IMPROVING ON BOLD PLAY WHEN
THE GAMBLER IS RESTRICTED

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Abstract

Suppose that a gambler starts with a fortune in \((0, 1)\) and wishes to attain a fortune of 1 by making a sequence of bets. Assume that whenever the gambler stakes an amount \(s\), the gambler’s fortune increases by \(s\) with probability \(w\) and decreases by \(s\) with probability \(1 - w\), where \(w < \frac{1}{2}\). Dubins and Savage showed that the optimal strategy, which they called ‘bold play’, is always to bet \(\min\{f, 1 - f\}\), where \(f\) is the gambler’s current fortune. Here we consider the problem in which the gambler may stake no more than \(\ell\) at one time. We show that the bold strategy of always betting \(\min\{\ell, f, 1 - f\}\) is not optimal if \(\ell\) is irrational, extending a result of Heath, Pruitt, and Sudderth.

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1. Introduction and background

Suppose that a gambler starts with a fortune in \((0, 1)\) and wishes to attain a fortune of 1 by making a sequence of bets. If the gambler’s current fortune is \(f\) then the gambler may stake any amount less than or equal to \(f\). The gambler wins the amount of the stake with probability \(w\) and loses the stake with probability \(1 - w\). Following [8], we refer to this game as red-and-black. Clearly the gambler should never stake more than \(1 - f\), which is enough to ensure that the gambler will reach the goal if the bet is won. The strategy in which the gambler always stakes \(\min\{f, 1 - f\}\) is called bold play.

In [8], Dubins and Savage developed a general theory for gambling problems. For red-and-black, they showed that if \(0 < w < \frac{1}{2}\), which means that the game is subfair, then bold play is the optimal strategy, in the sense that it maximizes the probability that the gambler will eventually reach the goal. Their proof is also given in [2, Chapter 7] and [10, Chapter 24]. See [1] for some computations comparing the probability that a gambler will reach the goal using bold play to the probability that a gambler will reach the goal using other strategies.

This result has been extended in several ways. Dubins and Savage [8] also considered primitive casinos, in which the gambler loses the stake \(s\) with probability \(1 - w\) and wins \(s(1 - r)/r\) with probability \(w\), where \(0 < r < 1\). Note that the game in which \(r = \frac{1}{2}\) is red-and-black. They showed that bold play is optimal when the game is subfair, which in this case means \(w < r\). Chen [5] considered red-and-black with inflation, in which the goal is not to reach 1 but to reach \((1 + \alpha)^n\) after \(n\) bets, for some \(n\). He showed that bold play is optimal when \(w \leq \frac{1}{2}\). A different extension is to incorporate a discount factor, so that the gambler receives
Wilkins [18] showed that if bold play fails to be optimal for all than bold play for sufficiently small probability that the gambler wins three bets. Therefore, first betting that the gambler can win four bets before going bankrupt becomes very small relative to the more likely to achieve the goal after winning three or fewer bets. Consequently, for sufficiently small can be at most 1\text{-}1/2, not just for very small w. The case of rational \( \ell \) remains open except when \( \ell = 1/n \) for some \( n \geq 3 \). Note that when \( \ell \) is rational and \( 1/2 < \ell < 1/4 \), it is not even known whether or not bold play can be improved upon for very small w.

To state our result more precisely, define the function \( s: [0, 1] \to [0, 1] \) by \( s(f) = \min(\ell, f, 1 - f) \). We think of \( s(f) \) as the bold stake for a gambler whose fortune is \( f \). Denote by \( X_k \) the gambler’s fortune after \( k \) bets, when the gambler plays boldly. Note that \( (X_k)_{k=0}^\infty \) is a Markov chain whose transition probabilities are given by

\[
P(X_{k+1} = f + s(f) \mid X_k = f) = w, \tag{1}
\]

\[
P(X_{k+1} = f - s(f) \mid X_k = f) = 1 - w. \tag{2}
\]
Theorem 1. Suppose that \( w < \frac{1}{2} \) and \( \ell \) is irrational. Then there exist \( f \in (0, 1) \) and \( \varepsilon \in (0, s(f)) \) such that

\[
wQ(f + s(f) - \varepsilon) + (1 - w)Q(f - s(f) + \varepsilon) > Q(f).
\]

If a gambler begins with a fortune of \( f \) and stakes \( s(f) - \varepsilon \) then the gambler’s fortune after one bet will be \( f + s(f) - \varepsilon \) with probability \( w \) and \( f - s(f) + \varepsilon \) with probability \( 1 - w \). Consequently, the left-hand side of (3) is the probability that the gambler will eventually reach the goal using the strategy of first staking \( s(f) - \varepsilon \) and playing boldly thereafter, while the right-hand side of (3) is the probability that the gambler will reach the goal using (only) bold play. Therefore, (3) implies that the strategy of first staking \( s(f) - \varepsilon \) and then playing boldly is superior to bold play and, hence, bold play is not optimal.

2. Proof of Theorem 1

In this section, we will prove Theorem 1. The key to the proof will be the following proposition. Here, and throughout the rest of the paper, all logarithms are assumed to be base 2; that is, we write \( \log n \) instead of \( \log_2 n \).

Proposition 1. Let \( S = \{ f : P(X_k = 1 - \ell \mid X_0 = f) > 0 \} \). That is, \( S \) is the set of all \( f \) such that a gambler who starts with a fortune of \( f \) and plays boldly could have a fortune of exactly \( 1 - \ell \) after a finite number of bets.

(i) Suppose that \( f \in S \). Then there exists a constant \( C > 0 \) such that if \( 0 < \varepsilon < \ell \) then

\[
Q(f) - Q(f - \varepsilon) \geq C(1 - w)^{-\log \varepsilon}.
\]

(ii) Suppose that \( f \notin S \). For all \( C > 0 \), there exists a \( \delta > 0 \) such that if \( 0 < \varepsilon < \delta \) then

\[
Q(f) - Q(f - \varepsilon) \leq C(1 - w)^{-\log \varepsilon}.
\]

(iii) If \( \ell \) is irrational then there exists an \( f \in (\ell, 1 - \ell) \) such that \( f - \ell \in S \) and \( f + \ell \notin S \).

Proposition 1 implies that \( Q(f) - Q(f - \varepsilon) \) is larger when \( f \in S \) than when \( f \notin S \). In other words, the difference between having a fortune of \( f \) and having a fortune of \( f - \varepsilon \) matters more to the gambler when \( f \in S \) than it does when \( f \notin S \). Proposition 1(iii) states that when \( \ell \) is irrational, we can find an \( f \) such that \( f - s(f) \in S \) and \( f + s(f) \notin S \). We will show that if a gambler starts with a fortune slightly below \( f \) then it is better to make slightly less than the bold stake, so that the fortune will not fall below \( f - s(f) \) if the bet is lost. This will imply Theorem 1.

An important tool for the proof of Proposition 1 is a coupling construction in which we follow two gamblers simultaneously. We present this construction in Section 2.1. We prove Proposition 1(i), 1(ii), and 1(iii) in Sections 2.2, 2.3, and 2.4 respectively. Then, in Section 2.5, we show how Theorem 1 follows from Proposition 1.

2.1. A coupling construction

Throughout this and the next two subsections, we consider two Markov chains \( (X_k)_{k=0}^{\infty} \) and \( (Y_k)_{k=0}^{\infty} \). We define \( X_0 = f_1 \) and \( Y_0 = f_2 \), where \( f_1 \geq f_2 \). Both chains evolve with the transition probabilities given by (1) and (2). Consequently, we can think of \( X_k \) as the fortune,
after \( k \) bets, of a gambler whose initial fortune is \( f_1 \), while \( Y_k \) is the fortune, after \( k \) bets, of a gambler whose initial fortune is \( f_2 \).

We assume that these sequences are coupled, so that both gamblers win and lose the same bets. To construct this coupling, we work with the probability space \( (\Omega, \mathcal{F}, P) \) defined as follows. Let \( \Omega = [0, 1]^{\infty} \) and denote sequences in \( \Omega \) by \( \omega = (\omega_1, \omega_2, \ldots) \), so that \( \omega \rightarrow \omega_i \) is the \( i \)th coordinate function. Let \( \mathcal{F}_i \) be the trivial \( \sigma \)-field and, for positive integers \( k \), let \( \mathcal{F}_k \) be the \( \sigma \)-field generated by the first \( k \) coordinate functions. Let \( \mathcal{F} = \sigma(\mathcal{F}_1, \mathcal{F}_2, \ldots) \) be the product \( \sigma \)-field. Let \( P \) be the product probability measure with the property that \( P(\omega_i = 1) = w \) and \( P(\omega_i = 0) = 1 - w \) for all \( i \). We then say that the two gamblers win the \( i \)th bet if \( \omega_i = 1 \) and lose the \( i \)th bet if \( \omega_i = 0 \). In particular, for \( k \geq 0 \), we define \( X_{k+1}(\omega) = X_k(\omega) + s(X_k(\omega)) \) and \( Y_{k+1}(\omega) = Y_k(\omega) + s(Y_k(\omega)) \) if \( \omega_{k+1} = 1 \), and \( X_{k+1}(\omega) = X_k(\omega) - s(X_k(\omega)) \) and \( Y_{k+1}(\omega) = Y_k(\omega) - s(Y_k(\omega)) \) if \( \omega_{k+1} = 0 \).

We now make some remarks pertaining to this construction.

**Remark 1.** Since \( s(f) = \min\{\ell, f, 1 - f\} \), we see that if \( f \leq g \) then \( s(g) - s(f) \leq g - f \). Therefore, \( f + s(f) \leq g + s(g) \) and \( f - s(f) \leq g - s(g) \). It then follows by induction and the construction of the sequences \( (X_k)_{k=0}^{\infty} \) and \( (Y_k)_{k=0}^{\infty} \) that \( X_k \geq Y_k \) for all \( k \). Likewise, the fact that \( |s(g) - s(f)| \leq g - f \) implies that \( X_k - Y_k \leq 2^k(f_1 - f_2) \), for all \( k \).

**Remark 2.** The fact that \( X_k \geq Y_k \) for all \( k \) means that if \( Y_k = 1 \) then \( X_k = 1 \). Since \( Q(f_1) = P(X_k = 1 \text{ for some } k) \) and \( Q(f_2) = P(Y_k = 1 \text{ for some } k) \), it follows that \( Q(f_1) \geq Q(f_2) \). That is, the function \( f \mapsto Q(f) \) is nondecreasing.

**Remark 3.** Note that

\[
E[X_{k+1} \mid \mathcal{F}_k] = w(X_k + s(X_k)) + (1 - w)(X_k - s(X_k))
= X_k + (2w - 1)s(X_k)
\leq X_k,
\]

where the last inequality holds because \( w < \frac{1}{2} \). Therefore, \( (X_k)_{k=0}^{\infty} \) is a supermartingale with respect to \( (\mathcal{F}_k)_{k=0}^{\infty} \). By the same argument, \( (Y_k)_{k=0}^{\infty} \) is a supermartingale with respect to \( (\mathcal{F}_k)_{k=0}^{\infty} \). By the martingale convergence theorem (see [9, Chapter 4]), there exist random variables \( L_1 \) and \( L_2 \) such that \( X_k \rightarrow L_1 \) almost surely (a.s.) and \( Y_k \rightarrow L_2 \) a.s. as \( k \rightarrow \infty \). If \( 0 < \varepsilon < \ell \) then \( s(f) > \varepsilon \) for \( f \in [\varepsilon, 1 - \varepsilon] \). It follows that \( L_1 \) and \( L_2 \) must be \( [0, 1] \)-valued random variables. Furthermore, it is easy to see that, for sufficiently large \( k \), \( X_k = 1 \) and \( Y_k = 1 \) on \( [L_1 = 1] \) and \( [L_2 = 1] \), respectively. Thus, \( Q(f_1) = P(L_1 = 1) \) and \( Q(f_2) = P(L_2 = 1) \), from which it follows that \( Q(f_1) - Q(f_2) = P(L_1 = 1 \text{ and } L_2 = 0) \).

### 2.2. Proof of Proposition 1(i)

We begin with the following lemma, in which we compute the gambler’s probability of reaching the goal starting from a sequence of fortunes approaching 1.

**Lemma 1.** For all \( n \geq 0 \), we have \( Q(1 - 2^{-n}\ell) = 1 - (1 - w)^n(1 - Q(1 - \ell)) \).

**Proof.** The statement is obvious when \( n = 0 \). Suppose that the result holds for some \( n \geq 0 \). Since \( s(1 - 2^{-(n+1)}\ell) = 2^{-(n+1)}\ell \), a gambler whose fortune is \( 1 - 2^{-(n+1)}\ell \) will, after the next bet, have a fortune of 1 with probability \( w \) and a fortune of \( 1 - 2^{-n}\ell \) with probability \( 1 - w \).
Thus, by the Markov property,
\[
Q(1 - 2^{-(n+1)}\ell) = w + (1 - w)Q(1 - 2^{-n}\ell) \\
= w + (1 - w)(1 - (1 - w)^n(1 - Q(1 - \ell))) \\
= 1 - (1 - w)^{n+1}(1 - Q(1 - \ell)).
\]

The lemma now follows by induction on \(n\).

**Proof of Proposition 1(i).** Let \(f_1 = f\) and \(f_2 = f - \varepsilon\), where \(0 < \varepsilon < \ell\). Since \(f \in S\), there exists a positive integer \(k\) such that if \(B\) denotes the event that \(X_k = 1 - \ell\) and \(X_{k+1} = 1\), then \(P(B) > 0\). Note that, for \(0 \leq j < k\), we have \(X_{j+1} - Y_{j+1} \geq X_j - Y_j\) unless either \(X_j > 1 - \ell\) and \(X_{j+1} = 1\) or \(Y_j < \ell\) and \(Y_{j+1} = 0\). Therefore, if \(B\) occurs then \(X_k \leq 1 - \ell - \varepsilon\) and, thus, \(X_{k+1} \leq 1 - \varepsilon\). Combining this observation with Remarks 2–3, we obtain
\[
Q(f) - Q(f - \varepsilon) = P(L_1 = 1 \text{ and } L_2 = 0) \\
\geq P(B)P(L_2 = 0 \mid B) \\
\geq P(B)(1 - Q(1 - \varepsilon)).
\]

Choose a nonnegative integer \(n\) such that \(2^{-(n+1)}\ell < \varepsilon \leq 2^{-n}\ell\), which implies that \(n \leq \log \ell - \log \varepsilon\). By Lemma 1,
\[
Q(1 - \varepsilon) \leq Q(1 - 2^{-(n+1)}\ell) \\
= 1 - (1 - w)^{n+1}(1 - Q(1 - \ell)) \\
\leq 1 - (1 - w)^{1+\log \ell - \log \varepsilon}(1 - Q(1 - \ell)).
\]

Thus, \(Q(f) - Q(f - \varepsilon) \geq C(1 - w)^{-\log \varepsilon}(1 - Q(1 - \ell))\), where \(C = P(B)(1 - w)^{1+\log \ell}(1 - Q(1 - \ell))\).

**2.3. Proof of Proposition 1(ii)**

Our next step is to prove Proposition 1(ii), which gives an upper bound for \(Q(f) - Q(f - \varepsilon)\) when \(f \notin S\). We will compare the sequences \((X_k)_{k=0}^\infty\) and \((Y_k)_{k=0}^\infty\) when \(f_1 = f\) and \(f_2 = f - \varepsilon\). Although \((X_k - Y_k)_{k=0}^\infty\) is not a supermartingale, we will be able to construct a supermartingale by considering the differences between the gamblers’ fortunes at a sequence of stopping times. It will then follow that the gamblers’ fortunes remain similar enough for us to obtain the desired upper bound on \(Q(f) - Q(f - \varepsilon)\) when \(f \notin S\).

Given \(f\) and \(f^*\) such that \(0 \leq f^* \leq f \leq 1\), we define
\[
h(f, f^*) = \begin{cases} 
    1 & \text{if } f = f^*, \\
    s(f) - s(f^*) & \text{otherwise.}
\end{cases}
\]

Note that \(-1 \leq h(f, f^*) \leq 1\) for all \(f\) and \(f^*\). If \(\ell \leq f^* \leq f \leq 1 - \ell\) then \(s(f) = s(f^*) = \ell\), which means that \(h(f, f^*) = 0\). If \(f^* \geq \ell\) and \(f \geq 1 - \ell\) then \(h(f, f^*) \leq 0\), while if \(f^* \leq \ell\) and \(f \leq 1 - \ell\) then \(h(f, f^*) \geq 0\). Also, recall that \(\omega_k = 1\) if the gamblers win the \(k\)th bet, and that \(\omega_k = 0\) if the gamblers lose the \(k\)th bet. We have
\[
X_{k+1}(\omega) - Y_{k+1}(\omega) = \begin{cases} 
    (1 + h(X_k(\omega), Y_k(\omega)))(X_k(\omega) - Y_k(\omega)) & \text{if } \omega_{k+1} = 1, \\
    (1 - h(X_k(\omega), Y_k(\omega)))(X_k(\omega) - Y_k(\omega)) & \text{if } \omega_{k+1} = 0.
\end{cases}
\]
Define
\[ W_k = (1 - u)^{-\log(X_k - Y_k)} = (X_k - Y_k)^{-\log(1 - w)}. \]  
(5)

By (4), we have
\[ E[W_{k+1} | \mathcal{F}_k] = w(1 + h(X_k, Y_k))^{-\log(1 - w)} W_k + (1 - w)(1 - h(X_k, Y_k))^{-\log(1 - w)} W_k \]
\[ = g(h(X_k, Y_k)) W_k, \]  
(6)

where
\[ g(x) = w(1 + x)^{-\log(1 - w)} + (1 - w)(1 - x)^{-\log(1 - w)}, \]
for \(-1 \leq x \leq 1\). Note that
\[ g'(x) = -\log(1 - w)(w(1 + x)^{-\log(1 - w) - 1} - (1 - w)(1 - x)^{-\log(1 - w) - 1}). \]

Suppose that \(0 < x < 1\). Since \(0 < -\log(1 - w) < 1\), we have \((1 + x)^{-\log(1 - w) - 1} < 1\) and \((1 - x)^{-\log(1 - w) - 1} > 1\). Therefore,
\[ g'(x) \leq -\log(1 - w)(w - (1 - w)) < 0. \]

Since \(g(0) = 1\), it follows that \(0 < g(x) < 1\), for \(x \in (0, 1]\).

We now introduce four lemmas that will help us to define a supermartingale.

**Lemma 2.** Suppose that \(1 - \ell \leq f_2 \leq f_1 \leq 1\). Then \(E[W_1] = W_0\).

**Proof.** We have \(s(f_1) = 1 - f_1\) and \(s(f_2) = 1 - f_2\). Therefore, \(h(f_1, f_2) = -1\). Since \(g(-1) = 1\), it follows from (6) that \(E[W_1] = W_0\).

**Lemma 3.** Suppose that \(f_2 \leq f_1 < 1 - \ell\). Define a stopping time \(R\) as follows. If \(h(f_1, f_2) \geq \frac{1}{2}\), then let \(R = 0\). If \(h(f_1, f_2) < \frac{1}{2}\) then let \(R(\omega) = \inf\{j : \omega_j = 1 \text{ or } h(X_j(\omega), Y_j(\omega)) \geq \frac{1}{2}\}\). Let \(L = \lceil 1 + (1 - 2\ell)/\ell \rceil\), where \(\lfloor \cdot \rfloor\) is the integer-part function. Then \(R \leq L\) and \(E[W_R] \leq W_0\).

**Proof.** Proceeding by contradiction, suppose that \(R(\omega) > L\) for some \(\omega\). Then the gamblers must lose the first \(L\) bets. However, by the definition of \(L\), any gambler who starts with a fortune of at most \(1 - \ell\) and then loses \(L\) consecutive bets has a fortune of at most \(\ell\). Therefore, there exists a \(j \leq L\) such that \(0 < X_j \leq \ell\). Since \(X_j \geq Y_j\), it follows that \(Y_j \leq \ell\) and, thus, \(s(X_j) = X_j\) and \(s(Y_j) = Y_j\). However, this means that \(h(X_j, Y_j) = 1\) and, thus, \(R \leq j\), i.e. a contradiction. Hence, \(R \leq L\).

For \(j < R\), we have \(Y_j \leq X_j < 1 - \ell\) and, therefore, \(0 \leq h(X_j, Y_j) \leq 1\). Since \(g(x) \leq 1\) for \(x \in [0, 1]\), we have, with the aid of (6),
\[
E[W_{j+1} | \mathcal{F}_j] = W_j \mathbb{I}_{\{R \leq j\}} + E[W_{j+1} | \mathcal{F}_j] \mathbb{I}_{\{R > j\}}
\]
\[
= W_j \mathbb{I}_{\{R \leq j\}} + g(h(X_j, Y_j)) W_j \mathbb{I}_{\{R > j\}}
\]
\[
\leq W_j \mathbb{I}_{\{R \leq j\}},
\]
where \(\mathbb{I}_{\{\cdot\}}\) is the indicator function and \(x \wedge y\) denotes \(\min\{x, y\}\). Therefore, \((W_j \mathbb{I}_{\{R \leq j\}})_{j=0}^{\infty}\) is a supermartingale with respect to \((\mathcal{F}_j)_{j=0}^{\infty}\). Note that \(0 \leq W_j \mathbb{I}_{\{R \leq j\}} \leq 1\) for all \(j\), so the optional stopping theorem (see [9, Chapter 4]) gives \(E[W_R] \leq W_0\).
Lemma 4. Suppose that \( f_2 \leq f_1 < 1 - \ell \). Let

\[
T(\omega) = \inf\{ j \geq 1 : \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \geq \frac{1}{2} \}.
\]

Let \( L = [1 + (1 - 2\ell)/\ell] \) as in Lemma 3. Then \( T \leq L + 1 \) and \( E[W_T] \leq \alpha W_0 \), where

\[
\alpha = 1 - (1 - g(\frac{1}{2}))(1 - w)^{2L}.
\]

Proof. Define the stopping time \( R \) as in Lemma 3. Then \( T = R \) if and only if the gamblers win the \( R \)th bet; otherwise \( T = R + 1 \). Clearly \( T \leq L + 1 \) by Lemma 3. Let \( A \) be the event that the gamblers win the \( R \)th bet. Then

\[
E[W_T] = E[E[W_T | \mathcal{F}_R]] = E[E[R_A + E[W_{R+1} | \mathcal{F}_R] I_{A^c}]].
\]

By the strong Markov property and (6), \( E[W_{R+1} | \mathcal{F}_R] = g(h(X_R, Y_R)) W_R \). If the gamblers lose the \( R \)th bet then \( h(X_R, Y_R) \geq \frac{1}{2} \). Therefore, since \( g \) is decreasing on \([0, 1]\),

\[
E[W_{R+1} | \mathcal{F}_R] I_{A^c} \leq g(\frac{1}{2}) W_R I_{A^c}.
\]

Thus,

\[
E[W_T] \leq E[W_R I_A + g(\frac{1}{2}) W_R I_{A^c}] = E[W_R - (1 - g(\frac{1}{2})) W_R I_{A^c}]. \tag{7}
\]

If \( A^c \) occurs then the gamblers lose the first \( R \) bets, and \( h(X_j, Y_j) < \frac{1}{2} \) for all \( j < R \). If \( h(X_j, Y_j) < \frac{1}{2} \) and the gamblers lose the \((j + 1)\)th bet, then \( X_{j+1} - Y_{j+1} \geq (X_j - Y_j)/2 \). Thus, on \( A^c \) we have \( X_R - Y_R \geq 2^{-R}(f_1 - f_2) \geq 2^{-L}(f_1 - f_2) \). Therefore,

\[
E[W_R I_{A^c}] \geq E[(2^{-L}(f_1 - f_2))^{\log(1-w)} I_{A^c}]
\]

\[
= 2^{L \log(1-w)} W_0 P(A^c)
\]

\[
= P(A^c)(1 - w)^L W_0. \tag{8}
\]

Since \( A^c \) occurs when the gamblers lose the first \( L \) bets, we have \( P(A^c) \geq (1 - w)^L \). Thus, since \( E[W_R] \leq W_0 \) by Lemma 3, combining (7) and (8) gives

\[
E[W_T] \leq E[W_R] - (1 - g(\frac{1}{2}))(1 - w)^{2L} W_0
\]

\[
\leq (1 - (1 - g(\frac{1}{2}))(1 - w)^{2L}) W_0 = \alpha W_0,
\]

which completes the proof.

Lemma 5. Suppose that \( f_2 < 1 - \ell \) and \( 1 - \ell \leq f_1 < 1 - \ell/2 \). Define the stopping time \( T \) by

\[
T(\omega) = \begin{cases} 1 & \text{if } \omega_1 = 1, \\ \inf\{ j \geq 2 : \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \geq \frac{1}{2} \} & \text{otherwise.} \end{cases}
\]

Let \( N(\omega) = 1 - \alpha \) if \( \omega_1 = 1 \) and let \( N(\omega) = 1 \) if \( \omega_1 = 0 \). Then \( T \leq L + 2 \) and \( E[NW_T] \leq W_0 \).

Proof. Let \( A \) be the event that the gamblers win the first bet, which means that \( \omega_1 = 1 \). We have

\[
E[NW_T] = E[E[NW_T | \mathcal{F}_1]] = E[(1 - \alpha) W_1 I_A + E[W_T | \mathcal{F}_1] I_{A^c}].
\]
If the gamblers lose the first bet then \(X_1 = f_1 - s(f_1) = 2f_1 - 1 < 1 - \ell\). Therefore, by Lemma 4 and the Markov property, we have \(T \leq L + 2\) and \(E[W_T \mid \mathcal{F}_T] 1_{A^c} \leq \alpha W_T 1_{A^c}\). Thus,

\[
E[NW_T] \leq E[(1 - \alpha)W_T 1_{A} + \alpha W_T 1_{A^c}]
\]

\[
= w(1 - \alpha)((1 + h(f_1, f_2))(f_1 - f_2))^{\log(1 - w)} + (1 - w)\alpha(1 - h(f_1, f_2))^{\log(1 - w)}
\]

\[
= [w(1 - \alpha)^{\log(1 - w)} + (1 - w)\alpha^{\log(1 - w)}]W_0
\]

\[
= \left(\frac{w}{1 - w}(1 - \alpha) + \alpha\right)W_0
\]

\[
\leq W_0.
\]

which completes the proof.

By combining Lemmas 2, 4, and 5, we can obtain Proposition 2, below, in which we construct the supermartingale needed to prove Proposition 1(ii). We first inductively define a sequence of stopping times \((T_k)_{k=0}^{\infty}\). Let \(T_0 = 0\). Given \(T_k\), we define \(T_{k+1}\) according to the following rules.

1. If \(Y_{T_k}(\omega) \geq 1 - \ell\) then let \(T_{k+1}(\omega) = T_k(\omega) + 1\).

2. If \(X_{T_k}(\omega) < 1 - \ell\) then let

   \[T_{k+1}(\omega) = \inf\{j \geq T_k(\omega) + 1: \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \geq \frac{1}{2}\}\].

3. Suppose that \(Y_{T_k}(\omega) < 1 - \ell\) and \(1 - \ell < X_{T_k}(\omega) < 1 - \ell/2\). If \(\omega T_k(\omega) + 1 = 1\), meaning the gamblers win the \((T_k + 1)\)th bet, then let \(T_{k+1}(\omega) = T_k(\omega) + 1\). Otherwise, let

   \[T_{k+1}(\omega) = \inf\{j \geq T_k(\omega) + 2: \omega_j = 1 \text{ or } h(X_{j-1}(\omega), Y_{j-1}(\omega)) \geq \frac{1}{2}\}\].

4. If \(Y_{T_k} < 1 - \ell\) and \(X_{T_k} \geq 1 - \ell/2\) then let \(T_{k+1} = T_k\).

**Proposition 2.** Define the sequence of stopping times \((T_k)_{k=0}^{\infty}\) as above. For \(k \geq 0\), let

\[B_k = \mathbb{S}\{j \in \{0, 1, \ldots, k - 1\}: X_j < 1 - \ell\},\]

where \(\mathbb{S}\) denotes the cardinality of the set \(S\). Let \(N_k = 1 - \alpha\) in the event that, for some \(j \leq T_k\), we have \(Y_{j-1} < 1 - \ell \leq X_{j-1}\) and the gamblers win the \(j\)th bet. Otherwise, let \(N_k = 1\). Define \(Z_k = \alpha^{-B_k}N_k W_{T_k}\). Then \((Z_k)_{k=0}^{\infty}\) is a supermartingale with respect to the filtration \((\mathcal{F}_T)_{k=0}^{\infty}\). Furthermore, \(T_{k+1} \leq T_k + L + 2\) for all \(k \geq 0\).

**Proof.** Let \(A_{1,k}\) be the event that \(Y_{T_k} \geq 1 - \ell\) and let \(A_{2,k}\) be the event that \(X_{T_k} < 1 - \ell\). Let \(A_{3,k}\) be the event that \(Y_{T_k} < 1 - \ell\) and \(1 - \ell < X_{T_k} < 1 - \ell/2\). Let \(A_{4,k}\) be the event that \(Y_{T_k} < 1 - \ell\) and \(X_{T_k} \geq 1 - \ell/2\). Note that, for all \(k\), exactly one of these four events occurs. We consider the four cases separately.
We will need one more lemma. Let
\[ Y(T_k) = \begin{cases} 1 & \text{if } T_k + 1 \leq T_k + 1, \\ \alpha & \text{if } T_k + 1 > T_k + 1, \\ \alpha - b_k N_k W_{T_k} & \text{if } T_k + 1 = T_k + 1, \end{cases} \]

Lemma 6. Let \( f \neq S \) and let \( N \) be a positive integer. Then there exist a positive integer \( M \) and a positive real number \( \delta \) such that if \( f_1 = f \) and \( f_2 = f - \varepsilon \), where \( 0 < \varepsilon < \delta \), then the following statements hold.

(i) If \( X_k \geq 1 - \ell \) and \( Y_k = 1 - \ell \) then \( k > M(L + 2) \), where \( L = \lfloor 1 + (1 - 2\ell) / \ell \rfloor \).

(ii) Let \( D_k = \{ j \in [0, 1, \ldots, k - 1] : X_T < 1 - \ell \text{ or } X_T = 1 \} \). Then \( D_M \geq N \).

Proof. Let \( R_k \) be the set of all possible values of \( X_k \), and let \( R_k = \bigcup_{j=0}^{k} R_j \). Note that \( R_k \) is a finite set because there are only \( 2^k \) possible outcomes for the first \( k \) bets. For all \( g \in [0, 1] \), let \( v(g) \) be the number of consecutive bets that a gambler whose fortune is \( g \) must lose for the
Therefore, \( Y_k \leq 2k^k < 2^{M(L+2)}\delta = \theta \). Therefore, \( Y_k \geq 1 - \ell \). This proves the first part of the lemma.

To prove the second part we claim that, for \( i = 0, 1, \ldots, N = 1 \),

\[
D_{M_i + V_{M_i(L+2)} + 1} \geq D_{M_i + 1}. \tag{12}
\]

To see how (12) implies the second part of the lemma, first note that \( D_{M_0} = D_0 = 0 \). Suppose that \( D_{M_i} \geq i \) for some \( i \geq 0 \). Then \( D_{M_{i+1}} = D_{M_i + V_{M_i(L+2)} + 1} \geq D_{M_i + 1} \geq i + 1 \) by (12). Hence, by induction, (12) implies that \( D_M \geq N \). Thus, we need only to prove (12). First, suppose that either \( XT_m < 1 - \ell \) or \( XT_m = 1 \). Then \( D_{M+1} = D_m + 1 \). Since \( (D_i)_{i=0}^\infty \) is a nonincreasing sequence and \( V_{M_i(L+2)} \geq 0 \), we have (12).

Thus, it remains only to prove (12) when \( 1 - \ell \leq X_{T_m} < 1 \). Write \( v \) for \( (X_{T_m})_j \). Note that \( v \leq V_{T_m} \), and \( T_m \leq M(L + 2) \) by Proposition 2. Therefore,

\[
T_m + v \leq T_m + V_{T_m} \leq M(L + 2) + V_{M_i(L+2)} \leq (M + V_{M_i(L+2)})(L + 2) \leq M(L + 2).
\tag{13}
\]

We now consider two cases. First, suppose that the gamblers lose the bets \( T_m + 1, \ldots, T_m + v \). Then \( X_j \geq 1 - \ell \) for \( T_m \leq j \leq T_m + v - 1 \) and \( X_{T_m+1} < 1 - \ell \). Also, by (13) and the first part of the lemma, we have \( Y_j \geq 1 - \ell \) for \( T_m \leq j \leq T_m + v - 1 \). Therefore, by the definition of the sequence \( \{T_j\}_{j=0}^\infty \), we have \( T_{M+k} = T_m + k \) for \( 1 \leq k \leq v \). It follows that \( X_{T_m+v} < 1 - \ell \), which means that \( D_{M_{i+1}} = D_{M_i + 1} \). Thus, \( D_{M_i + V_{M_i(L+2)} + 1} \geq D_{M_i + 1} + 1 \), which is (12). Finally, we consider the case in which, for some \( j \in \{1, \ldots, v\} \), the gamblers lose the bets \( T_m \), \ldots, \( T_m + j - 1 \) but win the bet \( T_m + j \). Then \( X_{T_m+j} = X_{T_m+j} = 1 \) and \( D_{M_i + V_{M_i(L+2)} + 1} \geq D_{M_i + 1} \), which is (12).

Proof of Proposition 1(ii). Fix \( C > 0 \) and \( f \neq S \). Since \( 0 < x < 1 \), there exists a positive integer \( N \) such that

\[
\alpha^{-N} (1 - \alpha)(1 - w)^{-\log(\ell/2)} \geq C^{-1}.
\]

Define \( M \) and \( \delta \) as in Lemma 6 and fix \( \varepsilon \in (0, \delta) \). Define \( (T_k)_{k=0}^\infty \) and \( (Z_k)_{k=0}^\infty \) as in Proposition 2, with \( f_1 = f \) and \( f_2 = f - \varepsilon \).

By Remark 3, there exist random variables \( L_1 \) and \( L_2 \) such that \( X_k \to L_1 \) a.s. and \( Y_k \to L_2 \) a.s. as \( k \to \infty \), and \( Q(f) - Q(f - \varepsilon) = P(L_1 = 1 \text{ and } L_2 = 0) \). Let \( A \) be the event that \( L_1 = 1 \) and \( L_2 = 0 \). Then there is an integer-valued random variable \( K \) such that, on the event \( A \), we have \( X_{T_K} \geq 1 - \ell/2 \) and \( Y_{T_k} < 1 - \ell \). By Lemma 6(i), in the event \( A \), we have \( T_K > M(L + 2) \) and, thus, \( K \geq M \). It also follows from Lemma 6(i) that if \( X_{T_l} \leq 1 \) for \( j \leq M \) then \( Y_{T_l} \geq 1 \) and, therefore, \( L_2 = 1 \). Consequently, in the event \( A \), we can see from the definitions of \( (B_i)_{i=0}^\infty \) and \( (D_i)_{i=0}^\infty \) that \( B_M = D_M \) and, thus, using Lemma 6(ii), \( B_K \geq B_M = D_M \geq N \).
Since $(Z_k)_{k=0}^\infty$ is a nonnegative supermartingale, it follows from the martingale convergence theorem (see [9, Corollary 4.2.11]) that there exists a random variable $Z$ such that $Z_k \to Z$ a.s. and $E[Z] \leq E[Z_0]$. On the event $A$, if $j > K$ then $T_j = T_K$ and, thus, $Z_j = Z_K = Z$. Hence, using (5),

\[
Z 1_A = Z_K 1_A \\
= \alpha^{-B_k} N_k (1 - \omega)^{-\log(X_{T_k} - Y_{T_k})} 1_A \\
\geq \alpha^{-N} (1 - \alpha)(1 - \omega)^{-\log(\ell/2)} 1_A \\
\geq C 1_A.
\]

It follows that $E[Z] \geq C^{-1} P(A)$. Thus, $Q(f) - Q(f - \epsilon) = P(A) \leq C E[Z] \leq C E[Z_0] = C (1 - \omega)^{-\log \epsilon}$, as claimed.

2.4. Proof of Proposition 1(iii)

Let $D_1^j = S \cap [0, \ell]$ and $D_2^j = S \cap [1 - \ell, 1]$. Define a sequence of stopping times $(t_k)_{k=0}^\infty$ by $t_0 = 0$ and $t_{k+1} = \inf \{ n > t_k : X_n \in [0, \ell] \cup [1 - \ell, 1] \}$, for all $k \geq 0$. Then define

\[
D_k = \{ f : P(X_{t_j} = 1 - 1 \text{ for some } j \leq k \mid X_0 = f) > 0 \}.
\]

Let $D_1^j = D_k \cap [0, \ell]$ and $D_2^j = D_k \cap [1 - \ell, 1]$. Note that $D_1^j = \bigcup_{k=0}^\infty D_1^k$ and $D_2^j = \bigcup_{k=0}^\infty D_2^k$.

We have $D_1^0 = \emptyset$ and $D_2^0 = [1 - \ell]$. For $k \geq 1$,

\[
D_1^k = \{ f \in [0, \ell] : P(X_{t_j} \in D_{k-1} \mid X_0 = f) > 0 \} \cup D_1^k_{k-1},
\]

\[
D_2^k = \{ f \in [1 - \ell, 1] : P(X_{t_j} \in D_{k-1} \mid X_0 = f) > 0 \} \cup D_2^k_{k-1}.
\]

Suppose that $X_0 = f$. If $f \in (\ell, 1 - \ell)$ then $s(f) = \ell$ for all $k < t_1$. Therefore, $X_{t_j} = f + n\ell$ for some $n \in \mathbb{Z}$. If, instead, $f \in [0, \ell]$ then $s(f) = f$, in which case either $X_1 = X_{t_1} = 0$ or $X_1 = 2f$. If $X_1 = 2f$ then $X_{t_j} = 2f + n\ell$ for some $n \in \mathbb{Z}$, where $n = 0$ if $2f \in [0, \ell] \cup [1 - \ell, 1]$. Likewise, suppose that $f \in [1 - \ell, 1]$. Then $s(f) = 1 - f$, so either $X_1 = X_{t_1} = 1$ or $X_1 = 2f - 1$. If $X_1 = 2f - 1$ then $X_{t_j} (f) = 2f - 1 + n\ell$ for some $n \in \mathbb{Z}$, where $n = 0$ if $2f - 1 \in [0, \ell] \cup [1 - \ell, 1]$.

We claim that if $f \in D_1^1 \cup D_2^1$ then there exist integers $a, b$, and $c$ such that $f = 2^{-c}(a + b\ell)$. Furthermore, we claim that if $f \neq 1 - \ell$ then we can choose $a, b$, and $c$ such that $c \geq 1$, $a \geq 1$, $a$ or $b$ is odd, and $a \geq 2$ if $f \in D_2^2$. We will prove these claims by induction on $k$. Note that $D_0 = [1 - \ell]$ so, for $f \in D_0$, we can take $a = 1$, $b = -1$, and $c = 0$. Now, suppose that our claims hold when $f \in D_{k-1}$, where $k \geq 1$. To show that our claims hold when $f \in D_k$, we consider two cases.

First, suppose that $f \in D_1^k \setminus D_{k-1}$. Then $P(X_{t_j} = g \mid X_0 = f) > 0$ for some $g \in D_{k-1}$. Since $0 \notin S$, we must have $2f + n\ell = g$ or, equivalently, $f = (g - n\ell)/2$ for some $n \in \mathbb{Z}$ and $g \in D_{k-1}$. If $g = 1 - \ell$ then $f = (1 - (n + 1)\ell)/2$, so $f = 2^{-c}(a + b\ell)$, where $a = 1$, $b = -(n + 1)$, and $c = 1$. If $g \neq 1 - \ell$ then $g = 2^{c-1}(a + b\ell)$, where $c \geq 1$, $a \geq 1$, and $a$ or $b$ is odd. Then $f = 2^{-(c+1)}(a + b\ell - 2^n\ell) = 2^{-(c+1)}(a + b - 2^{n+1}\ell)$. Note that $c + 1 \geq 1$, $a \geq 1$, and $b - 2n$ is odd if $b$ is odd, so $a$ or $b - 2n$ is odd.

Next, suppose that $f \in D_2^k \setminus D_{k-1}$. Then $P(X_{t_j} = g \mid X_0 = f) > 0$ for some $g \in D_{k-1}$. Since $1 \notin S$, we have $2f - 1 + n\ell = g$ or, equivalently, $f = (1 + g - n\ell)/2$ for some $n \in \mathbb{Z}$ and $g \in D_{k-1}$. If $g = 1 - \ell$ then $f = (2 - (n + 1)\ell)/2$. If $n + 1$ were even then $f = 1 - m\ell$ would hold for some positive integer $m$; since $D_2^2 \subseteq [1 - \ell, 1]$, we would have $f \in [1 - \ell, 1]$, which is
a contradiction because $1 \notin D^2_k$ and $1 - \ell \in D_{k-1}$. Therefore, $n+1$ is odd, so $f = 2^{-c}(a + b\ell)$, where $c = 1$, $a = 2$, and $b$ is odd. If, instead, $g \neq 1 - \ell$ then $g = 2^{-c}(a + b\ell)$, where $c \geq 1$, $a \geq 1$, and $a$ or $b$ is odd. Then $f = 2^{-c+1}(2^c + a + b\ell - 2^n n\ell) = 2^{-c+1}[2^c(a + b\ell - 2^n n\ell)]$. Note that $c+1 \geq 1$, $2^c + a \geq 2$, and either $2^c + a$ or $b - 2^n n$ is odd because $2^c$ and $2^n n$ are even and either $a$ or $b$ is odd. It now follows, by induction, that our claims hold for all $f \in D^1 \cup D^2$.

Since $\ell < \frac{1}{2}$, we can choose a positive integer $m$ such that $1 - m\ell \in (\ell, 2\ell]$. We can then choose positive integers $d$ and $n$ such that $2^{-d}(1 - m\ell) < 1 - 2\ell$ and $2^{-d}(1 - \ell) + n\ell \in (1 - 2\ell, 1 - \ell)$. Let $f = 2^{-d}(1 - m\ell) + n\ell$. Note that $f \in (\ell, 1 - \ell)$ and that $f - \ell = 2^{-d}(1 - m\ell) + (n - 1)\ell$. Suppose that a gambler who starts with a fortune of $f - \ell$ loses the first $n - 1$ bets, then wins the next $d + m - 1$ bets. After the $n - 1$ losses, the gambler’s fortune will be $2^{-d}(1 - m\ell)$. Then, after $d$ wins, the fortune will be $1 - m\ell$. After $m - 1$ additional wins, the gambler’s fortune will be $1 - \ell$. Consequently,

$$P(X_{n+m+d-2} = 1 - \ell \mid X_0 = f - \ell) > 0,$$

which means that $f - \ell \in S$. We now show by contradiction that $f + \ell \notin S$, which will complete the proof. Suppose that $f + \ell \in S$. Since $f + \ell > 1 - \ell$, there exist integers $a$, $b$, and $c$ such that $a > 2$, $c \geq 1$, $a$ or $b$ is odd, and $f + \ell = 2^{-c}(a + b\ell)$. We also have $f + \ell = 2^{-d}(1 + (2^n(n + 1) - m)\ell)$. Therefore, $2^d(a + b\ell) = 2^d(1 + (2^n(n + 1) - m)\ell)$ and, so, $2^d a - 2^c = 2^d(2^n(n + 1) - m) - 2^d b\ell$. Since $\ell$ is irrational, we must have $2^d a - 2^c = 2^d (2^n(n + 1) - m) - 2^d b = 0$. Thus, $2^d a = 2^c$ and, since $a \geq 2$, it follows that $a$ is even and $c > d$. Therefore, $b$ is odd and $b = 2^{-d}(2^n(n + 1) - m)$, which is a contradiction.

### 2.5. Obtaining Theorem 1 from Proposition 1

Suppose that $\ell$ is irrational. By Proposition 1(iii), there exists an $f_0 \in (\ell, 1 - \ell)$ such that $f_0 - \ell \in S$ and $f_0 + \ell \notin S$. Let $f = f_0 - \varepsilon$, where $0 < \varepsilon < \ell$ and $\varepsilon$ is small enough that $f \in (\ell, 1 - \ell)$. We will show that, for sufficiently small $\varepsilon$, we have

$$wQ(f + \ell - \varepsilon) + (1 - w)Q(f + \ell + \varepsilon) > Q(f),$$

which implies Theorem 1 because $s(f) = \ell$. Note that

$$wQ(f + \ell - \varepsilon) + (1 - w)Q(f + \ell + \varepsilon) = wQ(f_0 + \ell - 2\varepsilon) + (1 - w)Q(f_0 - \ell).$$

Since $f \mapsto Q(f)$ is nondecreasing, we have

$$Q(f) = wQ(f + \ell) + (1 - w)Q(f - \ell) \leq wQ(f_0 + \ell) + (1 - w)Q(f_0 - \ell - \varepsilon).$$

Since $f_0 - \ell \in S$, it follows from Proposition 1(i) that there exists a constant $C > 0$ such that

$$Q(f_0 - \ell) - Q(f_0 - \ell - \varepsilon) \geq C(1 - w)^{-\log \varepsilon}.$$ 

Let $C_0 = C(1 - w)$. Since $f_0 + \ell \notin S$, Proposition 1(ii) implies that, for sufficiently small $\varepsilon$, we have

$$Q(f_0 + \ell) - Q(f_0 + \varepsilon) \leq C_0(1 - w)^{-\log 2\varepsilon} = C(1 - w)^{-\log \varepsilon}.$$ 

Let $B = C(1 - w)^{-\log \varepsilon}$. Equations (15)–(18) imply that

$$wQ(f + \ell - \varepsilon) + (1 - w)Q(f + \ell + \varepsilon) - Q(f) \geq -wB + (1 - w)B = (1 - 2w)B > 0,$$

for sufficiently small $\varepsilon$, which gives (14).
Improving on bold play when the gambler is restricted

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