) \arrow[dr, "\mathscr{V}(-{,}Sb)"] & \\
S\mathscr{V}(A,B) \arrow[rr, "\theta^S"] & & \mathscr{V}(A,SB) \arrow[r, "S"] & \mathscr{V}(SA,S^2B) \arrow[rr, "\mathscr{V}(-{,}\mu')"'] & & \mathscr{V}(SA,SB)
\end{tikzcd}$$

Diagram 2

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