ON FINITE GROUPS OF THE FORM ABA

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Introduction. The class of finite groups G of the form ABA, where A and B are subgroups of G, is of interest since it includes the finite doubly transitive groups, which admit such a representation with A the subgroup fixing a letter and B of order 2. It is natural to ask for conditions on A and B which will imply the solvability of G. It is known that a group of the form AB is solvable if A and B are nilpotent. However, no such general result can be expected for ABA-groups, since the simple groups $PSL(2,2^n)$ admit such a representation with A cyclic of order $2^n + 1$ and B elementary abelian of order 2^n. Thus G need not be solvable even if A and B are abelian.

In **(3)** Herstein and the author have shown that G is solvable if A and B are cyclic of relatively prime orders; and in **(2)** we have shown that G is solvable if A and B are cyclic and A possesses a normal complement in G. The present paper is devoted to a proof of the following result:

THEOREM A. *If $G = ABA$, where A and B are cyclic subgroups of G, and if A is its own normalizer in G, then G is solvable.*

If G_0 is a subgroup of G containing A, then it is easy to see that $G = AB_0A$ with $B_0 \subseteq B$. Furthermore a homomorphic image $\bar{G}$ of G is of the form $\bar{A}\bar{B}\bar{A}$, and it can be shown that $N(\bar{A}) = \bar{A}$ if $N(A) = A$. Thus it is natural to attempt to prove Theorem A by induction on the order of G. In order to carry out the inductive argument, one must first determine the structure of all solvable groups which satisfy the hypotheses of Theorem A; and the bulk of the paper (Part I) is taken up with this problem. Our main result is the following:

THEOREM B. *Let $G = ABA$, where A and B are cyclic subgroups of G and $N(A) = A$, and assume that G is solvable. Then $G = AT$, where $T = [G,G]$, and T is the direct product of three A-invariant subgroups T_1, T_2, T_3, which satisfy the following conditions:*

(I) *T_1 is a 2-group; if $T_1 \neq 1$, then $A \cap T_1 \neq 1$;*

(II) *$T_2 = MQ$ where $M \lhd T_2$ and Q is a q-group, q a prime, either M is a 2-group and $q = 7$ or M is abelian of type (m,m), $(m,6) = 1$, and $q = 3$; if $T_2 \neq 1$,* then *$A \cap T_2 \neq 1$;*

(III) *T_3 is nilpotent of class 1 or 2 and $A \cap T_3 = 1$.*

The proof of Theorem B relies heavily upon the properties of regular ϕ-groups which were developed in **(2)** and especially upon the structure of

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regular ϕ-groups of prime power order. These properties are listed in § 1. In addition to this, we need bounds for the order of automorphisms of certain non-abelian p-groups, which include the extra-special p-groups as defined by Hall and Higman in **(7)**. These bounds will be determined in §§ 1 and 2. In the course of the proof of Theorem B we shall also obtain much more precise information concerning the structure of the "exceptional" groups T_1, T_2.

The proof of Theorem A from Theorem B utilizes the transfer of G into certain A-invariant p-Sylow subgroups P of G, where $p|o(A)$. Using Theorem B and our induction assumption, we are able to show by means of the Hall-Wielandt theorem that the p-Sylow subgroup of A is mapped isomorphically in the transfer of G into P. This argument works very smoothly if G possesses no subgroups of the form T_1 or T_2, but requires considerable modification if such subgroups are present.

Throughout the paper we shall write simply $G = ABA$, provided G is an ABA-group in which A,B are cyclic and A is its own normalizer.

In a subsequent paper we hope to treat the class of groups of the form ABA, where A and B are cyclic, but A is not necessarily its own normalizer.

PART I

The Structure of Solvable ABA-groups

1. ϕ-groups of prime power order. We recall from **(2)** that a group T is called a *ϕ-group* if T possesses an automorphism ϕ such that every element of T can be expressed in the form $\phi^i(g\phi^r(g)\phi^{2r}(g)\ldots\phi^{(j-1)r}(g))$ (we denote this expression by $\phi^i([g]_r^j)$) for some fixed element g in T and some fixed integer r, for suitable choice of integers i and j. g is called a *ϕ-generator* of T, and r the *ϕ-index* of T.* If ϕ leaves only the identity element of T fixed, T is called a *regular* ϕ-group. In particular, if $\phi^r = 1$, every element of T is of the form $\phi^i(g^j)$. In this case we say that T is of ϕ-index 0.

In Theorem 10 of **(2)**, we showed that T is a ϕ-group if and only if the holomorph G of T and ϕ is of the form ABA, where a generator a of A induces by conjugation the automorphism ϕ of T and where B is generated by the element ga^{-r}. It is clear that T will be a regular ϕ-group if and only if $N(A) = A$. Throughout the paper if $G = ABA$, we shall denote by ϕ the automorphism of G induced by conjugation by a generator a of A. Thus if an ABA-group G possesses a normal A-complement T, then T is a regular ϕ-group. The principal result of **(2**, Theorem 9**)** asserts that a regular ϕ-group T is nilpotent of class 1 or 2.

In Theorems 6 and 8 of **(2)**, we have determined the structure of a regular ϕ-group of prime power order rather precisely. As we shall make repeated use of this structure, we shall restate these results here. The following properties

*In **(2)** we have used the terms index and generator of P under ϕ; ϕ-index and ϕ-generator seem preferable, since they avoid possible confusion with the customary use of these terms in the theory of groups.

of a regular ϕ-group of prime power order are either explicitly contained in Theorems 6 and 8 of **(2)** or are easily derived from them.

If P is a regular ϕ-group of order p^n and ϕ-index r, P contains a normal* subgroup F invariant under ϕ such that

(1a) F is either elementary abelian, cyclic of order p^e, or of type (p^e, p^e). $F = 1$ if and only if P is elementary abelian, ϕ has order relatively prime to p and ϕ^r leaves only the identity element of P fixed.

(1b) ϕ acts irreducibly on $F_1 = \Omega_1(F)$.

(1c) $\bar{P} = P/F$ is elementary abelian, the image $\bar{\phi}$ of ϕ on $\bar{P}$ has order relatively prime to p and $\bar{\phi}^r$ is without non-trivial fixed elements.

(1d) if $k =$ order of ϕ on F_1 and $rs = \phi$-index of F_1, then $k|rs$. Thus F_1 is of ϕ-index 0.

(1e) If P is abelian, $P = H \times K$, where H, K are invariant under ϕ, $H \supseteq F$, ϕ has order kp^c on H for some c and order relatively prime to p on K.

We shall call F the *ϕ-nucleus* of P.

The preceding results depend crucially upon the following inequalities:

(1f) If ϕ has order h and ϕ^r is without non-trivial fixed elements on P, then $h^2/r > o(P)$; if P is of ϕ-index 0, and g is a ϕ-generator of P of order s, then $hs > o(P)$.

In § 1 we shall establish several further properties of regular ϕ-groups of prime power order, which we shall need for our subsequent work. In **(2)** we conjectured that if P has ϕ-index r and ϕ^r leaves only the identity element of P fixed, then P is in fact abelian. We shall include a proof of this conjecture when P has odd prime power order. The proof depends upon the following lemma, which is due to John Thompson.

LEMMA 1.1. *Let P be a p-group whose centre C and factor group $\bar{P} = P/C$ are both elementary abelian of the same order p^n. Suppose G has an automorphism ϕ which acts irreducibly on C and whose image $\bar{\phi}$ on $\bar{P}$ acts irreducibly on $\bar{P}$. Assume further that ϕ and $\bar{\phi}$, regarded as linear transformations, have the same characteristic polynomials on C and $\bar{P}$. Then the order of $\bar{\phi}$ is less that p^{n-1}.*

Proof. The associated Lie ring L of P is the Cartesian sum of two additive groups L_1 and L_2, with $L_1 \cong \bar{P}$ and $L_2 \cong C$. Regarding L as a vector space over the prime field k_p with p elements, $\bar{\phi}$ and ϕ induce linear transformations of L_1 and L_2 respectively, which we denote by the same letters. If $[x, y]$ denotes the Lie product in L, it follows from the definition of L that for any two elements x, y in L_1

$$[x\bar{\phi}, y\bar{\phi}] = [x, y]\phi. \tag{1}$$

*Theorem 8 of **(2)** asserts actually that F is in the centre of P. There is, however, an error in the proof. A correct proof, when p is an odd prime, will be given below in Lemma 1.5. It will also be shown that P is of class $\leqslant 2$ even when $p = 2$, although in this case F need not be in the centre of P. This will complete the proof of Theorem 9 of **(2)**.

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It follows from (1) that the elements of the form $[x, y]$, x, y in L_1 generate a subspace of L_2 invariant under ϕ. Since ϕ acts irreducibly on L_2 and P is non-abelian, the elements $[x, y]$ span L_2.

Let K_p^* be the algebraic closure of K_p and let $L^* = L^*_1 \oplus L^*_2$ be the corresponding Lie ring over K_p^*. Since the characteristic polynomial of $\bar{\phi}$ on L_1 is irreducible, its characteristic roots are of the form $\alpha, \alpha^p, \alpha^{p^2}, \ldots, \alpha^{p^{n-1}}$, for some element α of K_p^* of order k, where k = order of $\bar{\phi}$. Since $\bar{\phi}$ is completely reducible over K_p^*, L^*_1 has a basis $x_0, x_1, \ldots, x_{n-1}$ such that

$$x_i\bar{\phi} = \alpha^{p^i}x_i, \qquad i = 0, 1, 2, \ldots, n-1. \tag{2}$$

Using (1) and (2) we see that

$$[x_0, x_i]\phi = [x_0\bar{\phi}, x_i\bar{\phi}] = [\alpha x_0, \alpha^{p^i}x_i] = \alpha^{1+p^i}[x_0, x_i].$$

If $[x_0, x_i] = 0$ for all i, $\dim Z(L^*) > n$, where $Z(L^*)$ denotes the centre of L^*. But then $\dim Z(L) > n$, contrary to the fact that $o(C) = p^n$. Hence $[x_0, x_i] \neq 0$ for some i. Since ϕ has the same characteristic roots as $\bar{\phi}$, we conclude that $\alpha^{1+p^i} = \alpha^{p^j}$ for some j and hence

$$1 + p^i \equiv p^j (\text{mod } k) \text{ with } 0 < i \leqq n-1, 0 \leqq j \leqq n-1 \tag{3}$$

which is clearly impossible if $k > p^{n-1}$. Since $\bar{\phi}$ acts irreducibly on $\bar{P}$, $(k, p) = 1$, so that in fact $k < p^{n-1}$.

We also require some additional properties of ϕ-groups which we shall need later in the paper as well as in the present section.

LEMMA 1.2. *Let P be an elementary abelian regular ϕ-group of order p^n and of ϕ-index r, and assume $P = P_1 \times P_2$, where $P_i \neq 1$ and P_i is invariant under ϕ, $i = 1, 2$. If ϕ has order k_i on P_i, then $k_1 \neq k_2$. Furthermore, if ϕ^r leaves only the identity element of P_1 fixed, then $k_1 \nmid k_2$.*

Proof. Assume $k_1 \mid k_2$ and ϕ^r leaves only the identity element of P_1 fixed. Thus ϕ has order k_2 on P, and we may assume $r | k_2$. Let $x = x_1x_2$ with $x_i \in P_i$, $i = 1, 2$ be a ϕ-generator of P of ϕ-index r. Now $[x_2]_r^j = 1$ if and only if k_2/r divides j. Since $[x]_r^j = [x_1]_r^j[x_2]_r^j$, $z = [x_1]_r^{k_2/r}$ must be a ϕ-generator of P_1. Since ϕ^r leaves only the identity element of P_1 fixed and $k_1 \mid k_2$, $z = 1$ and hence $P_1 = 1$, a contradiction.

If $k_1 = k_2$, we need only show that ϕ^r has no non-trivial fixed elements on P_1. In the contrary case, ϕ^r leaves some subgroup $F_1 \neq 1$ of P_1 fixed. If $F_1 = P_1$, $r = k_1$ and ϕ^r is the identity on P whence every element of P is of the form $\phi^i(x^j)$. But this implies that ϕ acts irreducibly on P, which is not the case. On the other hand, if $F_1 \subset P_1$, set $\bar{P} = P/F_1 = \bar{P}_1 \times \bar{P}_2$. Since P is elementary abelian, F_1 is the ϕ-nucleus of P, so that by (1c) $\bar{\phi}^r$ leaves only the identity element of $\bar{P}$ fixed, and we reach a contradiction as in the preceding paragraph.

LEMMA 1.3. *Let P be a regular ϕ-group of order p^m, p a prime, and let F be the*

ϕ-nucleus of P. Then $P = HK$, where H,K are ϕ-invariant subgroups of P satisfying the following conditions:

(a) *H and K commute elementwise;*
(b) *$H \supseteq F$ and $H \cap K = \Omega_1(F)$;*
(c) *if ϕ has order k_1 on $\Omega_1(F)$, then ϕ has order k_1p^e on H for some e;*
(d) *the image of ϕ^{k_1} on $K/\Omega_1(F)$ leaves only the identity fixed;*
(e) *either $H = F$ or K is elementary abelian.*

Proof. We first show that (e) is a consequence of the remaining conditions. Set $F_1 = \Omega_1(F)$. Since $\tilde{K} = K/F_1$ is elementary abelian, it follows from (d), if K is abelian, that K is elementary abelian. Suppose K is non-abelian, and let k be the order of $\tilde{\phi}$ on $\tilde{K}$. Let $x, y \in K$ be such that $[x, y] = z \neq 1$. Applying ϕ^k, it follows at once that $\phi^k(z) = z$. Since $z \in F_1$ and ϕ acts irreducibly on F_1, we conclude that $k_1 \mid k$.

Assume now that (e) is false, in which case $H \supset F$ and K is non-abelian. Then $\bar{P} = P/F = \bar{H} \times \bar{K}$, where $\bar{\phi}$ leaves each factor invariant, has order k_1 on $\bar{H}$, and k on $\bar{K}$. If P is of ϕ-index r, $\bar{\phi}^r$ leaves only the identity element of $\bar{P}$ fixed. But then by Lemma 1.2, $k_1 \nmid k$, a contradiction.

Now let $\tilde{P} = P/F_1$. If $\tilde{F} = 1$, then $\tilde{P}$ is elementary abelian, $\tilde{\phi}$ has order prime to p on $\tilde{P}$, and $\tilde{\phi}^r$ leaves only the identity element of $\tilde{P}$ fixed. It follows therefore from Lemma 1.2 that $\tilde{P} = \tilde{H} \times \tilde{K}$, where each factor is $\tilde{\phi}$-invariant, either $\tilde{H} = 1$ or $\tilde{\phi}$ has order k_1 on $\tilde{H}$, and $\tilde{\phi}^{k_1}$ leaves only the identity element of $\tilde{K}$ fixed. If H, K are the inverse images of $\tilde{H}, \tilde{K}$ respectively, then ϕ has order $h = k_1p^e$ on H and $H \cap K = F_1$. But then $\tilde{\phi}^h$ leaves only the identity element of $\tilde{K}$ fixed, and it follows that the elements $y^{-1}\phi^h(y)$, $y \in K$, include a set of coset representatives of F_1 in K. If $y \in K$, $x \in H$, then $yxy^{-1} = x' \in H$. Applying ϕ^h to this relation, we readily conclude that $y^{-1}\phi^h(y)$ centralizes H for all y in K. Since $F_1 \subseteq Z(P)$, it follows at once that H, K commute elementwise. Thus the lemma holds if $F = 1$.

If $\tilde{F} \neq 1$, then by induction $\tilde{P} = \tilde{H}\tilde{K}$, where $\tilde{H},\tilde{K}$ satisfy the conditions of the lemma. Hence, if H denotes the inverse image of $\tilde{H}$ in P, then ϕ has order k_1p^e on H. Let K_1 be the inverse image of K in P. Then $K_1 \cap F = \Omega_2(F)$. If $K_1 \subset P$, it follows again by induction that $K_1 = \Omega_2(F)K$, where $\Omega_2(F) \cap K = F_1$ and K is ϕ-invariant. Thus $P = HK$, and $H \cap K = F_1$. Since $\tilde{\phi}^{k_1}$ leaves only the identity element of K/F_1 fixed, it follows as in the preceding case that H and K commute elementwise.

Suppose finally that $K_1 = P$. Then again as in the case $\tilde{F} = 1$, it follows that $F \subseteq Z(P)$. But then $\mathrm{cl}(P) \leqslant 2$ and $[P, P] \subseteq F_1$. Thus $\tilde{P} = P/F_1 = \tilde{F} \times \tilde{K}$, where each factor is $\tilde{\phi}$-invariant. The lemma now follows with $H = F$ and K the inverse image of $\tilde{K}$.

LEMMA 1.4. *Under the assumptions of the preceding lemma, if p is odd and F is abelian on at most two generators, then H is abelian.*

Proof. By induction $\tilde{H} = H/F_1$ is abelian. If $\tilde{H}$ is cyclic, H is clearly abelian.

If $\tilde{H}$ is of type (p^m, p^m) then $[H, H]$ is cyclic and contained in F_1. But in this case $o(F_1) = p^2$, and since ϕ acts irreducibly on H, it follows that $[H, H] = 1$. Hence H is abelian. Thus $\tilde{H} = \tilde{F} \times \tilde{H}_1$, $\tilde{H}_1 \neq 1$. If $\tilde{F}_1 = \Omega_1(\tilde{F})$, then $\tilde{\phi}$ either has order k_1 or $k_1 p$ on $\tilde{F}_1\tilde{H}_1$. Since $\tilde{H}_1 \neq 1$, $\tilde{\phi}$ cannot have order k_1 on $\tilde{F}\tilde{H}_1$ by Lemma 1.2. For the same reason $o(\tilde{H}_1) = o(\tilde{F}_1)$. In particular, it follows that $\bar{\phi}$ has the same characteristic polynomial on $\bar{H} = H/F$ as ϕ has on F_1.

If H is non-abelian, we consider the Lie ring L associated with H; L is represented as the direct sum of two additive groups L_1, L_2 with $L_1 \cong F_1$ and $L_2 \cong \bar{H}$. It follows now as in Lemma 1.1 that

$$1 + p^i \equiv p^j \pmod{k_1} \tag{4}$$

with $0 \leqq i, j \leqq n - 1$, where $o(F_1) = p^n$. We note that in this case $i = 0$ is possible. The only solution of this congruence is $n = 2, i = 0, j = 1$, whence $k_1 = p - 2$. But $k_1 | p^2 - 1$ and hence $k_1 = 3$. On the other hand, F_1 has ϕ-index 0 and hence $k_1 p = 3p > o(F_1) = p^2$, which is impossible unless $p = 2$.

LEMMA 1.5. *If P is a regular ϕ-group of order p^m, then* $\mathrm{cl}(P) \leqq 2$. *Furthermore if p is odd, the ϕ-nucleus F of P is contained in $Z(P)$.*

Proof. If F is elementary abelian, $\mathrm{cl}(P) \leqq 2$ since then $F \subseteq Z(P)$ and P/F is elementary abelian. Hence we may assume that F is abelian on at most two generators. If p is odd, it follows at once from the preceding two lemmas that $F \subseteq Z(P)$. Since P/F is elementary abelian, $\mathrm{cl}(P) \leqq 2$. On the other hand, if $p = 2$, write $P = HK$, where H, K satisfy the conditions of Lemma 1.3. Since H, K commute elementwise, it suffices to prove $\mathrm{cl}(H) \leqq 2$. Now ϕ has order $3 \cdot 2^e$ on H for some e, and hence $\phi_1 = \phi^{2e}$ is an automorphism of H of order 3 leaving only the identity element fixed. But then a result of Neumann **(8)** implies that $\mathrm{cl}(H) \leqq 2$.

LEMMA 1.6. *Let P be a regular ϕ-group of order p^m with ϕ-nucleus F. If P contains a ϕ-invariant abelian subgroup P_1 such that $P_1 \cap F = 1$, then $P_1 \subseteq Z(P)$.*

Proof. Write $P = HK$, where H, K satisfy the conditions of Lemma 1.3. It follows as in the proof of Lemma 1.4 that H contains no ϕ-invariant subgroups disjoint from F and hence $P_1 \subseteq K$. Without loss we may assume $K = P$. In particular, $F = \Omega_1(F)$. We can decompose $\bar{P} = P/F$ into the direct product of minimal $\bar{\phi}$-invariant subgroups $\bar{P}_i$, $i = 1, 2, \ldots, t$. The lemma follows at once by induction if $t > 2$. If $t = 1$, then $P = FP_1$ is abelian; so we may assume that $t = 2$ and that the inverse image of $\bar{P}_1 = F \times P_1$. Let h_i be the order of $\bar{\phi}$ on $\bar{P}_i$ and k_1 the order of ϕ on F. By Lemma 1.2 $h_1 \nmid h_2$; and by the same lemma $h_1 \nmid k_1$. Hence there exists an integer w not divisible by k_1 such that $\phi_1 = \phi^w$ acts trivially on the inverse image P_2 of $\bar{P}_2$ in P. Now if $x_i \in P_i$, $i = 1, 2$, then $[x_1, x_2] = z \in F$. Applying ϕ_1 to this relation, we conclude that P_2 centralizes all elements of P_1 of the form $x_1^{-1}\phi_1(x_1)$. Since ϕ acts irreducibly on P_1 and ϕ_1 is not trivial on P_1, P_1 centralizes P_2 and hence $P_1 \subseteq Z(P)$.

THEOREM 1. *Let P be a regular ϕ-group of order p^n, p odd, and of ϕ-index r, and assume that ϕ^r leaves only the identity element of P fixed. Then P is abelian.*

Proof. Let F be the ϕ-nucleus of P, and assume first that F is elementary abelian, in which case $F \subseteq Z(P)$. By (1b), ϕ acts irreducibly on F and by (1d) $k \mid rs$, where k is the order of ϕ on F and rs is the ϕ-index of F. Thus every element of F is of the form $\phi^i(x^j)$. If the elements x^j, $0 < j \leqq p-1$ lie in d distinct orbits of ϕ, then clearly $d|p - 1$. Since each of these orbits contains k elements, it follows, if $o(F) = p^m$, that

$$k = (p^m - 1)/d, \text{ and } d|p - 1. \tag{5}$$

Let $\bar{P}_i$ be the minimal $\bar{\phi}$-invariant subgroups of $\bar{P} = P/F$, $i = 1, 2, \ldots, t$, and let P_i be the inverse image of $\bar{P}_i$ in P. Denote by k_i the order of $\bar{\phi}$ on $\bar{P}_i$. Suppose first that some P_i is elementary abelian and that the order h_i of ϕ on P_i is relatively prime to p. Then $P_i = F \times K_i$, where K_i is ϕ-invariant. By Lemma 1.6 $K_i \subseteq Z(P)$. By induction P/K_i is abelian and hence $[P, P] \subseteq F \cap K_i = 1$. Thus P is abelian. On the other hand, if P_i is elementary abelian and $p \mid h_i$ or if P_i is abelian, but not elementary abelian, it is easy to see that $k_i = k$. Hence we may suppose that for each i either P_i is non-abelian or $k_i = k$.

If some P_i, say P_1, were non-abelian, then for suitable x_1, x_2 in P_1, $[x_1, x_2] = z \neq 1$ in F. Applying ϕ^{k_1} to this relation we conclude readily that $\phi^{k_1}(z) = z$ and hence that $k \mid k_1$. It follows that for any abelian P_i $k_i \mid k_1$, and this is impossible by Lemma 1.2. Thus either all P_i are non-abelian or all P_i are abelian. In the latter case we must have $t = 1$, since otherwise $k_1 = k_2 = k$, contrary to Lemma 1.2. Thus we may suppose that all P_i are non-abelian. Furthermore, it follows as in Lemma 1.2 that ϕ must have order $k_i p$ on P_i for some i, say $i = 1$.

Let $o(\bar{P}_1) = p^n$ and let $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$ be a basis for $\bar{P}_1$ such that

$$\bar{\phi}(\bar{y}_i) = \bar{y}_{i+1}, \; i = 1, 2, \ldots, n-1$$

and

$$\bar{\phi}(\bar{y}_n) = \bar{y}_1^{c_1}\bar{y}_2^{c_2} \ldots \bar{y}_n^{c_n}.$$

Regarding $\bar{\phi}$ as a linear transformation, its characteristic polynomial $\bar{f}(X)$ is given by

$$\bar{f}(X) = X^n - c_n X^{n-1} - \ldots - c_2 X - c_1. \tag{6}$$

Choose representative y_i of $\bar{y}_i$ such that $\phi(y_1) = y_{i+1}$, $i = 1, 2, \ldots, n - 1$ and $\phi(y_n) = z_0 y_1^{c_1} y_2^{c_2} \ldots y_n^{c_n}$, $z_0 \in F$.

Now $\phi^{k_1}(y_1) = zy_1$, where $z \neq 1$ in F since ϕ has order $k_1 p$ on P_1. Applying ϕ^i to this equation we find that

$$\phi^{k_1}(y_1) = \phi^{i-1}(z) y_i \qquad i = 1, 2, \ldots, n. \tag{7}$$

In particular, for $i = n$, and using (7), we obtain

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$$\phi^n(z)\phi^n(y_1) = \phi^n(\phi^{k_1}(y_1)) = \phi^{k_1}(\phi^n(y_1))$$
$$= \phi^{k_1}(z_0 y_1{}^{c_1} y_2{}^{c_2} \ldots y_n{}^{c_n}) = z^{c_1}\phi(z^{c_2}) \ldots \phi^{n-1}(z^{c_n})\phi^n(y_1),$$

whence

$$\phi^n(z) = z^{c_1}\phi(z^{c_2}) \ldots \phi^{n-1}(z^{c_n}). \tag{8}$$

If $f(X)$ denotes the characteristic polynomial of ϕ on F, it follows from (6) and (8) and the irreducibility of ϕ on F that $f(X) \mid \bar{f}(X)$. But $\bar{\phi}$ acts irreducibly on $\bar{P}_1$ and so $\bar{f}(X)$ is irreducible over the integers mod p. It follows at once that $f(X) = \bar{f}(X)$ and that $m = n$ and $k = k_1$. Lemma 1.1. now implies that $k < p^{n-1}$ in contradiction to (5).

Suppose finally that F is not elementary abelian. Then $P = HK$, where H, K satisfy the conditions of Lemma 1.3. If P is non-abelian, then K must be non-abelian by Lemma 1.4 since p is odd, and it follows as in the first part of the proof that the order k of ϕ on F_1 divides the order of $\tilde{\phi}$ on $\tilde{K} = K/F_1$. But if $\tilde{P} = P/F_1$, $\tilde{\phi}^r$ leaves only the identity element of $\Omega_1(\tilde{F})$ fixed, $\tilde{\phi}$ has order k on $\Omega_1(\tilde{F})$, and $\Omega_1(\tilde{F})$ centralizes $\tilde{K}$. This contradicts Lemma 1.2.

We remark that the assumption $p \neq 2$ was used only in the case F abelian of type (p^e, p^e), $e > 1$. Thus Theorem 1 holds without restriction on p if F is elementary abelian.

We conclude this section with one further result on ϕ-groups which we shall need.

LEMMA 1.7. *Let P be an elementary abelian ϕ-group of order p^{2n}, and assume ϕ has order $p^n + 1$. Then $p = 2$ and $n = 1$.*

Proof. Our conditions imply that ϕ acts irreducibly on P. Let g be a ϕ-generator of P of ϕ-index r, and suppose first that $h \nmid r$, where $h = p^n + 1$. We may assume $r \mid h$. Since ϕ is irreducible on P, ϕ^r leaves only the identity element of P fixed, and hence $[g]_r^{h/r} = 1$. Since P is a ϕ-group, this implies $h^2/r > o(P)$, whence

$$(p^n + 1)^2 > rp^{2n}. \tag{9}$$

(9) implies that $r = 1$ if p is odd and that $r \leqq 2$ if $p = 2$. But h is odd if $p = 2$ and since $r \mid h$, we conclude that $r = 1$ for all p. Suppose first that p is odd. Then for $s > h/2$ we have $[g]_1^s = [g]_1^h[\phi^s(g)\phi^{s+1}(g) \ldots \phi^{h-1}(g)]^{-1}$ whence

$$[g]_1^s = \phi^s([g^{-1}]_1^{h-s}). \tag{10}$$

$\phi^{\frac{1}{2}h}$ is an automorphism of P of order 2 without non-trivial fixed elements, and hence $\phi^{\frac{1}{2}h}(x) = x^{-1}$ for all x in P. It follows at once from (10) that $[g]_1^s = \phi^{s+h/2}([g]_1^{h-s})$, and consequently the elements $[g]_1^j$ lie in at most $\frac{1}{2}h$ distinct orbits. Thus $\frac{1}{2}h \cdot h > o(P)$, and consequently $(p^n + 1)^2 > 2p^{2n}$, which is impossible.

If $p = 2$, it follows as in (10), since $g^{-1} = g$, that $[g]_1^s = \phi^s([g]_1^{h-s})$. The non-identity elements of P thus lie in at most $\frac{1}{2}(h - 1) = 2^{n-1}$ orbits, and consequently $(2^n + 1)2^{n-1} \geqq 2^{2n} - 1$, which implies $n = 1$.

On the other hand, if $h \mid r$, P is of ϕ-index 0, whence $(p^n + 1)p > p^{2n}$, which implies $n = 1$. However, if p is odd, $\phi^{h/2}(g) = g^{-1}$ and hence the elements $\phi^i(g^j)$ lie in at most $1 + \frac{1}{2}(p - 1)$ orbits, and we obtain the stronger inequality $\frac{1}{2}(p^n + 1)(p + 1) > p^{2n}$, which is impossible.

2. Some preliminary lemmas.

We begin with several lemmas.

LEMMA 2.1. *If a group G admits an automorphism ϕ which leaves a normal abelian subgroup H of G elementwise fixed and is such that the image of ϕ on G/H is without non-trivial fixed elements, then $H \subseteq Z(G)$.*

Proof. If $x \in G$, $z \in H$, we have

$$xzx^{-1} = z',\ z' \in H. \tag{11}$$

Applying ϕ yields $\phi(x)z\phi(x^{-1}) = z'$, which together with (11) implies $x^{-1}\phi(x) \in C(H)$ for all x in G. Since H is abelian, $x^{-1}\phi(x)y \in C(H)$ for all x in G, all y in H.

If $g \in G$, its image $\bar{g}$ in $\bar{G} = G/H$ is of the form $\bar{x}^{-1}\bar{\phi}(\bar{x})$, $\bar{x} \in \bar{G}$, since $\bar{\phi}$ leaves only the identity element of $\bar{G}$ fixed. Thus $g = x^{-1}\phi(x)y$ for suitable elements x in G, y in H. Thus $H \subseteq Z(G)$, as asserted.

LEMMA 2.2. *If G is assumed to be abelian in Lemma 2.1, then G contains a subgroup K invariant under ϕ such that $G = H \times K$.*

Proof. Let $\theta(x) = x^{-1}\phi(x)$. Since G is abelian, θ is an endomorphism of G, whence by Fitting's lemma, $G = H_1 \times K$ where θ is nilpotent on H_1 and an automorphism on K. Since $\theta(x) = 1$ if $x \in H$, $H \subseteq H_1$. If $\bar{\theta}$ denotes the image of θ on $\bar{G} = G/H = \bar{H}_1 \times \bar{K}$, our hypotheses imply that $\bar{\theta}$ is an automorphism on $\bar{G}$. Since $\bar{\theta}$ is nilpotent on $\bar{H}_1$, necessarily $\bar{H}_1 = 1$ and hence $H_1 = H$. Since $\phi(x) = x\theta(x)$, $x \in K$ implies $\phi(x) \in K$, whence K is invariant under ϕ.

LEMMA 2.3. *Let A be a cyclic subgroup of a group G such that $N(A) = A$ and for any subgroup A_0 of A, $A_0 \subseteq Z(N(A_0))$. Assume further than G contains a normal subgroup H such that $A \cap H \subseteq Z(H)$. Then if $\bar{G} = G/H$ and $\bar{A}$ denotes the image of A in $\bar{G}$, we have $N(\bar{A}) = \bar{A}$.*

Proof. Let $A = (a)$ be of order h, and let $A \cap H = (a^r)$ with $r \mid h$. If x is a representative in G of $\bar{x}$ in $N(\bar{A})$, then we have

$$xax^{-1} = a^{\lambda}z \text{ for some integer } \lambda \text{ and some } z \text{ in } H. \tag{12}$$

Since $H \lhd G$, $x^{-1}zx = y$, y in H, and hence

$$a^{-1}xa = a^{\lambda-1}xy. \tag{13}$$

Let $K = AH$. Since A is Abelian, $A \cap H$ is in the centre of K. Set $K' = K/A \cap H$ and let $A' = (a')$, H', y' be the residues of A, H, y in K'. Clearly $N_{K'}(A') = A'$, $H' \lhd K'$, $K' = A'H'$, and $A' \cap H' = 1$. If ϕ' denotes the automorphism of H' induced by conjugation by a', ϕ' leaves only the identity

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element of H' fixed. Hence there exists an element t' in H' such that $\phi'(t') = y'^{-1}t'$. If t is a representative of t' in H, we conclude that

(14) $$a^{-1}ta = y^{-1}ta^{rd} \text{ for some integer } d.$$

We now obtain from (13) and (14)

$$a^{-1}xta = (a^{-1}xa)(a^{-1}ta) = (a^{\lambda-1}xy)(y^{-1}ta^{rd}),$$

whence

(15) $$(xt)^{-1}a^{\lambda}(xt) = a^{1-rd}.$$

By hypothesis (a^{λ}) lies in the centre of its normalizer, and consequently (15) implies $\lambda \equiv 1 \pmod r$. Thus $a^{\lambda-1} \in A \cap H$.

On the other hand, as in the derivation of (14) there is an element t_1 in H such that $a^{-1}t_1a = a^{rc}t_1z^{-1}$ for some integer c. Thus $a^{-1}t_1xa = (a^{-1}t_1a)\ (a^{-1}xa) = (a^{rc}t_1z^{-1})\ (a^{\lambda-1}xz)$. Since by hypothesis $A \cap H \subseteq Z(H)$ and $a^{\lambda-1} \in A \cap H$, it follows that $t_1xax^{-1}t_1^{-1} = a^{rc+\lambda}$. Hence $t_1x \in N(A) = A$, whence $x \in A$. Thus $N(A) = A$, as asserted.

If $A \cap H = 1$, $r = 0$ and (15) implies that $xt \in N(A)$, giving $\bar{x} \in \bar{A}$ and $N(\bar{A}) = \bar{A}$ at once. We thus have the following corollary.

Corollary. *Let A be a cyclic subgroup of a group G such that $N(A) = A$. If G contains a normal subgroup H such that $A \cap H = 1$ and $\bar{G} = G/H$, then $N(\bar{A}) = \bar{A}$, where $\bar{A}$ denotes the image of A in $\bar{G}$.*

We shall also need some properties of automorphisms of an extra-special p-group P, as defined by Hall and Higman in **(7)**. In their paper the only automorphisms ϕ of P which are considered are of order a power of a prime $q \neq p$ and the holomorph of P and ϕ is represented on a vector space V over the field with q elements. Many of their results can be carried through if ϕ has arbitrary order prime to p and if the representations of the holomorph of ϕ and P are taken in the complex numbers. In particular, the following lemma holds:

Lemma 2.4. *Let P be an extra-special p-group of order p^m and assume that P admits an automorphism ϕ of order k prime to p which acts trivially on $Z(P)$ and such that the image $\bar{\phi}$ of ϕ on $\bar{P} = P/Z(P)$ acts irreducibly. Then $k \leqq p^{\frac{1}{2}(m-1)} + 1$.*

We shall need one other similar result.

Lemma 2.5. *Let P be an extra-special p-group of order p^m and assume that P admits an automorphism ϕ of order k prime to p which acts trivially on $Z(P)$ and assume that $\bar{P} = \bar{P}_1 \times \bar{P}_2$, that $\bar{\phi}$ leaves $\bar{P}_i$ invariant and acts irreducibly on $\bar{P}_i$ and that $\bar{\phi}$ has the same minimal polynomial on $\bar{P}_i$, $i = 1, 2$. Then $k \leqq p^{\frac{1}{2}(m-3)} + 1$.*

Proof. We proceed as in Lemma 1.1 and consider the Lie ring $L = L_1 \oplus L_2$ associated with P over the field K_p with p elements and its extension $L^* = L^*_1 \oplus L^*_2$ over the algebraic closure K_p^* of K_p. Now $L_2 \cong \bar{P}$, and since $\bar{\phi}$

has the same characteristic polynomial on $\bar{P}_1$ and $\bar{P}_2$, it follows as in Lemma 1.1 that we can find a basis $x_1, \ldots, x_{2^n}$ of L_2 such that

$$x_i\bar{\phi} = \alpha^{p^i}x_i \text{ and } x_{n+i}\bar{\phi} = \alpha^{p^i}x_{n+i}, \qquad i = 1, 2, \ldots, n,$$

where $n = \frac{1}{2}(m-1)$ and α is a primitive kth root of unity in K_p^*.

Now for some x_i, x_j, $[x_i, x_j] = z \neq 0$ in L_1; and it follows that

$$z\phi = \alpha^{p^a+p^b}z$$

where a, b, denote the residues of $i, j \pmod{n}$. Since ϕ acts trivially on L_1,

$$\alpha^{p^a+p^b} = 1.$$

Thus $k \mid (1 + p^c)$, where $c \leqq n - 1$, and the lemma follows.

3. Applications to ABA-groups. We shall now apply the results of the preceding sections to obtain our first structure theorem for ABA-groups. We begin with the following lemma:

LEMMA 3.1. *Let $G = ABA$ and assume $G = AP$, where $P \lhd G$, $o(P) = p^m$, $p \geqq 5$, $A \cap P = Z(P)$ and $o(A \cap P) = p$. Then $G = A$.*

Proof. Set $\bar{G} = G/A \cap P = \bar{A}\bar{B}\bar{A} = \bar{A}\bar{P}$ so that $\bar{P} \lhd \bar{G}$ and $\bar{A} \cap \bar{P} = 1$. Since clearly $N(\bar{A}) = \bar{A}$, $\bar{P}$ is a regular $\bar{\phi}$-group where $\bar{\phi}$ is the image of ϕ on $\bar{P}$ and we may assume $\bar{P} \neq 1$. Let $\bar{F}$ be the $\bar{\phi}$-nucleus of $\bar{P}$.

Assume first that $\bar{F} = 1$, in which case $\bar{P}$ is the direct product of minimal $\bar{\phi}$-invariant subgroups $\bar{P}_i$, $i = 1, 2, \ldots, t$, on each of which $\bar{\phi}$ has order k_i prime to p. By Lemma 1.2, $k_1 \nmid k_i$, $i > 1$. Let P_1 be the inverse image of $\bar{P}_i$ in P, and assume $t > 1$. It follows by induction that $Z(P_i) \supset A \cap P$ and hence that each P_i is abelian. If $x_1 \in P_1$ and $x_i \in P_i$, $i > 1$, then $[x_1, x_i] = z \in A \cap P$. Now $\phi_1 = \phi^{pk_i}$ acts trivially on P_i, and hence if we apply ϕ_1 to this relation, we readily conclude that $x_1^{-1}\phi_1(x_1) \in C(x_i)$. Since x_1, x_i are arbitrary, and $k_1 \nmid k_i$, it follows that P_1 centralizes P_i for all i. Thus $P_1 \subseteq Z(P)$, a contradiction. Hence $t = 1$.

Let $A = A'A_p$, where A' has order k prime to p. We may assume that no non-trivial subgroup of A' is normal in G, since otherwise the lemma follows by induction. Hence $k = k_1$. Now P is an extra-special p-group, A' centralizes $Z(P)$, and $\bar{A}'$ acts irreducibly on $\bar{P}$. It follows therefore from Lemma 2.4 that

$$k \leqq p^n + 1, \tag{16}$$

where $n = \frac{1}{2}(m-1)$.

If $r = \bar{\phi}$-index of $\bar{P}$, $\bar{\phi}^r$ leaves only the identity element of $\bar{P}$ fixed, and hence $k^2/r > p^{2n}$. Since $k \mid (p^{2n} - 1)$, it follows therefore from (16) that $k = p^n + 1$. But then by Lemma 1.7, $p = 2$, contrary to hypothesis.

The same argument applies if $\bar{P}$ is elementary and the order k of $\bar{\phi}$ on $\bar{P}$ is prime to p, but $\bar{F} \neq 1$. In this case we conclude that $\bar{P} = \bar{F}$. Since $\bar{\phi}$ acts

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irreducibly on $\bar{F}$, we again obtain (16). Since $\bar{F}$ is of $\bar{\phi}$-index 0, $kp > p^{2n}$, which together with (16) implies $n = 2$ and $k = p + 1$. This yields a contradiction as above.

In the general case, let F be the inverse image of $\bar{F}$ in P. Since $\bar{F} \subseteq Z(\bar{P})$ and $o(Z(P)) = p, \mho^1(F) \subseteq Z(P)$, whence $\mho^1(F) = A \cap P$, and it follows that $\bar{F}$ is elementary abelian. Furthermore, we may assume that the image $\tilde{\phi}$ of $\bar{\phi}$ on $\tilde{P} = \bar{P}/\bar{F}$ acts irreducibly on P; otherwise the lemma follows readily by induction. Also F is abelian by induction.

The case $\bar{P}$ elementary abelian and $\bar{\phi}$ of order prime to p has already been considered; hence if $k_1 =$ order of $\bar{\phi}$ on $\bar{F}$ and $k_2 =$ order of $\tilde{\phi}$ on $\tilde{P}$, we must have $k_1 \mid k_2$. Furthermore, the order h of ϕ on F is either k_1 or $k_1 p$. If $x \in F$ and $y \in P$, then $yxy^{-1} \in F$ and consequently $\phi^h(yxy^{-1}) = yxy^{-1}$. But then $y^{-1}\phi^h(y) \in C(x)$ for all y in P. If $k_1 < k_2$, the elements $\tilde{y}^{-1}\phi^h(\tilde{y})$ generate $\tilde{P}$ and hence $x \in Z(P)$, a contradiction. We conclude that $k_1 = k_2$.

Since $\tilde{\phi}^r$ leaves only the identity element of $\tilde{P}$ fixed, $r \nmid k_2$ and therefore $\bar{\phi}^r$ leaves only the identity element of $\bar{P}$ fixed. Hence by Theorem 1 $\bar{P}$ is abelian. But then $\mho^1(P) \subseteq Z(P)$, whence $\bar{P}$ is elementary abelian. Thus P is an extra-special group and $\bar{\phi}$ has order $k_1 p$ on $\bar{P}$. But then $A' P$ satisfies the conditions of Lemma 2.5, and hence

$$(17) \qquad k_1 \leqq p^{n-1} + 1, \text{ where } p^n = o(\bar{F}).$$

On the other hand, since $\bar{F}$ is of $\bar{\phi}$-index 0, we must have $(p^n - 1)/(p - 1) | k_1$, which together with (17) implies that either $n = 1$ and $k_1 = 2$ or $n = 2$ and $k_1 = p + 1$. If $n = 2$, Lemma 1.7 shows that $p = 2$, contrary to assumption. If $n = 1$, $\tilde{\phi}$ has order 2 on $\tilde{P}$ and $o(\tilde{P}) = p$. Since $\tilde{\phi}^r$ leaves only the identity element of $\tilde{P}$ fixed, we may assume $r = 1$. If $\tilde{y}$ is a $\tilde{\phi}$-generator of $\tilde{P}$, then every element of $\tilde{P}$ must be of the form $\tilde{\phi}^i([\tilde{y}]_1^j)$. But the only elements of this form are $1, \tilde{y}, \tilde{y}^{-1}$ since $\tilde{\phi}$ has order 2. Thus $p = 3$, contrary to assumption.

We shall now prove the following theorem.

THEOREM 2. *Let $G = ABA$ and assume that G contains a normal subgroup P of order p^m, $p \geqq 5$, such that $G = AP$. Then the commutator subgroup of G is a unique normal complement of A in G.*

Proof. The proof will be by induction on $o(G)$. Let P_1 be a minimal subgroup of the centre of P normal in G. Thus either $P_1 \subset A$ or $P_1 \cap A = 1$. If $\bar{G} = G/P_1 = \bar{A}\bar{B}\bar{A} = \bar{A}\bar{P}$, $N(\bar{A}) = \bar{A}$ by the corollary of Lemma 2.3 in case $P_1 \cap A = 1$. The same conclusion clearly holds if $P_1 \subset A$. Hence by induction $\bar{G} = \bar{A}\bar{P}^*$, where $\bar{P}^* \lhd \bar{G}$, $\bar{P}^* \cap \bar{A} = 1$, and $\bar{P}^* = [\bar{G}, \bar{G}]$. If $P_1 \cap A = 1$, the inverse image P^* of $\bar{P}^*$ is a normal complement for A in G. Clearly $P^* \supseteq [G, G]$. On the other hand, if $x \in P^*$, $axa^{-1}x^{-1} = \phi(x)x^{-1}$. Since $N(A) = A$, ϕ leaves only the identity element of P^* fixed, and hence the elements $\phi(x)x^{-1}$ exhaust P^*. Thus $P^* = [G, G]$.

We may therefore suppose that $P_1 \subset A$ and that P contains no subgroup $\neq 1$ which is normal in G and disjoint from A. In this case we have $G = AP^*$,

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with $P^* \lhd G$, $A \cap P^* = P_1$ cyclic of order p, and $P_1 \subseteq Z(P^*)$. It follows from Lemma 2.2 that $Z(P^*) = P_1 \times P_2$ where $P_2 \cap A = 1$ and P_2 is invariant under A, whence normal in G. Thus $P_2 = 1$ and $P_1 = Z(P^*)$. The hypothesis of Lemma 3.1 is satisfied so that $G = A$, and the theorem is proved.

4. ABA-groups associated with the primes $p = 2$ and 3. To complete the description of ABA-groups G of the form AP with $P \lhd G$, and $N(A) = A$, we consider finally the case in which P is a 2-group or 3-group. We begin with the following lemma.

LEMMA 4.1. *Let $G = ABA = AP$, where P is a 2-group normal in G. Then P contains at most one A-invariant abelian subgroup of type* $(2, 2)$. *Furthermore any subgroup of A which is normal in G is in the centre of G.*

Proof. If K is an A-invariant abelian subgroup of type $(2, 2)$, no proper subgroup of K can be invariant under A, for otherwise we clearly have $N(A) \supset A$. Hence if P_1 denotes a minimal A-invariant subgroup of $Z(P)$, either $P_1 \cap K = 1$ or $P_1 = K$. Let $\bar{G} = G/P_1 = \bar{A}\bar{P}$. If $P_1 \subset A$, $N(\bar{A}) = \bar{A}$; if $P_1 \not\subset A$, the minimality of P_1 implies that $P_1 \cap A = 1$ so that $N(\bar{A}) = \bar{A}$ by the corollary of Lemma 2.3. Hence by induction $\bar{P}$ contains at most one $\bar{A}$-invariant abelian subgroup of type $(2, 2)$. The lemma follows at once unless P_1 itself is of type $(2, 2)$. But in this case P cannot contain another such subgroup K for then $P_1K = P_1 \times K$ would be a regular ϕ-group on which ϕ has order 3, and this is impossible by Lemma 1.2.

Let $A_0 \lhd G$, $A_0 \subseteq A$. Let L be a maximal A-invariant normal subgroup of P. We may assume that $AL \subset AP$, since otherwise A_0 is in the centre of G by induction on $o(P)$. In any case A_0 is in the centre of AL by induction. If $\tilde{G} = G/L = \tilde{A}\tilde{P}$, repeated application of Lemma 2.3 shows that $N(\tilde{A}) = \tilde{A}$ and hence that the image $\tilde{\phi}$ of ϕ leaves only the identity element of $\tilde{P}$ fixed. Since $A_0 \subseteq Z(L)$, it follows as in the proof of Lemma 2.1 that $x^{-1}\phi(x)$ centralizes A_0 for all $x \in P$. But there exist a set of coset representatives of L in P of the form $x^{-1}\phi(x)$. Thus $A_0 \subseteq Z(G)$.

Our main result for $p = 2$ is the following:

THEOREM 3. *Let $G = ABA = AP$, where P is a 2-group normal in G. Then either A has a normal complement in G or P contains two subgroups T_1, T_2 normal in G such that*

(a) *$G = A(T_1 \times T_2)$;*

(b) *A does not possess a normal complement in AT_1;*

(c) *$A \cap T_2 = 1$, T_2 contains no A-invariant abelian subgroup of type* $(2, 2)$, *and furthermore T_2 contains every A-invariant subgroup of P which is disjoint from A and which contains no A-invariant abelian subgroup of type* $(2, 2)$*;*

(d) $6 \mid o(A)$.

Proof. The proof will be made by induction on $o(P)$. We add to our induction hypotheses the following assertion:

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(e) $T_1 = QQ'$, where $Q, Q' \lhd G$, ϕ has order $3 \cdot 2^s$ on Q, $A \cap Q' = 1$, and if $Q' \neq 1$, the order of ϕ on Q' is divisible by 3, but is not of the form $3 \cdot 2^s$.

We note first of all that (b) and (e) imply (d). In fact $A \cap T_1 \neq 1$ by (b), whence $2 \mid o(A)$, and it follows at once from (e) that $3 \mid o(A)$.

Let P_1 be a minimal A-invariant subgroup of the centre of P and set $\bar{G} = G/P_1 = \bar{A}\bar{P}$. As in the preceding lemma, $N(\bar{A}) = \bar{A}$. We distinguish two cases.

Case 1. P contains no subgroup normal in G disjoint from A. Thus $P_1 \subset A$. Suppose first that $\bar{A}$ has a normal complement $\bar{P}^*$ in $\bar{G}$. We may suppose $\bar{P}^* = \bar{P}$, since otherwise the theorem follows by induction. Now $\bar{P}$ is a regular $\bar{\phi}$-group. Let $\bar{F}$ be its $\bar{\phi}$-nucleus and write $\bar{P} = \bar{H}\bar{K}$, where $\bar{H}, \bar{K}$ satisfy the conditions of Lemma 1.3. Suppose first that $\bar{F}$ is elementary abelian and $o(\bar{F}) = 2^n > 4$. Let F be the inverse image of $\bar{F}$ in P. If F is non-abelian, F is an extra-special group. Since $\bar{\phi}$ acts irreducibly on $\bar{F}$, it follows as in the proof of Lemma 3.1 that $\bar{\phi}$ has order $k = 2^{\frac{1}{2}n} + 1$ on $\bar{F}$, whence $n = 2$ by Lemma 1.7. Thus F is abelian. Let H, K be the inverse image of $\bar{H}, \bar{K}$ in P. It follows now as in Lemma 1.3 that F is in the centre of K. Since P contains no A-invariant normal subgroups disjoint from A, $K \subset P$ and hence $\bar{F} \subset \bar{H}$.

Now $\bar{\phi}$ has order $2k$ on $\bar{H}$, and hence $\bar{\phi}$ has the same characteristic polynomial on $\bar{F}$ as $\tilde{\phi}$ has on $\tilde{H} = \bar{H}/\bar{F}$. By the remark following Theorem 1, $\bar{H}$ must be elementary abelian. But H is non-abelian; otherwise $F \subseteq Z(P)$. Hence H is extra-special, and we may apply Lemma 2.5 as in the proof of Lemma 3.1 to conclude that $\bar{\phi}$ has order $k = 2^{\frac{1}{2}n} + 1$ on F. Thus $n = 2$ by Lemma 1.7, a contradiction.

On the other hand, if $\bar{F} = 1$, essentially the same arg[illegible] shows that no minimal $\bar{\phi}$-invariant subgroup of $\bar{P}$ has order greater th[illegible]. It follows therefore from Lemma 1.2 that either $\bar{P} = 1$ or $o(\bar{P}) = 4$. In the first case, $G = A$ and the theorem is obvious. In the second case, P must be a quaternion group and the theorem follows with $T_1 = Q = P$, and $T_2 = 1$.

We may therefore assume that $\bar{F} \neq 1$ is abelian of type $(2^e, 2^e)$. Let $\bar{F}_1 = \Omega_1(\bar{F})$ and let F_1 be the inverse image of $\bar{F}_1$ in H. If $F_1 \subseteq Z(H)$, then again as in Lemma 1.3, $F_1 \subseteq Z(P)$, a contradiction. Thus $A \cap H \subseteq [H, H]$ and A does not possess a normal complement in AH. If we set $H = Q$, then ϕ has order $3 \cdot 2^s$ on Q for some s.

Suppose $\bar{K}$ contains a minimal $\bar{\phi}$-invariant abelian subgroup $\bar{K}_1$ disjoint from $\bar{F}_1$. Since $o(\bar{K}_1) > 4$, it follows as above that the inverse image K_1 of $\bar{K}_1$ is abelian. But then $K_1 \subseteq Z(P)$, a contradiction. Thus $\bar{F}_1 = \Omega_1(\bar{K})$. If $\bar{K} = \bar{F}_1$, the theorem follows with $T_1 = H$, $T_2 = 1$; so assume $\bar{K} \supset \bar{F}_1$. Then $\bar{K}$ is non-abelian. If $K \subset P$, it follows by induction from (e) that $Q' = [AK, AK]$ is disjoint from A. Hence the theorem holds with $T_1 = P$, $T_2 = 1$.

Assume finally that $K = P$, in which case $\bar{F} = \bar{F}_1$ and F is a quaternion group. If $x \in F$, $y \in P$, then $[x, y] = z \in A \cap F$. Applying ϕ^6 to this relation,

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we find that F centralizes all elements of P of the form $y^{-1}\phi^6(y)$, $y \in P$. Since these form a set of coset representatives of F in P, we conclude that $Q = FK_1$, where $K_1 = C(F) \cap P \lhd P$ and $K_1 \cap F = A \cap F$. But then K is abelian, a contradiction.

Case 2. $P_1 \cap A = 1$. If $\bar{A}$ has a normal complement in $\bar{G}$, A obviously has one in G. Hence we may assume by induction that $\bar{G} = \bar{A}(\bar{T}_1 \times \bar{T}_2)$, where $\bar{T}_1$, $\bar{T}_2$ satisfy the conditions of the theorem. Let H_1, H_2 be the inverse images of $\bar{T}_1$, $\bar{T}_2$ in P.

Assume first that $o(P_1) \neq 4$ and hence that H_2 contains no A-invariant abelian subgroup of type $(2, 2)$. If $\bar{T}_2 \neq 1$, it follows by induction that $H_1 = T_1 \times P_1$, where T_1 is invariant under A and again as in Lemma 1.3 T_1 and H_2 commute elementwise. Thus $G = A(T_1 \times H_2)$. Clearly T_1 satisfies (b) and (e) and H_2 contains every A-invariant subgroup of P disjoint from A and contains no A-invariant subgroup of type $(2, 2)$. The theorem follows.

On the other hand, if $\bar{T}_2 = 1$, we may assume $\bar{T}_1 = \bar{P}$. Hence $\bar{P} = \bar{Q}\bar{Q}'$, where $\bar{Q}, \bar{Q}'$ satisfy (e). Let Q_1, Q_1' be the inverse images of $\bar{Q}, \bar{Q}'$ in P. Let $\bar{K}$ be a minimal $\bar{A}$-invariant subgroup of $\bar{Q}$ and K its inverse image in P. Either $\bar{K} \subset \bar{A}$ or $\bar{K}$ is abelian of type $(2, 2)$. In the first case it follows from the minimality of P_1 that $K = P_1 \times L$, where L is A-invariant (in fact, $L \subset A$). In the second case, K is abelian and the same conclusion follows since $o(P_1) \neq 4$. Now if $y \in Q_1$ and $z \in L$, we have

$$yzy^{-1} = z'x, \text{ where } z' \in L, x \in P_1. \tag{18}$$

Applying ϕ^m to (18), where $m = 3\cdot 2^s$ = order of $\bar{\phi}$ on $\bar{Q}$, we conclude readily that $\phi^m(x) = x$ and hence that $x = 1$, since ϕ does not have order 3 on P_1 and no proper subgroup of P_1 is A-invariant. Thus $L \lhd AQ_1$. If $\tilde{A}\tilde{Q}_1 = AQ_1/L$ and $\tilde{P}_1$ denotes the image of P_1 in $\tilde{A}\tilde{Q}_1$, we conclude by induction if $\tilde{A}$ does not have a normal complement in $\tilde{A}\tilde{Q}_1$ and from Lemma 1.3 if $\tilde{A}$ has a normal complement in $\tilde{A}\tilde{Q}_1$ that $\tilde{Q}_1 = \tilde{P}_1 \times \tilde{Q}$, where $\tilde{Q}$ is invariant under $\tilde{A}$. It follows at once that $Q_1 = P_1 \times Q$, where Q is A-invariant.

Now Q_1' is a regular ϕ-group. If F is the ϕ-nucleus of Q_1', the minimality of P_1 implies that either $P_1 \subset F$ or $P_1 \cap F = 1$. In the first case we must have $P_1 = F$ since $o(P_1) \neq 4$. But then $Q_1'/P_1 = \bar{Q}'$ is elementary abelian and $\bar{\phi}$ has odd order on $\bar{Q}'$. Since $\bar{\phi}$ does not have order 3 on $\bar{\phi}'$, we conclude that $\bar{Q}'$ contains a minimal $\bar{A}$-invariant subgroup $\bar{K}$ such that $o(\bar{K}) > 4$. Since $\bar{K} \not\subset \bar{T}_2$, this contradicts (c), and hence $P_1 \cap F = 1$. But then Lemma 1.3 implies that $Q_1' = P_1 \times Q'$. Finally, if $x \in Q$, $x' \in Q'$, we have

$$[x, x'] = z \in P_1. \tag{19}$$

By (e) $\bar{\phi}$ has order $m'\cdot 2^s$ on $\bar{T}_1$, where m' = order of $\bar{\phi}$ on $\bar{Q}'$. Applying $\phi^{m'2s}$ to (19), we see that $\phi^{m'}(z) = z$. But it follows from Lemma 1.2 applied to Q_1'/F that the order of ϕ on P_1 does not divide m', and hence $z = 1$. We conclude that $G = A(T_1 \times P_1)$ where $T_1 = QQ'$ and the theorem follows.

Suppose finally that $o(P_1) = 4$. Now H_2 is a regular ϕ-group. Let F_2 be its ϕ-nucleus. If $P_1 \subseteq F_2$, then F_2 is abelian of type $(2^c, 2^c)$; and since $\bar{T}_2$ contains no $\bar{A}$-invariant abelian subgroup of type (2, 2), $P_1 = F_2$. In this case $\bar{T}_2$ is elementary abelian and $\bar{\phi}$ has odd order on $\bar{T}_2$. If K_2 denotes the maximal elementary abelian A-invariant subgroup of H_2, ϕ has odd order on K_2, since otherwise $\bar{T}_2$ would contain an $\bar{A}$-invariant abelian subgroup of type (2, 2). Hence $K_2 = P_1 \times T_2$ where T_2 is A-invariant and lies in $Z(H_2)$ by Lemma 1.6. It follows at once from the structure of H_2 that $H_2 = K_1 \times T_2$ where K_1 is A-invariant and every A-invariant subgroup of K_1 contains P_1. Furthermore, T_2 contains no A-invariant abelian subgroup of type (2, 2). On the other hand, if $P_1 \cap F_2 = 1$, this same conclusion holds with $K_1 = P_1$.

Set $T_1 = H_1 K_1$ so that $G = A(T_1 T_2)$ and $T_1 \cap T_2 = 1$. It is clear from the construction of T_2 that T_2 satisfies (c). Furthermore, $T_1 = QQ'$, where $Q'/P_1 = \bar{Q}'\bar{K}_1$. Clearly Q, Q' satisfy (e). Finally it follows as in Lemma 1.3 that T_1 and T_2 commute elementwise, and the theorem follows.

In Part II we shall need one additional property of T_1:

LEMMA 4.2. *Let $G = ABA = AT$, where $T \lhd G$, $o(T) = 2^n$ and ϕ has order $3 \cdot 2^s$ on T. Let H be an elementary abelian subgroup of T with $o(H) > 2$ if $Z(T) \subseteq A$ and $o(H) = 2$ if $Z(T) \subset A$; and assume that H centralizes B. Then either $H \subseteq Z(T)$ or $Z(T) \subseteq A$ and $H \subseteq Z(T)B$.*

Proof. The proof is by induction on $o(G)$. We may clearly assume that T is a 2-Sylow subgroup of G and that $o(A) = 3 \cdot o(A \cap T)$. Let P be a minimal A-invariant subgroup of $Z(T)$ and suppose first that $P \cap A = 1$. We may assume T is non-abelian and $H \not\subseteq P$. In particular, $T \neq (A \cap T)P$. Let $B = (b)$, where $b = ya^r$, $y \in T$. In order to carry out the induction we shall also allow the possibility $o(H) = 2$ when $Z(T) \not\subseteq A$, but $B \subset T$. Observe that if $H \cap P \neq 1$, $[H \cap P, B] = 1$ implies a^r acts trivially on P, whence $3 \mid r$ and $B \subseteq T$.

Let $\bar{G} = G/P = \overline{ABA} = \overline{AT}$. Then by induction $\bar{H} \subseteq \bar{Q}$, where $\bar{Q} \lhd \bar{G}$, $\bar{A} \cap \bar{Q} \lhd \bar{Q}$, and $o(\bar{Q}/\bar{A} \cap \bar{Q}) = 4$. Let Q be the inverse image of $\bar{Q}$ in T. Suppose first that $H \subseteq (A \cap Q)P$. If $o(H) > 2$, $H \cap P \neq 1$, whence $3 \mid r$; if $o(H) = 2$, then $3 \mid r$ by assumption. But then if $a_1 x \in H$, where $(a_1) = \Omega_1(A \cap Q)$ and $x \in P$, it follows that $[a_1, b] = 1$, whence $a_1 \in Z(G)$ and $H \subseteq Z(T)$. Hence we may assume that $H \not\subseteq (A \cap Q)P$.

If $\bar{Q} = (\bar{A} \cap \bar{Q}) \times \bar{F}$, where $\bar{F}$ is $\bar{A}$-invariant, it follows as above that ϕ^r acts trivially on $\bar{F}$. Thus F is of ϕ-index 0 and hence of type (4,4). This implies Q is non-abelian; otherwise $H \subseteq (A \cap Q)P$. Hence by induction $Q = T$. If $\bar{Q}$ is non-abelian, $\bar{Q}$ is the central product of $\bar{A} \cap \bar{Q}$ and a quaternion group $\bar{F}$, and by induction $Q = T$. Now if $B \subset Q$ and $o(B) > 4$, it follows in either case that $C(B) \cap Q \subseteq (A \cap Q)PB$. Since H is elementary, this yields $H \subseteq (A \cap Q)P$, which is not the case. On the other hand, if $o(B) = 2$, $P \subset A(b^2)A = A$, a contradiction. Thus $3 \mid o(B)$. This forces $C(B) \cap Q$ to lie in a conjugate of $A \cap Q$ and hence in $(A \cap Q)P$, which is not the case.

Assume now that $Z(T) \subset A$. If $3 \mid o(B)$, $C(B) \cap T$ lies in a conjugate of $A \cap T$. Since A is cyclic, this implies $H \subseteq Z(T)B$. We may therefore assume $B \subset T$. The lemma follows at once by induction if $Z(\bar{T}) \subset \bar{A}$; so suppose the contrary. Then by the first part of the proof, $\bar{H} \subseteq \bar{Q} = \Omega_1(Z(\bar{T}))$ and $\bar{Q} = (\bar{A}\ \bar{Q}) \times \bar{F}$, where $\bar{F}$ is $\bar{A}$-invariant. Let F, Q be the inverse images of $\bar{F}, \bar{Q}$ in T. Suppose F is a quaternion group. Since $AF = AB_1A$ with $B_1 \subseteq B$, $C(B) \cap Q = (A \cap Q)B_1$ and the lemma follows. On the other hand, if F is abelian, then $H \subseteq F$. If B centralizes F, then so does $\phi^i(B)$ for all i. But then $F \subseteq Z(T)$, which is not the case. We conclude that $C(B) \cap F = (A \cap F)B_1 \subseteq H$, thus completing the proof.

For $p = 3$, we have the following result.

THEOREM 4. *Let $G = ABA = AP$, where P is a 3-group normal in G. Then either A has a normal complement in G or G contains two normal 3-subgroups T_1, T_2 such that*

(a) *$G = A(T_1 \times T_2)$;*

(b) *$A \cap T_1 \subseteq Z(T_1)$, $T_1/Z(T_1)$ is elementary abelian of order 9, T_1 contains a maximal subgroup T_0 invariant under A which is the direct product of $A \cap T_1$ and a cyclic group;*

(c) *T_2 is elementary abelian and contains no A-invariant subgroups of order 3;*

(d) *T_1 does not contain a 3-Sylow subgroup of A.*

Proof. The proof is entirely analogous to that of Theorem 3. We shall use the same notation. If $P_1 \subset A$ and $\bar{G}$ possesses a normal $\bar{A}$-complement, it follows from the proof of Lemma 3.1 that G possesses a normal A-complement unless $\bar{P}$ contains an elementary abelian subgroup $\bar{H}_1$ of order 9 on which $\bar{\phi}$ has order 6. If $\bar{P} = \bar{H}\bar{K}$, this can only occur if $\bar{F}$ is cyclic, $\bar{H} \supset \bar{F}$, and $\bar{H}_1 = \Omega_1(\bar{H})$. But then by Lemma 1.3, $\bar{K}$ is elementary abelian and contains no $\bar{\phi}$-invariant subgroups of order 3. Its inverse image in P possesses a normal P_1 complement K which centralizes the inverse image H of $\bar{H}$. If H has a normal P_1-complement, then G has a normal A-complement. Otherwise the second possibility of the theorem holds with $T_1 = H$, $T_2 = K$. The final condition of the theorem follows from the fact that $\bar{\phi}$ has order 6 on $\Omega_1(\bar{H})$.

If $\bar{P} = \bar{T}_1 \times \bar{T}_2$, then $P = T_1 \times T_2$, where T_1 is the inverse image of $\bar{T}_1$ and T_2 is the normal P_1-complement contained in the inverse image of $\bar{T}_2$. We have only to verify (b). Now $A \cap T_1 \lhd T_1$ and T_1 admits an automorphism ϕ_1 of order 2 which fixes $A \cap T_1$ and is such that the image $\tilde{\phi}_1$ of ϕ_1 on $\tilde{T}_1 = T_1/A \cap T_1$ leaves only the identity element of $\tilde{T}_1$ fixed. This implies that $\tilde{T}_1$ is abelian. Furthermore by Lemma 2.1, $A \cap T_1 \subseteq Z(T_1)$. Thus $\mathrm{cl}(T_1) = 2$ and (b) follows at once.

Suppose next that $P_1 \cap A = 1$. If $\bar{G}$ has a normal $\bar{A}$-complement, then so does G. Hence we may assume $\bar{P}$ satisfies the second alternative of the theorem. If $o(P_1) > 3$, the theorem follows as in Case 2 of Theorem 3; while if $o(P_1) = 3$, it follows for the same reason that $G = A(T_1 \times T_2)$, where ϕ has order $2 \cdot 3^s$ on T_1, $\bar{T}_1$ satisfies (b), and T_2 satisfies (c). Again it remains

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to verify (b). If $\Omega_1(A \cap T_1) \lhd T_1$, it follows by induction and the argument of the preceding case that T_1 satisfies (b).

In the contrary case we must have $A \cap T_1 = \Omega_1(A \cap T_1)$. Let Z be the inverse image of $Z(\bar{T}_1)$ in T_1. If Z is abelian, then $[T_1, T_1]$ is cyclic and A-invariant. Since $A \cap T_1 \ntriangleleft T_1$, $[T_1, T_1] \cap A = 1$; and it follows at once that $A \cap T_1$ has a normal complement in T_1, which is not the case. Hence $P_1 = [Z, Z]$. Thus there exists x in Z, y in $A \cap T_1$ such that $[x, y] = z \neq 1$ in P_1. On the other hand, by the structure of $\bar{T}_1$, we can choose x so that $\bar{x} = \bar{x}_1{}^3$ for some $\bar{x}_1$ in $\bar{T}_1$. But then if x_1 is a representative of $\bar{x}_1$ in T_1, $[x_1, y] = z_1 \in P_1$; and it follows that $[x, y] = 1$, a contradiction.

5. Some special results on linear groups. Lemma 3.1 of **(2)** was the principal tool in the proof that a solvable regular ϕ-group is nilpotent **(2,** Theorem 1**)**. In analysing the structure of ABA-groups, we shall need some slight extensions of this result. For our present purposes, it will be more convenient to rephrase this lemma in terms of groups of linear transformations:

LEMMA 5.1. *Let $L = AQ$ be a linear group acting irreducibly on an m-dimensional vector space P over a field with p elements, where $A = (\phi)$ is cyclic, Q is an elementary abelian q-group for some prime $q \neq p$, and Q is a minimal normal subgroup of L. Assume further that Q does not have the unit representation as an absolutely irreducible constituent. Then if d denotes the order of ϕ on Q and h its order on P, we have $d \mid m$ and $h \mid d(p^{m/d} - 1)$.*

Remark. If G denotes the holomorph of L and P, the final condition of the lemma is simply the statement that no element $\neq 1$ of P lies in $Z(PQ)$. The minimality of Q in turn implies that PQ has a trivial centre.

We shall need a special case of this result:

LEMMA 5.2. *Under the hypotheses of Lemma 5.1, if the subspace P_0 of P left elementwise fixed by ϕ is one-dimensional, then $d = m = h$.*

Proof. If we take P_0 as the minimal subspace W of P in the proof of Lemma 3.1 of **(2)**, we conclude at once that ϕ^d is the identity on P. Furthermore, the same lemma shows that over the algebraic closure K_p^* of the ground field, the corresponding vector space P^* can be decomposed into the direct sum of subspace $P_1^*, P_2^*, \ldots, P_t^*$, each of dimension d, each invariant under ϕ, and such that the matrix Φ_i of ϕ on P_i^* with respect to a suitable basis assumes the form

$$\Phi_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 \\ & & & & \\ 0 & 0 & & \ldots & 1 \\ b_i & 0 & & \ldots & 0 \end{pmatrix}, \quad b_i \in K_p^*, \ i = 1, 2, \ldots, t. \tag{20}$$

Since $\phi^d = 1$ on P, $b_i = 1$ for all i, and hence we may assume that the P_i^* are actually subspaces of P. Now 1 is a characteristic root of each Φ_i, and hence ϕ leaves fixed some non-zero vector of each P_i^*, $i = 1, 2, \ldots, t$. But by hypothesis the subspace left elementwise by ϕ is 1-dimensional. Thus $t = 1$, and $d = m = h$.

From Lemma 5.1 we can derive a slight extension of Theorem 1 of **(2)**.

LEMMA 5.3. *Let $G = ABA = AT$, where $T \lhd G$. Assume $T = MQ$ where $M \lhd G$, $A \cap T \subset M$, Q is a q-group and if $\bar{G} = G/M = \bar{A}\bar{Q}$, that $\bar{Q}$ is a minimal normal subgroup of $\bar{G}$ and $N(\bar{A}) = \bar{A}$. Assume further that $Z(M)$ contains a p-subgroup P, $p \neq q$, such that $A \cap P = 1$ and P is a minimal normal subgroup of G. Then PQ is nilpotent.*

Proof. $C(Q) \cap P$ is invariant under A since $P \subseteq Z(M)$, and hence $C(Q) \cap P \lhd G$. In view of the minimality of P, we may assume $C(Q) \cap P = 1$. If PQ is A-invariant, PQ is a regular ϕ-group and hence is nilpotent by Theorem 1 of **(2)**. If PQ is not A-invariant, the proof of Theorem 1 of **(2)** goes through without essential change.

In fact, $\bar{G}$ may be regarded as a group of linear transformations on P and as such satisfies the hypotheses of Lemma 5.1. Furthermore $\bar{\phi}$ has order $d > 1$ on $\bar{Q}$ since $\bar{A} \cap \bar{Q} = 1$ and $N(\bar{A}) = \bar{A}$. In view of Lemmas 4.1 and 4.2 of **(2)**, it suffices to show that d divides the ϕ-index r_1 of P. By the proof of Theorem 10 of **(2)**, T contains an element g such that the elements $\phi^i([g]_r^j)$ include a set of coset representatives of $A \cap T$ in T. If t is the least integer such that $[g]_r^t \in (A \cap M)P$, r_1 may clearly be taken as a multiple of rt. On the other hand, $[\bar{g}]_r^t = 1$ and since $\bar{Q}$ is abelian, $\bar{\phi}^{rt}(\bar{g}) = \bar{g}$, whence $d \mid rt$. Thus $d \mid r_1$.

We shall also need a slight variation of this result.

LEMMA 5.4. *Lemma 5.3 holds under the alternative assumption that $\bar{Q}$ is a quaternion group and $\bar{A}$ does not centralize $\bar{Q}$.*

Proof. Clearly $C(Z(Q)) \cap P \lhd G$. If $C(Z(Q)) \cap P = P$, P is in the centre of $M^* = Z(Q)M$, and the conclusion follows at once from the preceding lemma with M^* playing the role of M. In the contrary case, Q and hence $\bar{Q}$ is represented faithfully on P.

If $\bar{A} = (\bar{a})$, $\bar{a}^3$ is in the centre of $\bar{G}$. Hence if P_1 denotes a minimal subgroup of P invariant under ϕ^3, P can be written as the direct product of subgroups P_i, $i = 1, 2, \ldots, n$, of the same order p^t, each invariant under ϕ^3, and on each of which ϕ^3 has the same minimal polynomial. In particular, if h denotes the order of ϕ on P, we have

$$h \mid 3(p^t - 1). \tag{21}$$

If $n = 1$, $P_1 = P$, and ϕ^3 acts irreducibly on P. If P is extended to a vector space P^* over the algebraic closure of the field with p elements, it follows that ϕ^3 is represented in P^* by a diagonal matrix with distinct characteristic roots. On the other hand, since $\bar{A} \cap \bar{Q}$ is not in the kernel of the representation of $\bar{G}$ on P, at least one of the absolutely irreducible constituents, say, χ, of $\bar{G}$ in P^* has degree >1. (In fact, it is easy to see that they are all of the same

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degree.) Since $\bar{a}^3$ is in the centre of $\bar{G}$, $\bar{a}^3$ is represented by a scalar matrix in the representation χ. It follows at once that ϕ^3 is represented in P^* by a diagonal matrix whose characteristic roots are not all distinct. This is a contradiction, and hence $n > 1$.

Now the order d of $\bar{\phi}$ on $\bar{Q}$ is either 3 or 6, and it follows as in Lemma 5.3 that the ϕ-index r_1 of P is a multiple of 3. Since one of the inequalities $h^2 > r_1 o(P)$ or $hp > o(P)$ must hold and $o(P) = p^{tn}$, it follows at once from (21) that $n \leqq 2$, whence $n = 2$.

Let $g = g_1 g_2$ be a ϕ-generator P, where $g_i \in P_i$, $i = 1, 2$. If $g \in P_i$, say $g \in P_1$, the elements $\phi^{3i}([g]^j_{r_1})$ are all in P_1 since P_1 is invariant under ϕ^3 and $3 \mid r_1$. Hence there are at most $3(p^t - 1)$ elements different from 1 of the form $\phi^i([g]^j_{r_1})$ in P, and consequently

$$(22) \qquad 3(p^t - 1) \geqq p^{2t} - 1,$$

and this is impossible since $p \neq 2$.

We may therefore assume that $g_1 \neq 1$, $g_2 \neq 1$. To reach a contradiction, we shall show that (22) holds. It clearly suffices to prove that there are at most $p^t - 1$ distinct elements different from 1 of the form $\phi^{3i}([g]^j_{r_1}) = \phi^{3i}([g_1]^j_{r_1})\phi^{3i}([g_2]^j_{r_2})$. Suppose $\phi^{3i}([g_1]^j_{r_1}) = \phi^{3k}([g_1]^m_{r_1})$ for some i, j, k, m. Since ϕ^3 acts irreducibly on P_1, P_2 with the same minimal polynomial, the corresponding relation with g_1 replaced by g_2 must hold, and we conclude at once that there are at most $p^t - 1$ elements of the required form.

6. Exceptional ABA-groups of types I, II, and III. We have seen in §4 that there exist ABA-groups G with $N(A) = A$ in which A does not have a normal complement. In this section we shall determine two further classes of ABA-groups which have this property. We begin with the following lemma.

LEMMA 6.1. *Let $G = ABA = AT$, where $T \lhd G$. Assume that $T = MQ$, where M is nilpotent and normal in G, Q is a q-group for some prime q, and if $\bar{G} = G/M = \bar{A}\bar{Q}$, then $N(\bar{A}) = \bar{A}$. Then if $L \subset M$ is normal in G and $\tilde{G} = G/L = \tilde{A}\tilde{B}\tilde{A}$, we have $N(\tilde{A}) = \tilde{A}$.*

Proof. The proof is by induction on $o(G)$. It clearly suffices to prove the lemma under the assumption that L is a minimal subgroup of M normal in G. Since this implies that L is abelian, the lemma will follow at once from Lemma 2.3 if we can show that every subgroup of A lies in the centre of its normalizer.

Let $A_0 \subset A$ and $G_0 = N(A_0)$. If $G_0 \subset G$, we may assume by induction that $A_0 \subseteq Z(G_0)$. Thus we need only consider the case in which $A_0 \lhd G$. Let P be a p-Sylow subgroup of M. Since M is nilpotent, $P \lhd G$. If $p \geqq 5$, it follows from Theorem 2 that $G_p = AP = AP^*$, where $P^* \lhd G_p$ and $A \cap P^* = 1$. The hypotheses of Lemma 2.1 are satisfied if we take A_0 for H and $A_0 P^*$ for G. Thus $A_0 \subseteq Z(A_0 P^*)$ and hence $A_0 \subseteq Z(G_p)$. On the other hand, if $p = 2$ or 3, Lemma 4.1 and Theorem 4 imply that $A_0 \subseteq Z(G_p)$. Thus $A_0 \subseteq Z(AM)$.

It follows that $\bar{G} = \bar{A}\bar{Q}$ acts as a group of automorphisms of A_0. Since $N(\bar{A}) = \bar{A}$, we can again apply Lemma 2.1 to conclude that the elements of $\bar{Q}$ induce the identity automorphism of A_0. Thus Q centralizes A_0, and $A_0 \subseteq Z(G)$.

The preceding argument can easily be adapted to give the following corollary:

COROLLARY. *Let $G = ABA$, and assume A contains a subgroup A_0 which is normal in G such that $\bar{G} = G/A_0$ satisfies the hypotheses of the preceding lemma. Then A_0 is in the centre of G.*

Remark. The lemma also obviously holds if $Q \subset A$.

THEOREM 5. *Let $G = ABA = AT$, where $T \lhd G$. Assume that $T = MQ$, where $M \neq 1$ is nilpotent and normal in G, Q is a q-group for some prime q, $Z(T) = 1$, $A \cap T \subset M$, and no normal subgroup of T lies properly between M and T. Then either*

(a) *M is a 2-group of order 2^{3s}, $o(A \cap M) = 2^s$ and $o(Q) = 7$; or*

(b) *M is an abelian group of type (t, t), where $2 \nmid t$, $3 \nmid t$, $o(A \cap M) = t$ and $o(Q) = 3$.*

Furthermore, if $\bar{\phi}$ denotes the image of ϕ on $\bar{G} = G/M = \bar{A}\bar{Q}$, then

(c) *ϕ has order $3 \cdot 2^s$ on T and $\bar{\phi}$ has order 3 on $\bar{Q}$ in case (a); and ϕ has order $2t$ on T and $\bar{\phi}$ has order 2 on $\bar{Q}$ in case (b); in either case $\bar{Q}$ does not have $\bar{\phi}$-index 0.*

(d) *There exists a q-Sylow subgroup Q^* of T such that $\phi(Q^*) = uQ^*u^{-1}$, $u \in A \cap M$, and no q-Sylow subgroup of T is A-invariant.*

(e) *In case (a) $\Omega_1(Z(M))$ has order 8.*

(f) *For any proper subgroup L of M normal in G, G/L satisfies the hypotheses of the theorem.*

Proof. Since M is nilpotent and $Z(T) = 1$, $(o(M), q) = 1$. If P is a minimal subgroup of $Z(M)$ normal in G, then P is elementary abelian of order p^m for some prime p. Furthermore $A \cap P \neq 1$, for otherwise by Lemma 5.3 PQ is nilpotent which is not the case. Also $P \not\subset A$, for by Lemma 6.1 $N(\bar{A}) = \bar{A}$ and then Lemma 2.1 forces Q to centralize P. Thus $m > 1$. Since $P \subseteq Z(M)$, $\bar{G}$ can be regarded as a group of linear transformations on P; and since $A \cap P$ is cyclic, the hypotheses of Lemma 5.2 are satisfied. Hence if ϕ has order h on P and $\bar{\phi}$ has order d on $\bar{Q}$, $d = m = h$.

By Lemma 2.2, $P = (A \cap P) \times P_0$ where P_0 is a regular ϕ-group of order p^{m-1}. Since ϕ has order m on P_0, $mp > p^{m-1}$ if P_0 is of ϕ-index 0. The only solutions of this inequality are $m = 2$ or $m = 3$ and $p \neq 2$. On the other hand, if P_0 is of ϕ-index $r_0 \neq 0$ and if ϕ acts irreducibly on P_0, then

$$m^2 > r_0 p^{m-1}, \tag{23}$$

which implies $p = 2$, $m \leqq 6$ or $p = 3$, $m = 2$. An even stronger inequality holds if ϕ does not act irreducibly on P_0.

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It follows readily from the proof of Lemma 5.2 that $f(X) = X^{m-1} + X^{m-2} + \ldots + X + 1$ is the characteristic polynomial of ϕ on P_0. Hence if $p = 2$ and m is even, 1 is a characteristic root of ϕ on P_0. Since this leads at once to the contradiction $N(A) \supset A$, the cases $p = 2$, $m = 2, 4$, or 6 are excluded. Lemma 1.7 shows that $p = 2$, $m = 5$ is also impossible. Thus $p = 2$, $m = 3$ or $m = 2$, and hence $o(P) = 8$ or p^2. If $o(P) = p^2$, then $p \neq 2$, for otheriwse ϕ leaves the generator of P_0 fixed. In particular, it follows that $(m,p) = 1$ in all cases.

Since $N(\bar{A}) = \bar{A}$ by the preceding lemma, $\bar{Q}$ is a regular $\bar{\phi}$-group. Our hypotheses imply that $\bar{\phi}$ acts irreducibly on $\bar{Q}$, and hence one of the inequalities $dq > o(\bar{Q})$ or $d^2 > o(\bar{Q})$ necessarily holds. Since $m = d = 2$ or 3, we conclude that $o(\bar{Q}) = q$ except possibly in the case $d = 3$, $q = 2$. But $p = 2$ if $m = 3$, whence $p = q$, a contradiction. Thus $o(Q) = q$.

We next establish (f). We may assume $P \subset M$, since otherwise (f) holds trivially. If $\tilde{G} = G/P = \tilde{A}\tilde{T} = \tilde{A}\tilde{M}\tilde{Q}$, it clearly suffices to show that $\tilde{G}$ satisfies the conditions of the theorem. Since $N(\tilde{A}) = \tilde{A}$ by the preceding lemma, we need only show that $Z(\tilde{T}) = 1$; so assume the contrary. Let $\tilde{K}$ be a minimal $\tilde{A}$-invariant subgroup of $Z(\tilde{T}) \cap \tilde{M}$ and let K be the inverse image of $\tilde{K}$ in G. We may clearly assume K is a p-group, for otherwise $Z(T) \neq 1$.

Now if $(y) = Q$ and $x \in K$, we have

$$[x, y] = z, z \in P. \tag{24}$$

Since $z \in Z(M)$, $[x^p, y] = 1$. But $x^p \in P \subseteq Z(M)$, and hence $x^p = 1$.

Consider first the case $p = 2$. Then clearly K is elementary abelian. If the order of ϕ on K is odd, then the holomorph of Q and ϕ is completely reducible on K, so that $K = P \times H$ where H is invariant under Q and ϕ. Clearly Q centralizes H. To obtain a contradiction, we need only show that H is in the centre of M. Since M is nilpotent, it suffices to show that H is in the centre of the 2-Sylow subgroup S of M.

Since S contains at most one A-invariant abelian subgroup of type (2, 2), H is not of type (2, 2) and ϕ does not have order 3 on H. Furthermore $H \cap A = 1$, since $A \cap P \neq 1$. Now by Theorem 3, $AS = A(S_1 \times S_2)$, where S_1, S_2 satisfy the conditions of Theorem 3. Our conditions imply that $H \subset S_2$. Since S_2 is a regular ϕ-group and H is a ϕ-invariant abelian subgroup of S_2, $H \subseteq Z(S_2)$ and hence $H \subseteq Z(S_1 \times S_2)$. Now $S = (A \cap S)S_1S_2$ and it follows from the minimality of K that $A \cap S$ centralizes H. Thus $H \subseteq Z(S)$, a contradiction.

If ϕ has even order on K, $\bar{K}$ must be of type (2, 2). By the proof of Theorem 3, S contains such a normal subgroup K only if AS has a normal A-complement. But then if S were non-abelian, $P_0 = [s, s] \cap P$ would be Q-invariant, which is not the case. Thus S is abelian, and we conclude that $Z(T) \neq 1$, a contradiction. Thus (f) holds if $p = 2$.

Suppose then that $p \neq 2$. If $x_1, x_2 \in K$, it follows readily from (24) that

$[[x_1, x_2], y] = 1$. Since $[x_1, x_2] \in P$, we conclude that K is abelian, and, as above, K is elementary abelian. The balance of the proof now parallels the case $p = 2$. Thus (f) holds in all cases.

We next prove that T contains a q-Sylow subgroup Q^* satisfying (d). By induction $\tilde{T}$ possesses such a q-Sylow subgroup and hence for some q-Sylow subgroup Q_1 of T we have $\phi(Q_1) = vQ_1v^{-1}$, where $v \in (A \cap M)P$. Since ϕ has order m on P and $(m, p) = 1$, it follows that $(A \cap M)P = (A \cap M) \times P_0$. If $v = uv_0$, where $u \in A \cap M$ and $v_0 \in P_0$, there exists an element w in P_0 such that $w\phi(w^{-1}) = v_0$, whence $\phi(wQ_1w^{-1}) = \phi(w)uv_0Q_1v_0^{-1}u^{-1}\phi(w^{-1}) = uwQ_1w^{-1}u^{-1}$. The subgroup $Q^* = wQ_1w^{-1}$ thus has the required property. Without loss of generality we may assume $Q^* = Q$.

This result will now be used to establish (a) and (b). Consider first the case $P = M$. Since $o(Q) = q$, Q acts regularly on P. If $p = 2$, $o(P) = 8$, and we must have $q = 7$. Suppose then that $o(P) = p^2$. Now AT is an ABA-group so that there exists a fixed element g in T and an integer r such that the elements $\phi^i([g]_r^j)$ include a set of coset representatives of $A \cap P$ in T. Write $g = xy$, where $x \in P$ and $(y) = Q$. Since $d = 2$, $\phi(y) = uy^{-1}u^{-1}$, where $u \in A \cap P$ by (d). If $d \nmid r$, $\phi^i([g]_r^j)$ is of the form w or $wy^{\pm 1}$, $w \in P$, for all i, j; and this gives immediately $q = 3$. On the other hand, if $d \mid r$, $[g]_r^j \in P$ if and only if $q \mid j$. But now since ϕ has order 2 on the abelian group P,

$$(25) \qquad [g]_r^q = (xy)(xuyu^{-1})(xu^2yu^{-2}) \ldots (xu^{q-1}yu^{-(q-1)}) = u^{-1}(xuy)^q u^{-(q-1)}.$$

Since y acts regularly on P, $(xuy)^q = 1$, and hence $[g]_r^q = u^{-q}$. Thus $A \cap P$ itself is the only coset of $A \cap P$ in P which is of the required form. Thus $q = 3$, as asserted. Since $p \neq q$, $q = 3$ implies $p \neq 3$. The remaining conditions of (a) and (b) have been established above in the case $P = M$. Furthermore, we have shown, when $d = 2$, that $d \nmid r$ and hence that $\bar{Q}$ does not have $\bar{\phi}$-index 0. The argument applies equally well if $d = 3$ and $q = 7$.

Assume next that $P \subset M$. By induction $\tilde{M}$ is either a 2-group of order $2^{3(s-1)}$, $o(\tilde{A} \cap \tilde{M}) = 2^{s-1}$ and $q = 7$ or $\tilde{M}$ is abelian of type (t', t'), $2 \nmid t'$, $3 \nmid t'$, $o(\tilde{A} \cap \tilde{M}) = t'$ and $q = 3$. In the first case $o(P) = 8$, M has order 2^{3s}, $o(A \cap M) = 2^s$. In the second case $o(P) = p^2$ with $p \neq 2, 3$, $o(M) = t^2$, where $t = pt'$ and $o(A \cap M) = t$. Furthermore, $[M, M]$ is cyclic, normal in G, and contained in P. But P is a minimal normal subgroup of G and is of type (p, p). Thus $[P, P] = 1$ and M is abelian. To prove M is in fact of type (t, t), we need only show that the p-Sylow subgroup of M is of type (p^c, p^c), and this follows at once from the fact that A is cyclic.

It follows for the same reason that $\Omega_1(Z(M)) = P$ in case (a). Thus (e) holds.

To prove (c) let k be the order of ϕ on T and set $t = 2^s$ in case (a). Then in both cases (a) and (b), it follows from (d) that $k \mid mt$. On the other hand, we clearly have $m \mid k$ and $(m, t) = 1$. Now $o(A) = mte$, for some integer e. If $k < mt$, it follows at once that $y = \phi^{ke}(y) = a^{ke}ya^{-ke}$, and hence that Q

centralizes a^{ke}. But clearly $a^{ke} \neq 1$ and lies in $A \cap M$, a contradiction. Thus $k = mt$. Since the final assertion of (c) has already been established, (c) holds.

The same argument shows that no q-Sylow subgroup of T is invariant under A, thus completing all parts of the theorem.

Theorems 3, 4, and 5 will serve to motivate the definitions of *exceptional* ABA-groups which we shall now make. In view of what is to follow, it will be necessary to include a slightly larger class of groups than those satisfying the conditions of these theorems.

DEFINITION. Let $G = ABA = AT$, where $T \lhd G$. Then G will be called an *exceptional* ABA*-group*

(a) of *type I* if T is a 2-group and G satisfies the hypotheses of Theorem 3 with $T = T_1 \neq 1$;

(b) of *type II* if $T = MQ$, M is a 2-group normal in G, Q is a 7-group, $A \cap T \subset M$, $C(M) \cap Q = Q_0$ is cyclic, $\tilde{G} = G/Q_0 = \tilde{A}\tilde{T}$ satisfies the hypotheses of Theorem 5;

(c) of *type III* if $T = MQ$, M is abelian of type (t, t), $(t, 3) = 1$, Q is a 3-group; if $Q_0 = C(M) \cap Q$, then $\tilde{G} = G/Q_0 = \tilde{A}\tilde{T}$ satisfies the hypotheses of Theorem 5; either $A \cap T \subset M$ and Q_0 is cyclic or $\bar{G} = G/M = \bar{A}\bar{Q}$, satisfies the hypotheses of Theorem 4 with $\bar{Q} = T_1$.

Furthermore if Q_0 is cyclic and disjoint from A, we require the following additional conditions in (b) and (c): if $\bar{G} = G/M = \bar{A}\bar{Q}$, then $\bar{\phi}$ has order $m \cdot q^s$ on $\bar{Q}$, where $m = 3$ if $q = 7$ and $m = 2$ if $q = 3$, and $\bar{Q}$ is not of $\bar{\phi}$-index 0.

Remarks. For exceptional groups of type III, we shall also allow the possibility that $M = 1$ and G satisfies the conditions of Theorem 4 with $Q = T_1$. The complexity of the definition of exceptional groups of types II and III arises from the need for T to be A-invariant. The problem is that the image $\bar{Q}_0$ of Q_0 does not possess an $\bar{A}$-invariant complement in $\bar{Q}$. If $\bar{Q} = \bar{H}\bar{K}$, where $\bar{H}$, $\bar{K}$ satisfy the conditions of Lemma 1.3, our requirements force $\bar{Q} = \bar{H}$ and $\bar{H} \supset \bar{F}$, where $\bar{F}$ is the $\bar{\phi}$-nucleus of $\bar{Q}$. Lemma 6.3 will give further clarification of this point.

We shall also call an A-invariant subgroup T of an ABA-group G an *exceptional subgroup* of G (of type I, II, or III) if $G^* = AT = AB^*A$ is an exceptional AB^*A-group (of type I, II, or III).

We next prove

LEMMA 6.2. *Let $G = ABA = AT$, where $T \lhd G$. Assume that $T = MQ$, where M is nilpotent and normal in G, Q is a q-group for some prime q, $(o(M), q) = 1$, and $A \cap T \subset M$. Let Q_1, Q_2 be two disjoint subgroups of Q such that $MQ_i \lhd G$, $i = 1, 2$. Then either Q_1 or Q_2 centralizes M.*

Proof. Let S_1 be a minimal subgroup of Q_1 such that $MS_1 \lhd G$. If S_1 centralizes M, $MS_1 = M \times S_1$ and since $o(M)$ is prime to q, $S_1 \lhd G$. If $G' = G/S_1 = A'T' = A'M'Q'$, $N(A') = A'$ since $A \cap S_1 = 1$, and the lemma follows at once by induction. We may thus suppose that S_1 does not centralize M.

Let L be a maximal subgroup of M normal in G such that $LS_1 = L \times S_1$ and set $\tilde{G} = G/L = \tilde{A}\tilde{T} = \tilde{A}\tilde{M}\tilde{Q}$, $\tilde{S}_1$ denoting the image of S_1 in $\tilde{Q}$. If $\tilde{P}$ is a minimal subgroup of $\tilde{M}$ normal in $\tilde{G}$, it follows from the maximality of L and the nilpotency of $\tilde{M}$, that $\tilde{S}_1$ does not centralize $\tilde{P}$. But then by the first part of the proof of Theorem 5, $o(\tilde{S}_1) = q$, and if $\bar{G} = G/M = \bar{A}\bar{Q}$, $\bar{\phi}$ has order m on the image $\bar{S}_1$ of S_1 in $\bar{Q}$, where $m = 3$ if $q = 7$ and $m = 2$ if $q = 3$.

If S_2 is a minimal subgroup of Q_2 such that $MS_2 \lhd G$ and $\bar{S}_2$ its image in $\bar{Q}$, we may similarly assume that $o(\bar{S}_2) = q$ and that $\bar{\phi}$ has order m on $\bar{S}_2$. But $\bar{S}_1\bar{S}_2 = \bar{S}_1 \times \bar{S}_2$ must be a regular $\bar{\phi}$-group, which is impossible by Lemma 1.2 since $\bar{\phi}$ has the same order on each factor. This contradiction establishes the lemma.

LEMMA 6.3. *Let $G = ABA = AT$, where $T \lhd G$. Assume that $T = MQ$ where M is nilpotent and normal in G, Q is a q-group for some prime q, $A \cap T \subset M$, and $M \cap Z(T) = 1$. Then $T = T^* \times Q^*$, where T^* us an exceptional subgroup of type II or III, $Q^* \subset Q$, and $Q^* \lhd G$.*

Proof. We may suppose $M \neq 1$ since otherwise the lemma holds trivially with $T^* = 1$. Our conditions imply that M has order prime to q. Let now S be a minimal subgroup of $Z(Q)$ such that $MS \lhd G$. We distinguish two cases.

Case 1. For any minimal subgroup P of M, normal in G, $P \cap C(S) = 1$, and only the identity element of Q centralizes M.

It follows as in the proof of Theorem 5 that $o(P) = p^m$, where $m = 3$, $q = 7$ if $p = 2$ and $m = 2$, $q = 3$ if $p \neq 2$, that $o(A \cap P) = p$, and that $o(S) = q$. Furthermore, as in the proof of (f) of Theorem 5, $M \cap Z(MS) = 1$. The minimality of S implies that $Z(MS) = 1$. Hence $T^* = MS$ is an exceptional subgroup of type II or III.

If $S_1 \subset Q$ is such that $S \cap S_1 = 1$ and $MS_1 \lhd G$, then $MS_1 = M \times S_1$ by Lemma 6.2. Our present assumption implies that the image $\bar{S}$ of S in $\bar{G} = G/M = \bar{A}\bar{Q}$ is the unique minimal subgroup of $\bar{Q}$, normal in $\bar{G}$.

Since $\bar{Q}$ is represented faithfully on P and $o(P) = 8$ or p^2, $\bar{Q}$ must be abelian and hence cyclic. If T^* is of type II, $o(Q) = 7$, or else $\Omega_1(Q)$ centralizes M. If T^* is of type III, the argument in Theorem 5 which showed that $q = 3$ can be repeated to show that $o(Q) = 3$. Thus $S = Q$, and the lemma follows with $T = T^*$.

Case 2. Either S centralizes some minimal subgroup of M normal in G or $C(M) \cap Q \neq 1$.

Now $C(M) \cap T \lhd G$. Since $(o(M), q) = 1$, $Q_0 = C(M) \cap Q$ is characteristic in $C(M) \cap T$ and hence is also normal in G. Thus if $Q_0 \neq 1$, Q contains a subgroup $\neq 1$ which centralizes M and is normal in G. We shall show that the same assertion holds if S centralizes P. We may clearly assume $C(M) \cap Q = 1$.

Let L be a maximal subgroup of M normal in G which is centralized by S and assume $L \subset M$. Set $G' = G/L = A'T' = A'M'Q'$, S' denoting the image of

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S in Q'. If P' is any minimal subgroup of M' normal in G', then $C(P') \cap S' = 1$. If $Q_0' = C(M') \cap Q'$ and Q_0 denotes the inverse image of Q_0' in $Q, S \cap Q_0' = 1$ and it follows from Lemma 6.2 that Q_0 centralizes M. Thus $Q_0 = 1$ and consequently $Q_0' = 1$. It follows now by Case 1 applied to G' that $Q' = S'$ and hence that $L \cap Z(T) \neq 1$, contrary to hypothesis. Thus $L = M$ and S centralizes M.

It remains therefore to prove the lemma under the assumption that S centralizes M. Let $\tilde{G} = G/S = \tilde{A}\tilde{T} = \tilde{A}\tilde{M}\tilde{Q}$. Since $A \cap S = 1$, $N(\tilde{A}) = \tilde{A}$. Since M has order prime to q, $\tilde{M} \cap Z(\tilde{T}) \neq 1$ implies $M \cap Z(T) \neq 1$. Thus $\tilde{M} \cap Z(\tilde{T}) = 1$, and it follows by induction that $\tilde{T} = \tilde{T}^* \times \tilde{Q}''$, where $\tilde{T}^*$ is an exceptional subgroup of type II or III, $\tilde{Q}'' \subset \tilde{Q}$ and $\tilde{Q}'' \lhd \tilde{G}$.

If $\tilde{T}^* = \tilde{M}\tilde{Q}'$ with $\tilde{Q}' \subset \tilde{Q}$, and if Q', Q'' are the inverse images of $\tilde{Q}'$, $\tilde{Q}''$ in T, then $Q' \cap Q'' = S$ and Q'' centralizes M. To complete the proof, we must show that one of the following two possibilities necessarily holds:

(a) $Q' = Q_1 \times S$ and $MQ_1 \lhd G$;

(b) $Q'' = Q_2 \times S$, $Q_2 \lhd G$, and MQ' is an exceptional subgroup. In the first case the lemma will follow with $T^* = MQ_1$ and $Q^* = Q''$; and in the second case with $T^* = MQ'$ and $Q^* = Q_2$.

Now $\bar{Q}$ is a regular $\bar{\phi}$-group and hence has the form $\bar{Q} = \bar{H}\bar{K}$, where $\bar{H}$, $\bar{K}$ satisfy the conditions of Lemma 1.3. Suppose first that $\bar{Q}' \subseteq \bar{H}$. Since $\bar{Q}'$ does not have $\bar{\phi}$-index 0, $\bar{Q}' \not\subset \bar{F}$, where $\bar{F}$ is the $\bar{\phi}$-nucleus of $\bar{Q}$. But then $\bar{Q}' = \bar{H} \supset \bar{F}$; and it follows from Lemma 1.3 that $\bar{K}$ is abelian and hence that $\bar{Q} = \bar{Q}' \times \bar{Q}_2$, where $\bar{Q}_2$ is $\bar{\phi}$-invariant. Thus (b) holds.

Suppose then that $\bar{Q}' \not\subset \bar{H}$. If $\tilde{Q}' = \bar{Q}'/\bar{S}$ has order greater than q, it follows from the structure of $\tilde{Q}'$ that $\bar{Q}' = \bar{H} \times \bar{S}$ and that $\bar{H} \supset \bar{F}$. But then $\bar{Q}'' \subset \bar{K}$ and $\bar{Q} = \bar{H} \times \bar{Q}''$; and (a) holds. Finally if $o(\tilde{Q}') = q$, we must have $\bar{Q}' = \bar{S} \times \bar{Q}_1$, where $\bar{Q}_1$ is $\bar{\phi}$-invariant; otherwise $o(\bar{S}) = q$, $\bar{\phi}$ has order mq on $\bar{Q}'$, where $m = 3$ if $q = 7$ and $m = 2$ if $q = 3$, and $\bar{Q}' = \bar{H}$, contrary to assumption.

7. Some properties of exceptional ABA-groups. To help illuminate the discussion we shall give an example of an exceptional ABA-group G of type III and of order $6p^2$. Thus $G = AT = AMQ$, where M is abelian of type (p, p), $o(Q) = 3$, and $o(A) = 2p$. If (x_1, x_2) is a basis for M, $(y) = Q$, and $(a) = A$, we may assume, in view of Theorem 5, that

$$x_1 = a^2,\ \phi(x_2) = ax_2a^{-1} = x_2^{-1} \text{ and } \phi(y) = x_1^k y^{-1} x_1^{-k} \tag{26}$$

for some integer k. First of all, we must have $k = \frac{1}{2}(p+1)$ since otherwise y centralizes $A \cap M$.

Furthermore

$$yx_1y^{-1} = x_1^{\alpha}x_2^{\beta}, \qquad yx_2y^{-1} = x_1^{\gamma}x_2^{\delta} \tag{27}$$

for suitable integers α, β, γ, δ.

Applying ϕ to (27) gives

$$y^{-1}x_1y = x_1^{\alpha}x_2^{-\beta}, \qquad y^{-1}x_2^{-1}y = x_1^{\gamma}x_2^{-\delta}. \tag{28}$$

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From (27) and (28) together we deduce that

(29) $$\alpha = \delta \text{ and } \alpha^2 - \beta\gamma = 1.$$

The condition that y induce an automorphism of M of order 3 gives in addition

(30) $$\alpha = \tfrac{1}{2}(p - 1).$$

Conversely conditions (26)—(30) with $k = \frac{1}{2}(p + 1)$ are sufficient to define a group G of the form $AT = AMQ$ such that $T \lhd G$, $Z(T) = 1$, and $N(A) = A$. Furthermore, the elements $[y]_1^{2j}$ are in the coset $(A \cap M)x_2^{\beta k j}$ of $A \cap M$, while the elements $[y]_1^{2j+1}$ are in the coset $(A \cap M)x_2^{-\beta k j}y$. It follows that the elements $\phi^i([y]_1^j)$ include a set of coset representatives of $A \cap M$ in T. Thus G is in fact an ABA-group with $B = (ya)$. Exceptional ABA-groups of type II can be similarly constructed.

We shall now determine a few of the properties of ABA-groups which contain exceptional subgroups of types II or III.

LEMMA 7.1. *Let G be an ABA-group containing an exceptional subgroup* $T \neq 1$ *of type II or III. Then* $2 \mid o(A)$, $2 \mid o(B)$, $6 \mid o(G)$, *and if* B_2 *denotes the* 2*-Sylow subgroup of* B, $B_2 \not\subset A$.

Proof. By assumption $G^* = AT = AB^*A$, $B^* \subset B$, is an exceptional AB^*A-group. It clearly suffices to prove the lemma for G^*, and hence without loss of generality we may assume $G^* = G$. If $T = MQ$ with $M \neq 1$, we may also assume that $Q_0 = 1$, and that no proper subgroup of M is normal in G, for otherwise the lemma follows by induction on $o(G)$.

Thus M is elementary abelian of order p^m, where $m = 3$ if $p = 2$ and $m = 2$ if $p \neq 2$. Furthermore $o(Q) = q$, where correspondingly $q = 7$ or 3. In either case $2 \mid o(A)$, $6 \mid o(G)$ by Theorem 5. If $p = 2$, $(a^6) \lhd G$ and by induction we may assume $o(A) = 6$. Thus $o(G) = 168$. Since G is an ABA-group, there must exist an element y in T and an integer r such that the elements $\phi^i([y]_r^j)$ include a set of coset representatives of $A \cap M$ in T. By Theorem 5, we may take $r = 1$, and hence the element $b = ya$ will be a generator of B. Now $b^3 = (ya)^3 = y\phi(y)\phi^2(y)a^3$, so that by the structure of T, $b^3 \in M$. If $b^3 \in A \cap M$, the set ABA will contain less than 168 distinct elements. Thus $o(B) = 6$ and $B_2 \not\subset A$.

Similarly, if T is of type III and $M \neq 1$, we may assume $o(G) = 6p^2$, $o(A) = 2p$. Again we have $B = (b)$, where $b = (ya)$ for some element y in T, and $b^2 = (ya)^2 = y\phi(y)a^2 \in M$. Thus $b^{2p} = 1$. Since $b^p = (b^{p-1})b$ and $b^{p-1} \in M$ since p is odd, $b^p \in A$ would imply that $b \in AM$ and the set ABA would be contained in AM, which is not the case. Hence $o(B) = 2p$ and $B_2 = (b^p) \not\subset A$.

Suppose finally that $M = 1$ and T is an exceptional 3-group. Then by Theorem 4, $\bar{G} = G/Z(T) = \bar{A}\bar{T} = \bar{A}\bar{B}\bar{A}$, where $\bar{T}$ is an elementary abelian $\bar{\phi}$-group of order 9 on which $\bar{\phi}$ has order 6. Again we may assume $o(\bar{A}) = 6$. Now $\bar{B} = (\bar{b})$, where $\bar{b} = \bar{y}\bar{a}$, $y \in \bar{T}$. Thus $\bar{b}^6 = 1$. If $\bar{b}^3 \in \bar{A}$, then $[\bar{y}]_1^3 \in \bar{A} \cap \bar{T} = 1$, which is not the case. Thus $2 \mid o(B)$ and $B_2 \not\subset A$, completing the proof.

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THEOREM 6. *An ABA-group G cannot contain exceptional subgroups of both type II and III.*

Proof. Suppose G contains an exceptional subgroup T_1 of type II and an exceptional subgroup T_2 of type III with $T_i \neq 1$, $i = 1, 2$. Then $G_i = AT_i = AB_iA$, $B_i \subset B$, is an exceptional AB_iA-group of type II if $i = 1$ and of type III of $i = 2$. If B_1^* denotes the 2-Sylow subgroup of B_i, it follows from the preceding lemma that $B_1^* \not\subset A$, $i = 1, 2$. Since B is cyclic, this implies $B_1^* \cap B_2^* \not\subset A$ and hence that $G_1 \cap G_2 \not\subset A$. We shall derive a contradiction by showing that, in fact, $G_1 \cap G_2 = A$.

By Theorem 5 and the definition of exceptional subgroups, ϕ has order 3.2^s7^k on T_1 for some s and k. Hence if $p \neq 2, 3, 7$, the p-Sylow subgroup A_p of A is normal in G_1 and is the p-Sylow subgroup of G_1. Now M_2 is abelian of type (t, t), where $2 \nmid t$, $3 \nmid t$. Let S_p be the p-Sylow subgroup of M_2 for some prime $p \mid t$. If $p \neq 7$, it follows at once that $S_p \cap G_1 \subset A_p$.

Suppose that $p = 7$ and $S_p \cap G_1 \not\subset A_p$. $S_p = (A_p \cap M_2) \times L_p$, where L_p is cyclic, invariant under A, and the order of ϕ on L_p is 2. Our assumptions imply that $L_p \cap G_1 \neq 1$. On the other hand, if $H_1 = C(M_1) \cap Q_1$, A_pH_1 is the unique maximal A-invariant p-group in G_1. Since $L_p \cap G_1$ is A-invariant we must have $L_p \cap G_1 \subset A_pH_1$. But by the structure of G_1, ϕ has order 3.7^c on H_1 and hence on A_pH_1, contrary to the fact that ϕ has order 2 on $L_p \cap G_1$. Thus $S_p \cap G_1 \subset A_p$ if $p = 7$, and we conclude that $M_2 \cap G_1 \subset A$.

If $M_2 \neq 1$, set $H_2 = C(M_2) \cap Q_2$; while if $M_2 = 1$, set $H_2 = Z(Q_2)$. In either case H_2 is A-invariant and hence so is $H_2 \cap G_1$. But, by the structure of T_1, A_3 is the only A-invariant 3-Sylow subgroup of T_1; and hence $H_2 \cap G_1 \subset A_3$. Thus $M_2H_2 \cap G_1 \subset A$.

Suppose finally that $x \in T_2 \cap G_1$, $x \notin M_2H_2$. Now $T_2 \cap G_1$ is A-invariant and contains $A \cap M_2H_2$. But by the structure of T_2, any A-invariant subgroup of T_2 which contains x and $A \cap M_2H_2$ necessarily contains a subgroup of M_2H_2 which properly contains $A \cap M_2H_2$. In particular, this must be true of $T_2 \cap G_1$, contrary to the fact that $M_2H_2 \cap G_1 = A \cap M_2H_2$. Thus $T_2 \cap G_1 \subset A$. Since $G_2 = AT_2$, $G_2 \cap G_1 = A$, and the theorem is proved.

We shall also need an analogous result for ABA-groups which contain exceptional subgroups of types I and III.

LEMMA 7.2. *If $G = ABA$ contains exceptional subgroups T_1, T_2 of types I and III respectively, then $Z(T_1) \subset A$ and a 3-Sylow subgroup of T_2 has order 3.*

Proof. If $G_1 = AT_1 = AB_1A$ and $G_2 = AT_2 = AB_2A$, it follows as in Theorem 6 that $G_1 \cap G_2 = A$. Furthermore, by Lemma 7.1 the 2-Sylow subgroup B_2^* of B_2 does not lie in A.

Now $T_1 = QQ'$, where Q, Q' satisfy condition (e) of Theorem 3. If $G_0 = AQ = AB_0A$, $B_0 = (b_0)$, where $b_0 = ya^r$, $y \in Q$. If k is the least integer such that $[y]_r^k = 1$, then $b_0^k \in A$. Furthermore, since ϕ has order $3 \cdot 2^s$ on Q, $k = 3 \cdot 2^m$ or 2^m according as $3 \nmid r$ or $3 \mid r$. Now the 2-Sylow subgroup B_0^* of B_0 must lie in A; otherwise $B_0^* \cap B_2^* \not\subset A$ and $G_1 \cap G_2 \supset A$. It follows that $b_0^3 \in A$.

If K is a maximal A-invariant normal subgroup of Q, we may assume without loss that $AK \subset AQ$, for otherwise we can replace Q by K in Theorem 3. It follows that $AK = A(b_0^3)A = A$, and hence that $K \subset A$. Since Q/K is elementary of order 4, $\Omega_1(K)$ is a quaternion group and $Z(Q) \subset A$. Now $A \cap Q' = 1$ and if $Q' \neq 1$, the proof of Theorem 3 shows that $Q \cap Q'$ contains an abelian subgroup of type (2, 2). Since Q contains no such A-invariant subgroup, $Q' = 1$ and $Z(T_1) \subset A$ as asserted.

The preceding argument shows that the 3-Sylow subgroup of $B_1(= B_0)$ does not lie in A. This forces the 3-Sylow subgroup of B_2 to lie in A, otherwise $G_1 \cap G_2 \supset A$. If $T_2 = M_2Q_2$, we consider $\bar{G}_2 = G_2/M_2(A \cap Q_2) = \bar{A}\bar{B}_2\bar{A} = \bar{A}\bar{Q}_2$. $\bar{Q}_2$ is a regular $\bar{\phi}$-group and $\bar{\phi}$ has order 2.3^s on $\bar{Q}_2$. If $\bar{B}_2 = (\bar{b}_2)$, where $\bar{b}_2 = \bar{y}_2\bar{a}^{r_2}$, $\bar{y}_2 \in \bar{Q}_2$, it follows as above that $\bar{b}_2{}^2 \in \bar{A}$. But then $[\bar{y}]_{r_2}^2 = 1$, and hence $o(\bar{Q}_2) = 3$, forcing $A \cap Q_2 = 1$ and $o(Q_2) = 3$.

8. Strongly factorizable *ABA*-groups. We shall call an ABA-group G *strongly factorizable* if $G = AT$, where $T = T_1 \times T_2 \times T_3$, each T_i is normal in G; and if $T_1 \neq 1$, then T_1 is an exceptional subgroup of type I, if $T_2 \neq 1$, then T_2 is an exceptional subgroup of type II or III, and $A \cap T_3 = 1$.

A number of consequences of this definition follow immediately from our previous results. First of all, T_3 is a regular ϕ-group and hence is nilpotent of class $\leqq 2$. Furthermore since A is cyclic, $T_1 \neq 1$ implies that either T_2 is of type III or $T_2 = 1$.

The definition also implies that G is solvable and that $T = [G, G]$. Finally, if $G = ABA = AM$, where M is nilpotent and normal in G, it follows from Theorems 2, 3, and 4 that G is in fact strongly factorizable.

Theorem B is an immediate corollary of the following theorem, which has been our main objective in Part I.

THEOREM B′. *If $G = ABA$ and G is solvable, then G is strongly factorizable*

The proof will be broken up into a sequence of lemmas.

LEMMA 8.1. *If $G = ABA$ is strongly factorizable, then so is every subgroup of G containing A and every homomorphic image of G.*

Proof. If G' is a subgroup of G containing A, $G' = AT'$, where $T' = G' \cap T$, $T' \lhd G'$. Clearly $T' = T_1' \times T_2' \times T_3'$, where $T_i' \subseteq T_i$, $i = 1, 2, 3$, and $A \cap T_3' = 1$. If $T_2 = MQ$, Q contains a maximal subgroup Q_0 which is A-invariant such that MQ_0 is nilpotent and $[T_2: MQ_0] = q$. It follows from Theorem 5 and the definition of exceptional subgroups of type II and III that either $T_2' = T_2$ or $T_2' \subseteq MQ_0$. In the latter case AT_2' possesses a normal 2-complement. Similarly either T_1' is an exceptional subgroup of type I or AT_1' possesses a normal 2-complement. It follows at once that G' is strongly factorizable.

If $G' = A'B'A'$ is a homomorphic image of G, we need only show that $N(A') = A'$, for the remaining parts of the definition of strong factorizability follow as above. Now $M^* = T_1 \times MQ_0 \times T_3$ is nilpotent and $[T: M^*] = q$.

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If $\bar{G} = G/M^* = \bar{A}\bar{Q}$, $N(\bar{A}) = \bar{A}$, so that the hypotheses of Lemma 6.1 are satisfied. Hence if $G' = G/L$ and $L \subseteq M$, $N(A') = A'$. If $L \subset T$, but $L \not\subset M^*$, then necessarily $T_2 \subseteq L$.

Since $G' = G/M/L/M$, it follows readily that $N(A') = A'$. If $L \subset A$, this conclusion is obvious. Since $L = (A \cap L)(T \cap L)$, $N(A') = A'$ in all cases.

LEMMA 8.2. *Let $G = ABA = AT$, $T \lhd G$, $T = PQ$, P a p-group normal in G, Q a q-group with $q \neq p$, and assume that G contains no normal subgroups which lie properly between P and T. Then G is strongly factorizable.*

Proof. The proof is by induction on $o(G)$. If $\bar{G} = G/P = \bar{A}$, $G = AP$ and G is strongly factorizable. The lemma also holds, as remarked above, if T is nilpotent; and it follows from Theorem 5 if $Z(T) = 1$. Hence we may assume that none of these conditions prevail.

Let P_1 be a minimal subgroup of $Z(T)$, normal in G. Since T is not nilpotent, $P_1 \subset P$. If $\bar{G} = G/P_1$, $N(\bar{A}) = \bar{A}$ by Lemma 6.1, and hence by induction $\bar{G} = \bar{A}\bar{T}$, where $\bar{T} = \bar{T}_1 \times \bar{T}_2 \times \bar{T}_3$ satisfies the required conditions.

If $\bar{T}$ is nilpotent, then so is T. We must therefore have $\bar{T}_2 = \bar{M}\bar{Q}$, with $\bar{M} \neq 1$. In particular, this implies $\bar{T}_1 = 1$; otherwise $p = 2$ and $\bar{T}_2$ if of type II, which is not possible for strongly factorizable groups, as pointed out above.

If $\bar{T}_3 \neq 1$, it follows by induction that the inverse image H_2 of $\bar{T}_2$ is of the form $P_1 \times T_2$, where T_2 is an exceptional subgroup of type II or III. Furthermore if H_3 is the inverse image of $\bar{T}_3$ in G, $H_3 \lhd G$ and $H_3Q = H_3 \times Q$. If $T_2 = MQ$, we have, for any x in M and any z in H_3, $[x, z] = z'$, $z' \in P_1$. Conjugating this relation by y in Q, it follows that $[x, y]$ commutes with z. But y acts regularly on M if $y \neq 1$, and hence $MH_3 = M \times H_3$. Thus $G = A(T_2 \times H_3)$. Since $p \neq 3$, the lemma now follows from Theorems 2 and 3. We may thus assume that $\bar{T}_3 = 1$ and hence that $G = AH_2$.

Now by the minimality of P_1 either $P_1 \subset A$ or $P_1 \cap A = 1$. Assume first that $P_1 \subset A$. Let $\bar{K} = \Omega_1(M)$ and let K be its inverse image in H_2. By Theorem 5 $\bar{K}$ is elementary abelian of order 8 of p^2 and $\bar{A} \cap \bar{K} \neq 1$. But this implies $P_1 \subset L \subset K$, where L is Q-invariant, contrary to the fact that $\bar{Q}$ leaves no proper subgroup $\neq 1$ of $\bar{K}$ invariant. Thus $P_1 \cap A = 1$.

Suppose first that $\bar{M}$ is a 2-group. Since $o(A \cap K) = 2$ we must have $K = \Omega_1(K)$, otherwise we reach a contradiction as above. A similar argument shows K is abelian, whence $K = (A \cap K) \times K_1$, where K_1 is A-invariant. Suppose K_1 were not of the form $P_1 \times L$, where L is A-invariant. Then $o(P_1) = 4$, $o(K_1) = 16$ and ϕ has order 6 on K_1. But the image $\bar{K}_1$ of K_1 in M is a regular $\bar{\phi}$-group, and by the structure of $\bar{T}_2$, its $\bar{\phi}$-index is a multiple of 3. Therefore K_1 is of ϕ-index 0 and ϕ acts irreducibly on $\Omega_1(K_1) = K_1$, a contradiction. Thus $K_1 = P_1 \times L$, where L is A-invariant. Furthermore, by Lemma 1.2 P_1 contains no A-invariant subgroups of type (2, 2). Hence if M' denotes the inverse image of $\bar{M}$ in G, Theorem 3 implies that $M' = P_1 \times M$, where M is A-invariant. Now $\Omega_1(M) = (A \cap K)L$ and $\Omega_1(M)$ is clearly Q-invariant. Thus $\Omega_1(M) \lhd G$, and it follows at once by induction applied to $G/\Omega_1(M)$ that $H_2 = P_1 \times MQ$. Hence the lemma holds if $\bar{T}_2$ is of type II.

Essentially the same argument applies if $\bar{T}_2$ is of type III, provided we can prove that K is abelian. Since $[K, K]$ is cyclic and $\Omega_1(K) = K$, this will necessarily be the case unless $o(P_1) = p$ and $K = (A \cap K)K_1$, where $A \cap K_1 = 1$, K_1 is elementary abelian of order p^2, and ϕ has order $2p$ on K_1. But this leads to a contradiction since again K_1 is of ϕ-index 0.

LEMMA 8.3. *Let $G = ABA = AT$, $T \lhd G$, $T = PQ$, P a p-group normal in G, Q a q-group with $q \neq p$ and $A \cap T \subset P$. Then G is strongly factorizable.*

Proof. Let $\bar{G} = G/P = \bar{A}\bar{Q}$, let $\bar{Q}_1$ be a minimal subgroup of the centre of $\bar{Q}$ invariant under $\bar{A}$, and let Q_1 be its inverse image in Q. We may assume $Q_1 \subset Q$ since otherwise the lemma follows from the preceding lemma.

If $Q_1 \lhd G$, we set $\tilde{G} = G/Q_1 = \tilde{A}\tilde{T}$. Since $A \cap Q_1 = 1$, $N(\tilde{A}) = \tilde{A}$ by the corollary of Lemma 2.3. Hence by induction $\tilde{G}$ is strongly factorizable, whence $\tilde{G} = A(\tilde{T}_1 \times \tilde{T}_2 \times \tilde{T}_3)$ where the subgroups $\tilde{T}_i$ satisfy the required conditions. Let H_i be the inverse image of $\tilde{T}_i$, $i = 1, 2, 3$, and let P_1 be the p-Sylow subgroup of H_1H_3. Since Q_1 is normal in PQ_1, Q_1 is in the centre of H_2H_3, and it follows at once that H_1H_3 is nilpotent. Thus $H_1H_3 = P_1 \times S$, where S is the q-Sylow subgroup of H_1H_3.

We may assume $\tilde{T}_2 \neq 1$ since otherwise the lemma follows immediately. Now the group H_2S satisfies the hypotheses of Lemma 6.3 and consequently $H_2S = T_2 \times Q'$, where T_2 is an exceptional subgroup of type II or III, $Q' \subset S$, and Q' is A-invariant. Our conditions also imply that $P_1T_2 = P_1 \times T_2$, and it follows at once that G is strongly factorizable.

We may therefore assume that $Q_1 \ntriangleleft G$. By induction $G_1 = APQ_1$ is strongly factorizable, and hence $G_1 = A(T_1 \times T_2 \times T_3)$ where the subgroups T_i have the appropriate properties. If $T_2 = 1$, Q_1 is in the centre of the nilpotent group T_1T_3. Since Q_1 is a q-Sylow subgroup of T_1T_3, it is A-invariant and hence normal in G, contrary to assumption. Thus $T_2 \neq 1$.

Now $T_1 \neq 1$ implies $p = 2$. But this is impossible since then T_2 would be of type II. Thus $T_1 = 1$. Furthermore $T_2 = MQ_1$, and $o(Q_1) = q$ by the minimality of $\bar{Q}_1$. Furthermore $P = (A \cap P)MT_3$. If $x \in T_3$ and $y \in Q$, the normality of P implies $yxy^{-1} = zx'$, $z \in (A \cap P)M$, $x' \in T_3$. Conjugating this relation by $y_1 \neq 1$ in Q_1 we conclude immediately that y_1 and z commute. But M is A-invariant and $A \cap M \neq 1$. Since A is cyclic, it follows if $z \neq 1$ that $z^i \in M$ for some integer i, with $z^i \neq 1$. But this is a contradiction since T_2 has a trivial centre. Thus $z = 1$ and hence $T_3 \lhd G$.

If $T_3 = 1$, $P = (A \cap P)M$. If $A \cap P \supset A \cap M$, it follows readily from the structure of M that Q_1 normalizes $A \cap M$, which is not the case. Thus $A \cap P = A \cap M$. If $T_3 \neq 1$, we can obtain the same conclusion by considering G/T_3, since $A \cap T_3 = 1$. Thus $P = M \times T_3$.

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Finally, if $z \in M$, and $y \in Q$, we have $yzy^{-1} = z'x$, $z' \in M$, $x \in T_3$. Conjugating this relation by $y_1 \neq 1$ in Q_1, we readily obtain $y[z, y_1]y^{-1} = [z', y_1]$. Since Q_1 acts regularly on M, it follows that $M \lhd Q$, and hence by Lemma 6.3 $MQ = MQ_2 \times Q'$ where each factor is A-invariant. Thus $G = A(MQ_2 \times Q'T_3)$, and since $A \cap Q'T_3 = 1$, G is strongly factorizable.

LEMMA 8.4. *Let $G = ABA$ and assume that G contains a normal subgroup P of prime power order such that $\bar{G} = G/P = \bar{A}\bar{B}\bar{A}$, is an exceptional $\bar{A}\bar{B}\bar{A}$-group. Then G is strongly factorizable.*

Proof. The proof will be made by induction on $o(G)$. Let $o(P) = p^m$ and $\bar{G} = \bar{A}\bar{T}$, where $\bar{T}$ is an exceptional subgroup. We first consider the case in which no proper subgroup of P is normal in G.

Assume $K \lhd G$, where K is a q-group. If $\tilde{G} = G/K = \tilde{A}\tilde{B}\tilde{A}$, we first show that $N(\tilde{A}) = \tilde{A}$. By the minimality of P, either $K \supset P$ or $K \cap P = 1$. If $K \supset P$, $\tilde{G}$ is a homomorphic image of $\bar{G}$, whence $N(\tilde{A}) = \tilde{A}$ by Lemma 8.1. If $K \cap P = 1$ and $\tilde{P}$ denotes the image of P in $\tilde{G}$, $\tilde{G}/\tilde{P}$ is a homomorphic image of $\bar{G}$ and hence $N(\tilde{A}) \subset \tilde{A}\tilde{P}$. But KP is nilpotent and hence $G_0 = AKP$ is strongly factorizable. If $\tilde{G}_0 = G_0/K$, it follows that $N_{\tilde{G}_0}(\tilde{A}) = \tilde{A}$, whence $N(\tilde{A}) = \tilde{A}$, as asserted.

Let H be the inverse image of $\bar{T}$ in G. We distinguish three cases.

Case 1. $\bar{T}$ of type I. We may assume $p \neq 2$ since otherwise the lemma follows immediately from Theorem 3. Thus $H = PQ$, where Q is a 2-Sylow subgroup of H. We may assume Q contains the inverse image of $\bar{A} \cap \bar{T}$. (Since $A \cap H$ need not be contained in P, the lemma is not a consequence of the preceding lemma.) We may assume $C(P) \cap Q = 1$, otherwise the lemma follows by induction or from the preceding lemma by considering $G/C(P) \cap Q$. Let $\bar{K}$ be a maximal subgroup of $\bar{T}$ normal in $\bar{G}$. Then $\bar{A}\bar{K} = \bar{A}\bar{K}_1$, where $\bar{K}_1$ is $\bar{A}$-invariant and either $\bar{A} \cap \bar{K}_1 = 1$ or $\bar{K}_1$ is an exceptional subgroup. If K_1 denotes the inverse image of K_1 in Q it follows either from the preceding lemma or by induction that $PK_1 = P \times K_1$, whence $K_1 = 1$ and $\bar{K} \subset \bar{A}$. Thus $\bar{Q} = (\bar{A} \cap \bar{Q})\bar{Q}_1$, where $\bar{A} \cap \bar{Q} = Z(\bar{Q})$ and $\bar{Q}_1$ is a quaternion group. Without loss we may assume $\bar{Q} = \bar{Q}_1$. Since $A \cap Q$ centralizes $A \cap P$ and $A \cap Q \subseteq Z(Q)$, the minimal nature of P implies that $A \cap P = 1$. But then the conditions of Lemma 5.4 are satisfied, and hence PQ is nilpotent.

Case 2. $\bar{T} = \bar{M}\bar{Q}$ is of type II. We assume $p \neq 2$, otherwise the lemma follows from Lemma 6.3. If M denotes a 2-Sylow subgroup of the inverse image of $\bar{M}$ in G, $G_0 = APM$ is strongly factorizable by induction. Hence $G_0 = A(P \times M_0)$, where $M = (A \cap M)M_0$. Since $C(P) \cap M \lhd G$, it follows from the structure of $\bar{T}$ that $M = M_0$. If $\tilde{G} = G/M = \tilde{A}\tilde{P}\tilde{Q}$, $\tilde{P}\tilde{Q}$ is nilpotent by Lemma 6.3, and the lemma follows at once.

Case 3. $\bar{T} = \bar{M}\bar{Q}$ is of type III. $\bar{M}$ is abelian of type (t, t) with $(t, 6) = 1$; and $\bar{Q}$ is a 3-group. Assume first that $\bar{M} \neq 1$. If $p \nmid t$, it follows as in case 2 that the inverse image of $\bar{A}\bar{M}$ in G has the form $G_0 = A(P \times M)$, where $M \lhd G$, except possibly if $p = 2$ and $7 \mid t$. In this case, it may happen that $G_0 = A(T_2 \times M)$, where T_2 is an exceptional subgroup of type II. But then the 7-Sylow subgroup S_0 of $C(P) \cap G_0$ has index 7 in a 7-Sylow subgroup S of G_0 and is normal in G. It follows that $[\bar{S} \cap \bar{M} : \bar{S}_0 \cap \bar{M}] = 7$ and $\bar{S}_0 \cap \bar{M} \lhd \bar{T}$, contrary to the structure of $\bar{T}$. Thus $G_0 = A(P \times M)$, where $M \lhd G$. By induction $\tilde{G} = G/M = \tilde{A}\tilde{P}\tilde{Q}$ is strongly factorizable. By the minimal nature of P, either $\tilde{P}\tilde{Q}$ is nilpotent or $\tilde{P}\tilde{Q}$ is an exceptional subgroup of type III. In either case the lemma follows at once.

If $p \mid t$, $p \neq 2, 3$. In this case $G_0 = A(P_0 \times M_0)$, where P_0, $M_0 \lhd G$ and P_0 is a p-group containing P. If $M_0 \neq 1$, the lemma follows easily by induction; hence we may assume $M_0 = 1$ and hence that $\bar{M}$ is a p-group. Furthermore we may assume that $\bar{A} \cap \bar{Q} \neq 1$; otherwise the lemma follows from the preceding one. Thus $G = AP_0Q$, where Q is a 3-group of class 2 and $A \cap Q \neq 1$. We may also assume $Z(Q)$ does not centralize P_0; otherwise the lemma follows by induction. Now $[P_0, P_0]$ is cyclic, and hence either P_0 is abelian or $[P_0, P_0] = P$ has order p. But in this case $Z(Q)$ centralizes P and consequently P_0. Thus P_0 is abelian. It follows now exactly as in the proof of Lemma 8.2 that $P_0 = P \times P_1$, where P_1 is normal in G and $A \cap P = 1$. The lemma follows at once by induction by considering G/P_1.

There remains the case $\bar{M} = 1$. Thus $G = APQ$, $\mathrm{cl}(Q) = 2$, and $A \cap Q \neq 1$. As in case 1 we may assume $C(P) \cap Q = 1$. Since $A \cap Q \subseteq Z(Q)$ and $A \cap Q$ centralizes $A \cap P$, it follows from the minimality of P that $A \cap P = 1$. Hence if $\bar{K}$, K, and K_1 are as in case 1, we must have $K_1 = 1$ and $\bar{K} \subseteq \bar{A}$. But by the structure of $\bar{Q}$, a maximal $\bar{A}$-invariant subgroup of $\bar{Q}$ does not lie in $\bar{A}$.

This completes the induction when no proper subgroup of P is normal in G.

Case 4. P is not a minimal normal subgroup of G. Let P_0 be a minimal subgroup of $Z(P)$ normal in G. If $\tilde{G} = G/P_0 = \tilde{A}\tilde{B}\tilde{A}$, $\tilde{G}$ is strongly factorizable by induction. Thus $\tilde{G} = A(\tilde{T}_1 \times \tilde{T}_2 \times \tilde{T}_3)$, the subgroups $\tilde{T}_i$ having the required properties. Let H_i be the inverse image of $\tilde{T}_i$ in G, $i = 1, 2, 3$. Under the hypotheses of the lemma, if $\tilde{T}_1 \neq 1$ and $\tilde{T}_2 \neq 1$, then $p \neq 2$.

Assume first that $\tilde{T}_2 \neq 1$. Then H_1H_3 is a p-group and P_0 is in its centre. P_0 must therefore be a minimal normal subgroup of AH_2, and it follows from Case 2 or 3 that AH_2 is strongly factorizable.

If $AH_2 = A(P_0 \times T_2)$ where T_2 is an exceptional subgroup, then $G = A(T_2H_1H_3)$ and H_1H_3 commutes elementwise with all elements of T_2 of order prime to p. The lemma follows immediately if $p \nmid \mathrm{o}(T_2)$. Let $T_2 = MQ$, and suppose next that Q is a p-group, in which case $p = 3$ or 7 and $H_1H_3 = H_3$. If $M \neq 1$, the lemma follows by considering G/M; while if $M = 1$, it follows from Theorem 4. Assume next that $p \mid \mathrm{o}(M)$. If $p = 2$, MH_1H_3 is a 2-group, $A \cap Q = 1$, and Lemma 6.3 applies. If $p \neq 2$, we may assume M is a p-group, or else the lemma follows by induction. Since $p \neq 2, 3$, A possesses a normal complement P^* in AH_3, which is normal in G, and centralized by Q. Furthermore $M = (A \cap M) \times M^*$, where M^* is A-invariant. Thus P^*M^* is a regular ϕ-group, whence by Lemma 1.6, $P^*M^* = P^* \times M^*$. Since $C(P^*) \lhd G$, we must have $P^*M = P^* \times M$, and the lemma follows.

On the other hand, if $AH_2 \neq A(P_0 \times T_2)$, H_2 is necessarily an exceptional subgroup and $H_1H_3 = H_3$. Thus $G = AH_2H_3$, $H_2 \cap H_3 = P_0$, and H_3 commutes with all elements of H_2 of order prime to p.

If $H_2 = MQ$, $P_0 \subseteq M$. As above, we may assume M is a p-group and $p \neq 2$. If $A \cap Q = 1$, the preceding lemma applies; so assume $A \cap Q \neq 1$. It follows now as in case 3 that $C(MH_3) \cap Q \neq 1$, and the lemma follows by induction.

Finally, if $\tilde{T}_2 = 1, G = AH_1H_3$. If $p = 2$, H_1H_3 is a 2-group and G is strongly factorizable. If $p \neq 2$, it follows readily that $AH_1 = A(P_0 \times T_1)$, where T_1 is an exceptional 2-group and T_1 centralizes H_3. Again G is strongly factorizable, and the lemma is proved.

With the aid of the preceding lemmas we shall now establish Theorem B'. The proof will be by induction on $o(G)$. Let P be a minimal normal subgroup of G. If $A_0 \subset A$ and $G_0 = N(A_0)$, G_0/A_0 is strongly factorizable by induction. It follows at once from the corollary of Lemma 6.1 that $A_0 \subseteq Z(G_0)$. Since P is an abelian p-group, Lemma 2.3 now yields $N(A) = A$, where $G = G/P = \bar{A}\bar{B}\bar{A}$. Thus by induction $\bar{G}$ is strongly factorizable so that $\bar{G} = \bar{A}(\bar{T}_1 \times \bar{T}_2 \times \bar{T}_3)$, where the subgroups $\bar{T}_i$ satisfy the required conditions. Let H_i be the inverse image of T_i in G, $i = 1, 2, 3$.

We shall distinguish three cases.

Case 1. $P \subset A$. By Lemma 2.1, $P \subseteq Z(H_3)$, whence H_3 is nilpotent. If $p \neq 2$, Lemma 8.4 implies H_1 is nilpotent, while if $p = 2$, $P \subseteq Z(H_1)$ since $o(P) = 2$. Thus H_1H_3 is nilpotent and it follows from Theorems 2 and 3 that $AH_1H_3 = A(T_1 \times T_2 \times T_3)$ is strongly factorizable. If $T_2 \neq 1$, then $p = 3$ and T_2 is an exceptional 3-group of type III. Furthermore, by Lemma 8.4, either $H_2 = P$, $H_2 = P \times T_2^*$, where T_2^* is an exceptional subgroup of type II or III, or $p = 3$ and $H_2 = T_2^*$ is an exceptional subgroup of type III.

If $T_2 \neq 1$, then by Theorem 6, either $H_2 = P$ or T_2^* is of type III. But in the latter case, it follows that a homomorphic image $\tilde{G}$ of $\bar{G}$ contains two $\tilde{\phi}$-invariant subgroups of order 3, each disjoint from A; and this is impossible by Lemma 1.2. Thus $H_2 = P$ and G is strongly factorizable. We may therefore assume $T_2 = 1$ and $H_2 \neq P$.

Suppose T_1T_3 is not a p-group and let S be an r-Sylow subgroup of T_1T_3, $r \neq p$. If $x \in S$, $y \in H_2$, we have $[x, y] = z \in P$. Since P centralizes S and H_2, $[x^p, y] = 1$ and it follows that S centralizes H_2. But then we conclude that G is strongly factorizable by considering G/S and applying induction. Hence we may assume T_1T_3 is a p-group, in which case the theorem follows from Lemma 8.4.

Case 2. $A \cap P = 1$. This time Lemma 8.4 gives $H_2 = P$ or $H_2 = P \times T_2$, where T_2 is an exceptional subgroup of types II or III. Furthermore H_1H_3 is nilpotent and $AH_1H_3 = A(T_1 \times T_3)$ is strongly factorizable. It follows as in the preceding paragraph that G is strongly factorizable.

Case 3. $P \not\subset A$, $A \cap P \neq 1$. We may suppose that no minimal normal subgroup of G lies in A or is disjoint from A.

Assume first that $\bar{T}_2 \neq 1$. Then $G' = AH_1H_3$ is strongly factorizable by induction. Suppose G' contained a normal subgroup L of order prime to p such that $A \cap L = 1$. Then L centralizes P and the image $\bar{L}$ of L in $\bar{G}$ centralizes $\bar{T}_2$, whence L centralizes H_2. Thus $L \lhd G$, contrary to assumption. Suppose next that G' contains an exceptional subgroup T' of type II or III. By Lemma 8.4 and Theorem 6, H_2 also contains an exceptional subgroup of the same type; and this leads to a contradiction as in case 1. We conclude that G' has the form $A(T_1 \times T_3)$, where T_3 is a p-group. If $p = 2$ or $T_1 = 1$, PT_1T_3 is a p-group and the theorem follows from Lemma 8.4. In the remaining case AH_2T_3 is strongly factorizable by induction, and the theorem follows at once.

Assume finally that $\bar{T}_2 = 1$. If $p \neq 2$, then Lemma 8.4 implies that $H_1 = P \times T_1$, where either $T_1 = 1$ or T_1 is exceptional of type I. Furthermore, it follows as in case 1 that H_3 centralizes T_1. This forces $T_1 = 1$, otherwise G contains a minimal normal subgroup which lies in A or is disjoint from A. Let $\bar{Q}$ be a q-Sylow subgroup of $\bar{T}_3$ with $q \neq 3$ or p and suppose $\bar{Q} \neq 1$. By Lemma 8.4 the inverse image of $\bar{Q}$ in G is nilpotent and again G contains a minimal normal subgroup which lies in A or is disjoint from A. Thus $o(\bar{T}_3) = p^c3^d$ and the theorem follows from Lemma 8.3 if $p \neq 3$ and from Theorem 4 if $p = 3$.

On the other hand, if $p = 2$, it follows as in the preceding paragraph that $o(\bar{T}_3) = 2^c7^d$. In this case Lemma 8.3 and Theorem 3 show that G is strongly factorizable. This completes the proof of Theorem B'.

Theorem B' has the following corollary.

COROLLARY. *Let $G = ABA$ be a non-strongly factorizable ABA-group of lowest possible order. Then G does not possess a non-trivial normal subgroup of prime power order.*

PART II

THE SOLVABILITY OF ABA-GROUPS

Having determined the structure of solvable ABA-groups, we turn now to the proof of Theorem A. In view of Theorem B', this is equivalent to showing that every ABA-group is strongly factorizable. Throughout Part II G will denote an ABA-group of least order which is not strongly factorizable. Hence all proper subgroups and homomorphic images of G which are themselves ABA-groups will be strongly factorizable. Furthermore, by the corollary of Theorem B', G contains no non-trivial normal subgroups of prime power order.

9. ABA-groups which possess a normal A-complement. Let $G = ABA$ and let p be a prime dividing $o(A)$. We shall call p *non-exceptional* if

(a) G contains an A-invariant p-Sylow subgroup P^*;

(b) If A_p is a p-Sylow subgroup of A, then $P^* = A_pP$, where $P \lhd P^*$ and $A_p \cap P^* = 1$;

(c) $N(X)$ possesses a normal A_p-complement for every A-invariant normal subgroup $X \neq 1$ of P^*.

Otherwise we call p *exceptional.*

THEOREM 7. *If p is non-exceptional, then G contains a normal subgroup K_p such that $G = A_pK_p$, $A_p \cap K_p = 1$.*

Proof. Let P^*, P be as above. If p is odd, it will suffice by the Hall-Wielandt theorem (**6**, Theorem 14.4.2) to find a weakly closed subgroup P_0 of P^* such that either $P_0 \subseteq Z_{p-1}(P^*)$ or P_0 is abelian, since $N(P_0)$ possesses a normal A_p-complement.

Now P is a regular ϕ-group. Let F be its ϕ-nucleus and set $\bar{A}\bar{P} = AP/F$. We know that $\bar{P}$ is elementary abelian and $\bar{\phi}$ has order prime to p on $\bar{P}$. Hence $\bar{A}_p$ centralizes $\bar{P}$ and $\bar{P}^* = P^*/F$ is abelian. In particular, P^* is abelian if $F = 1$, and we may take $P_0 = P^*$. If F is elementary abelian, $\mathrm{cl}(P^*) \leqq 2$ and we again may take $P_0 = P^*$. If F is cyclic or abelian on two generators, we write $P = HK$, where H, K satisfy the conditions of Lemma 1.3. It follows readily that K and $\Omega_1(H)$ lie in $Z_2(P^*)$ and hence $\Omega_1(P) \subseteq Z_2(P^*)$. Furthermore by the structure of H, $\Omega_1(A_p) \subseteq Z_2(P^*)$; thus $\Omega_1(P^*) \subseteq Z_2(P^*)$ and we may take $P_0 = \Omega_1(P^*)$.

This argument breaks down for $p = 2$. In this case we can apply the Hall-Wielandt theorem only if P_0 is a weakly closed subgroup of $Z(P^*)$. We shall show in fact that either F_1 is a weakly closed subbroup of P^* or $\Omega_1(P^*) \subseteq Z(P^*)$.

Suppose $F_1^x \subset P^*$. Since $x = a^ib^sa^j$ for suitable i, s, j and P^* is A-invariant, $F_1^{b^s} \subset P^*$. Since F_1 is A-invariant, it suffices to prove that $F_1^{b^s} = F_1$. Suppose first that for some z in F_1,

$$b^szb^{-s} = a_1z', \tag{31}$$

where $(a_1) = \Omega_1(A_p)$ and $z' \in P$.

Now $AP = AB_pA$ with $B_p = (b_p) \subseteq B$. Thus $b_p = ya^r$, for some y in P and some integer r, so that P is of ϕ-index r and y is a ϕ-generator of P. Consider first the case that ϕ^r leaves only the identity element of F_1 fixed and let k be the order of ϕ^r on F_1. Conjugating (31) by $b_p{}^i$ for $i = 0, 1, \ldots, k-1$, we obtain

$$b^s\phi^{ri}(z)b^{-s} = a_1z_i', \tag{32}$$

where $z_i' \in P$, $i = 0, 1, \ldots, k-1$.

Multiplying these relations together for $i = 0, 1, \ldots, k-1$, we obtain $1 = a_1{}^kz^*$, where $z^* \in P$. But this is impossible since k is prime to p and $A \cap P = 1$.

On the other hand, if ϕ^r is the identity on F_1, $b_p = ya^r$ centralizes z and consequently also a_1z'. Since $\bar{\phi}^r$ leaves only the identity element of $\bar{P} = P/F$ fixed, $z' \in F$, and hence $a_1z' \in Z(A_2F)$ by Lemma 4.2. Thus b_p centralizes z' and consequently also a_1. We conclude that a_1 centralizes the ϕ-generator y of P and hence lies in $Z(P^*)$. Now $P = HK$. If $H \supset F$, $\bar{\phi}$ has order k on $\bar{H} = H/F$ and $\bar{\phi}^r$ leaves only the identity element of $\bar{H}$ fixed. But since ϕ^r acts trivially on F_1, $k \mid r$ and $\bar{\phi}^r$ acts trivially on $\bar{H}$, a contradiction. Thus $H = F$. But then it follows that $\Omega_1(P^*) = (a_1)\Omega_1(K) \subseteq Z(P^*)$ and we may take $P_0 = \Omega_1(P^*)$.

Therefore we may assume that $F_1^{b^s} \subset P$. Hence for any z in F_1, we have

$$b^s z b^{-s} = z', \tag{33}$$

where $z' \in \Omega_1(P)$.

If ϕ^r leaves only the identity element of F_1 fixed, then it follows as in the preceding case that $[z']_r^k = 1$, where k is the order of ϕ on F_1. If $\tilde{P} = P/H = \tilde{K}$, it follows from (d) of Lemma 1.3 that $\tilde{\phi}^k$ leaves only the identity element of $\tilde{K}$ fixed and hence the same is true of $\tilde{\phi}^{kr}$. But $[\tilde{z}']_r^k = 1$, and this implies that $\tilde{\phi}^{kr}(\tilde{z}') = \tilde{z}'$. Thus $\tilde{z}' = 1$ and $z' \in \Omega_1(H)$.

We may assume that $Z' \notin F_1$ since otherwise $F_1^{b^s} = F_1$ and F_1 is weakly closed in P^*. But then $\Omega_1(H)$ is elementary abelian and ϕ has order $2k$ on $\Omega_1(H)$. Let k' be the order of $\tilde{\phi}^r$ on $\tilde{K} = K/F_1$ and set $y' = [y]_r^{k'}$. Then y' is a ϕ-generator of H of ϕ-index $r' = rk'$. Furthermore, k' is not a multiple of k and hence $\phi^{r'}$ leaves only the identity element of F_1 and consequently of F fixed. We first prove that r' is odd.

If we set $k_1 = k/(r', k)$, then $y_1 = [y']_{r'}^{k_1}$ is a ϕ-generator of F. Suppose $2 \mid r'$, and assume first that $F = F_1$. Then H is abelian and ϕ has order $2k$ on H. Thus $\phi^{r'}(y_1) = \phi^{r'}([y']_{r'}^{k_1}) = \phi^{r'}(y'\phi^{r'}(y') \ldots \phi^{r'(k_1-1)}(y')) = y_1$, contrary to the fact that $\phi^{r'}$ leaves only the identity element of F_1 fixed. If $F \subset F_1$, we obtain the same contradiction by considering $H/\mho_1(F)$. Thus r' is odd, as asserted.

Now $b_p^{k'} = y'a^{r'}$. Hence if we conjugate (33) by $b_p' = b_p^{k'k_1}$, we see that b_p' centralizes b^szb^{-s} and hence centralizes z'. Suppose first that $F = F_1$. Since $b_p' = y_1a^{r'k_1}$, we conclude that $\phi_1(z') = z'$, where $\phi_1 = \phi^{r'k_1}$. Since $k \mid r'k_1$, ϕ_1 acts trivially on F_1. Since the subgroup of H left elementwise fixed by ϕ_1 is invariant under ϕ, it follows, if $z' \notin F_1$, that ϕ_1 acts trivially on H. Since $r'k_1$ is odd, we conclude that ϕ has order k on H, contrary to the fact that ϕ has order $2k$ on H. On the other hand, if $F \supset F_1$, we obtain the same contradiction by considering $H/\mho^1(F)$.

Suppose finally that ϕ^r is the identity on F_1. Then as above $H = F$ and $\Omega_1(P) = \Omega_1(K) \subseteq Z(P)$. But then conjugating (33) by b_p, we conclude that $\phi^r(z') = z'$. Since $z' \in K$ and $\tilde{\phi}^r$ leaves only the identity element of $\tilde{K}$ fixed, $z' \in F_1$; and it follows that F_1 is weakly closed in P^*.

LEMMA 9.1. *If $p \mid o(A)$, but $p \nmid o(T)$ for any exceptional subgroup T of G, then p is non-exceptional.*

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Proof. Let P^* be a maximal A-invariant p-subgroup of G containing A_p. Since G contains no normal p-subgroups, $N(P^*) \subset G$ and hence $N(P^*)$ is strongly factorizable. Thus $N(P^*) = AT^* = A(T_1^* \times T_2^* \times T_3^*)$. By our hypothesis a p-Sylow subgroup P of T^* necessarily lies in the nilpotent group T_3^*, which is disjoint from A. By the maximality of P^*, we must have $P^* = A_pP$. Thus P^* is a p-Sylow subgroup of $N(P^*)$ and hence of G. Furthermore, P is A-invariant, normal in P^*, and $A_p \cap P = 1$.

Finally if $X \neq 1$ is any A-invariant normal subgroup of P^*, then $N(X)$ is strongly factorizable and hence $N(X) = AT'$, $T' \lhd N(X)$ and $A_p \cap T' = 1$. Since A is abelian, $N(X)$ possesses a normal A_p-complement. Thus P is non-exceptional, as asserted.

THEOREM 8. *Let $G = ABA$, and assume that every prime dividing $o(A)$ is non-exceptional. Then $[G, G]$ is a normal complement of A in G and is nilpotent of class 1 or 2. In particular, G is solvable. Furthermore the hypotheses are satisfied if $2 \nmid o(A)$ or $6 \nmid o(G)$.*

Proof. It follows readily from the preceding lemma that the assumptions of the theorem are satisfied if and only if G contains no exceptional subgroups. In particular, Theorem 3 and Lemma 7.1 show that this is the case if $2 \nmid o(A)$ or $6 \nmid o(G)$.

If G is solvable, Theorem B' implies that $G = AT$, $T \lhd G$, and $A \cap T = 1$. Since $N(A) = A$, we must have $T = [G, G]$; and since T is a regular ϕ-group, it is nilpotent of class 1 or 2.

Let then G be a non-solvable ABA-group of least order satisfying the conditions of the theorem. By Theorem 7, $G = A_pK_p$, where $K_p \lhd G$ and $A_p \cap K_p = 1$. If

$$T = \bigcap_{p \mid o(A)} K_p,$$

then $T \lhd G$, $G = AT$, and $A \cap T = 1$. Thus T is a regular ϕ-group, whence T and G are solvable, a contradiction.

10. Proof of Theorem A.

In view of Theorem 8, G must contain an exceptional subgroup T. Suppose $T = MQ$ if of type II or III with $M \neq 1$. Let M_1 be a minimal normal subgroup of AT and set $G^* = N(M_1)$. Then we have

LEMMA 10.1. *G^* contains a q-Sylow subgroup Q^* such that $N(Q^*) \subset G^*$. In particular, Q^* is a q-Sylow subgroup of G.*

Proof. By Theorem 5 we may assume $\phi(Q) = uQu^{-1}$, where $u \in A \cap M$. Thus if $o(A \cap M) = t$, $\phi^t(Q) = Q$ and since $(t, q) = 1$, Q is invariant under the q-Sylow subgroup A_q of A. Since G^* is strongly factorizable, $G^* = AT^*$, where $T^* \lhd G^*$ and $T^* = T_1^* \times T_2^* \times T_3^*$. Clearly $T \subseteq T_2^*$, and without loss we may assume $T = T_2^*$. If Q' denotes a q-Sylow subgroup of T_3, Q' is

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A-invariant and $Q^* = A_q QQ'$ is a q-Sylow subgroup of G^*. Let $Q_0 = C(M) \cap Q$. Then if $y \in Q \tilde{M} Q_0$, we have

$$\phi(y) = y^c z',\ z' \in MQ_0, \tag{34}$$

and $c = 2$ if $q = 7$, $c = 1$ if $q = 3$.

If now $x \in N(Q^*)$, we can write $x = a^i b^s a^j$. Since $\phi(Q^*) \subset MQ^*$ $b^s a^j Q^* a^{-j} b^{-s} \subset MQ^*$. In particular, $b^s \phi^j(y^d) b^{-s} \in MQ^*$ for all d. By (34) we can choose d so that $\phi^j(y^d) = yz$, $z \in MQ_0$, and hence

$$b^s yz b^{-s} \in MQ^*. \tag{35}$$

Now $AM_1 = AB_1A$, where $B_1 = (b_1) \subset B$, $b_1 = va^r$ for some v in M_1 and some integer r. By the structure of T, m divides r, where $m = 3$ if $q = 7$ and $m = 2$ if $q = 3$. Furthermore, ϕ has order m on M_1 and $M_1 \subseteq Z(M)$. By our minimal choice of M_1, M_1 is an elementary abelian p-group for some prime p and hence $b_1^p = (va^r)^p = v^p a^{rp} = a^{rp}$. Since G contains no normal subgroups of prime power order, $A \cap B = 1$ and consequently $a^{rp} = 1$. We conclude that $a^r \in A \cap M_1$ and hence that $b_1 \in M_1 \subseteq Z(M)$.

It follows now from (35) that $[b_1, b^s(yz)^i b^{-s}] = b^s[b_1, y^i]b^{-s} \in MQ^*$ for all i. But Q acts irreducibly on M_1 and hence $b^s M_1 b^{-s} \subseteq MQ^*$. Thus $xM_1x^{-1} \subset MQ^*$. But M_1 contains all elements of order p in MQ^*; therefore $xM_1x^{-1} = M_1$ and $x \in G^*$. Thus $N(Q^*) \subset G^*$, as asserted. Since Q^* is a q-Sylow subgroup of $N(Q^*)$, Q^* is a q-Sylow subgroup of G.

From this lemma we can derive the following extension of Theorem 7.

LEMMA 10.2. *If G contains an exceptional subgroup $T = MQ$ of types II or III such that $A \cap T \subset M$, then G contains a normal subgroup K_q such that $G = A_q K_q$ and $A_q \cap K_q = 1$.*

Proof. Let Q^* be as in Lemma 10.1 and let $\bar{A}\bar{Q}^* = AMQ^*/M$. Then $\bar{Q}^* = \bar{A}_q \bar{Q}\bar{Q}'$ and $\bar{Q}\bar{Q}'$ is a regular $\bar{\phi}$-group. If $\mathrm{cl}(\bar{Q}^*) \leqq 2$, then $\mathrm{cl}(Q^*) \leqq 2$. Since $N(Q^*) \subset G^*$ and q is prime to $o(M)$, $N(Q^*)$ contains a normal A_q-complement, and hence by the Hall-Wielandt theorem, so does G.

But now by the proof of Theorem 7, either $\mathrm{cl}(Q^*) \leqq 2$ or $\Omega_1(Q^*) \subseteq Z_2(Q^*)$; and hence we may assume that $\Omega_1(Q^*) \subseteq Z_2(Q^*)$. If $\Omega_1(Q^*)$ centralizes M, then $\Omega_1(Q^*)$ is A-invariant and it follows that $G' = N(\Omega_1(Q^*))$ is strongly factorizable and contains T. If $G' = AT'$, where $T' = T_1' \times T_2' \times T_3'$, we must have $T \subseteq T_2'$ and hence G' possesses a normal A_q-complement. Again the lemma follows from the Hall-Wielandt theorem.

On the other hand, the proof of Lemma 10.1 applies equally well to any subgroup of Q^* which does not centralize M. Hence in the remaining case, $N(\Omega_1(Q^*)) \subset G^*$ and the lemma follows as above.

LEMMA 10.3. *G does not contain an exceptional subgroup of type II.*

Proof. Suppose G contains an exceptional subgroup $T = MQ$ of type II.

Then by Theorem 6, G does not contain an exceptional subgroup of type III, and hence no exceptional subgroup of G has order divisible by 3. But $3 \mid o(A)$ by Theorem 5 and hence 3 is non-exceptional by Lemma 9.1. Thus by Theorem 7, we have $G = A_3K_3$, where $K_3 \lhd G$ and $A_3 \cap K_3 = 1$. Since $A \cap T \subset M$, the preceding lemma implies that $G = A_7K_7$, where $K_7 \lhd G$ and $A_7 \cap K_7 = 1$. If $L = K_3 \cap K_7$, then $L \lhd G$ and $A_3A_7 \cap L = 1$.

Let M_1, G^*, and Q^* be as in Lemma 10.1, and let A^* be the subgroup of A generated by the elements of order prime to 3 and 7. Then $G^* = AT^*$, where $T^* = [G^*, G^*]$, and $Q^* = A_7(Q \times Q')$. Now QQ' is a 7-Sylow subgroup of L and since $N(QQ') \subset G^*$, $N(QQ') \cap L \subset A^*T^*$. But $\bar{\phi}$ has order 3.7^s on $\bar{Q} = MQ/M$; hence A^* centralizes Q and A^*T^* possesses a normal Q-complement. Since $\mathrm{cl}(QQ') \leqq 2$, we conclude that $L = QH$, where $H \lhd L$ and $Q \cap H = 1$.

Now clearly $\phi(x) \in H$ for any element x of H of order prime to 7. Since Q' is a 7-Sylow subgroup of H, $\phi(x) \in H$ if $x \in Q'$. If x is any 7-element of H, then $x = ux'u^{-1}$, $x' \in Q'$ and $u \in H$. But then $\phi(x) = \phi(u)\phi(x')\phi(u^{-1})$, where $\phi(x') \in Q'$. Since $\phi(u) \in L$ and $H \lhd L$, it follows that $\phi(x) \in H$. We conclude that H is A-invariant. Since A_7Q' is a 7-Sylow subgroup of AH and $A_7Q' \subset Q^*$, $AH \subset G$, and consequently H is solvable by induction. Thus L and consequently G is solvable, a contradiction.

LEMMA 10.4. *G does not contain an exceptional subgroup of type III.*

Proof. Suppose G contains an exceptional subgroup $T = MQ$ of type III. Assume first that 2 is exceptional. Since G does not contain an exceptional subgroup of type II, it must then contain an exceptional subgroup T_1 of type I. We may therefore apply Lemma 7.2. First of all, this yields $A \cap T = A \cap M$, and hence by Lemma 10.2, $G = A_3K_3$, where $K_3 \lhd G$ and $A_3 \cap K_3 = 1$. Secondly we have $\Omega_1(T_1) \subseteq A$. Now it is easy to see that G possesses an A-invariant 2-Sylow subgroup R containing T_1, and hence by Theorem 3 $\Omega_1(A_2) \subseteq Z(R)$. In the next lemma we shall show that this forces $\Omega_1(A_2)$ to be weakly closed in R, so assume this. Now $G' = N(\Omega_1(R))$ is strongly factorizable. It follows at once that $G' \cap K_3$ possesses a normal A_2T_1-complement. But then by the Hall-Wielandt theorem applied to K_3, we have $K_3 = (A_2T_1)H$, where $H \lhd K_3$ and $A_2T_1 \cap H = 1$. As in the preceding lemma, H is A-invariant and $AH \subset G$. Thus H and hence G is solvable, a contradiction.

Hence 2 is non-exceptional. Therefore by Theorem 7, $G = A_2K_2$, where $K_2 \lhd G$ and $A_2 \cap K_2 = 1$. Suppose next that $M \neq 1$. If $Q^* = A_3(Q \times Q')$ and G^* are as in Lemma 10.1, Q^* is a 3-Sylow subgroup of G. If $A \cap T = A \cap M$, Lemma 10.2 yields $G = A_3K_3$, $K_3 \lhd G$ and $A_3 \cap K_3 = 1$. Let $L = K_2 \cap K_3$. Since $\bar{\phi}$ has order $2 \cdot 3^s$ on $\bar{Q} = MQ/Q$, it follows as in the preceding lemma that $L = QH$, where $H \lhd L$, H is A-invariant, and $AH \subset G$; again we reach a contradiction.

On the other hand, if $A \cap Q \neq 1$, it follows from Theorem 4 that $\Omega_1(Q^*) \subseteq Z_2(Q^*)$. But then the Hall-Wielandt theorem gives $K_2 = (A_3Q)H$, where $H \lhd K_2$. Once again H is A-invariant and $AH \subset G$, which leads to a contradiction.

Finally, if $M = 1$, G contains an A-invariant 3-Sylow subgroup Q^* containing Q, which by Theorem 4 has the form $A_3(Q \times Q')$, where Q' is abelian and A-invariant. Since $N(\Omega_1(Q^*))$ is strongly factorizable, we reach a contradiction as in the preceding case.

Finally we prove

LEMMA 10.5. *G does not contain an exceptional subgroup of type I.*

Proof. Suppose G contains an exceptional subgroup T_1 of type I. We may assume that a 2-Sylow subgroup R of G has the form $A_2(T_1 \times T_2)$, where T_1, T_2 satisfy the conditions of Theorem 3. By the preceding lemma, 3 is non-exceptional and hence $G = A_3K_3$, $K_3 \lhd G$, $A_3 \cap K_3 = 1$. It will suffice to show that $Z(R)$ contains a weakly closed subgroup, for then we shall reach a contradiction as in the first part of the proof of Lemma 10.4.

Now $AR = AB_pA$ with $B_p = (b_p) \subseteq B$. Thus $b_p = ya^r$ with y in R. Let $T_1 = QQ'$, where Q,Q' satisfy the conditions of Theorem 3 and let $Z_1 = \Omega_1(Z(Q))$. Then $Z_1 \subseteq Z(R)$ and $Z_1 = (A \cap Z_1) \times F_1$, where F_1 is A-invariant of order 1 or 4. Suppose first that $F_1 \neq 1$ and ϕ^r is the identity on F_1. If $Z_1' = Z_1^{b^s} \subset R$ for some s, it follows as in Theorem 7 that $Z_1' \subset Q$ and $[Z_1', B_p] = 1$. But then $Z_1' = Z_1$ by Lemma 4.2, and this implies that Z_1 is weakly closed in R.

Suppose next that $F_1 \neq 1$, ϕ^r leaves only the identity element of F_1 fixed, and $F_1' = F_1^{b^s} \subset R$. Again as in Theorem 7 we have $F_1' \subseteq Q$ and

$$z'\phi^r(z')\phi^{2r}(z') = 1,\ z' \in F_1'. \tag{36}$$

We shall prove by induction on $o(Q)$ that (36) forces $F_1' = F_1$, from which it will follow that F_1 is weakly closed in R. By induction we may assume that $F_1' \subseteq Q_1$, where $Q_1 \lhd AQ$, and $(A \cap Q_1)F_1$ is normal and of index 4 in Q_1. Set $AQ_1/F_1 = \bar{A}\bar{Q}_1$. If $\bar{Q}_1$ is the central product of $\bar{A} \cap \bar{Q}_1$ and a quaternion group, it is easy to see that (36) forces $\bar{F}_1' = 1$. Hence we may assume $\bar{Q}_1 = (\bar{A} \cap \bar{Q}_1) \times \bar{F}$ is elementary, where $\bar{F}$ is $\bar{A}$-invariant and $o(\bar{F}) = 4$. Let F be the inverse image of $\bar{F}$ in Q.$_1$ Since Q does not possess a normal A-complement, F is of ϕ-index 0 and hence abelian of type (4, 4). But clearly (36) implies $F_1' \subseteq F$, whence $F_1' = F_1$.

Suppose finally that $Z_1 \subset A$ and Z is not weakly closed in R. Then for some s, $Z_1' = Z_1^{b^s} \subset R$ and $Z_1' \neq Z_1$. As in the first case, $Z_1' \subseteq Q$ and $[Z_1', B_p] = 1$. Lemma 4.2 now implies that $Z_1' \subset Z_1B'$, where $B' \subseteq B \cap Q$ and $o(B') = 2$. Since B is abelian, it follows that b^s normalizes $H = Z_1B'$ and that b^{2s} centralizes H. Thus $b^s \in C^*(H)$, where $C^*(H)$ denotes the extended centralizer of H in G. But $C^*(H) \subseteq C(Z_1)$ and hence $Z_1' = Z_1$, a contradiction. The lemma is proved.

Lemmas 10.3, 10.4, and 10.5 show that G contains no exceptional subgroups. But then every prime dividing $o(A)$ is non-exceptional, and Theorem 8 shows that G must be solvable. This completes the proof of Theorem A.

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