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On Inductive Limit Type Actions of the Euclidean Motion Group on Stable UHF Algebras

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Abstract. An invariant is presented which classifies, up to equivariant isomorphism, C^*-dynamical systems arising as limits from inductive systems of elementary C^*-algebras on which the Euclidean motion group acts by way of unitary representations that decompose into finite direct sums of irreducibles.

1 Introduction

There is now a long history of classification results for C^*-dynamical systems of the following form: We have a C^*-algebra given as an inductive limit and actions of a group on each of the algebras in the inductive system such that the connecting maps are equivariant, resulting in an action on the limit algebra. Most of the results obtained so far have been for compact groups. Handelman and Rossmann [9, 10] classified actions of compact groups on AF algebras that left invariant some increasing sequence of finite dimensional subalgebras with dense union under the restriction that the actions of the group on the finite dimensional subalgebras arose from homomorphisms of the group into their unitaries. They called such actions locally representable. In [2, 11] this was extended to inductive systems with more complicated C^*-algebras, but a local representability condition was still required. In [8], the local representability condition was removed for the special case where the group is $\mathbb{Z}/2\mathbb{Z}$ and the algebras in the inductive system are finite dimensional. In the case of non-compact groups, AF flows were classified in [3, 4].

In this paper, we obtain the first such classification result for a group which is neither compact nor abelian, namely the Euclidean motion group. We shall make use of some common notations: for the fixed point subalgebra of a C^*-algebra A under the action α of the group G we shall write either A^α or A^G depending on whether the group or the action is being emphasized. We write $M(A)$ for the multiplier algebra of the C^* algebra A, and we shall sometimes find it convenient to write $(A, \alpha) \otimes (B, \beta)$ for the C^*-dynamical system $(A \otimes B, \alpha \otimes \beta)$.

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2 The Euclidean Motion Group and its Representations

In this section we recall a few facts about the Euclidean motion group. A full exposition of this material may be found in [14]. See also [5].

The Euclidean motion group, E, is the group of transformations of the plane generated by rotations about the origin and translations. If $a \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we denote by $t(a)$ the translation by the vector a, and by $r(\alpha)$ the counterclockwise rotation by α radians. We then have that $t(a)t(b) = t(a+b)$, $r(\alpha)r(\beta) = r(\alpha+\beta)$, and $r(\alpha)t(a) = t(e^{i\alpha}a)r(\alpha)$, so that any element of E may be written $t(a)r(\alpha)$ for some $a \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. E may be viewed as a subgroup of $GL_2(\mathbb{C})$ via the embedding $t(a)r(\alpha) \mapsto \left(\begin{smallmatrix} e^{i\alpha} & a \\ 0 & 1 \end{smallmatrix}\right)$. The translations form a normal subgroup of E, with the rotations as a complimentary subgroup, so we may also express E as $\mathbb{R}^2 \rtimes \mathbb{T}$, where the rotations act on $\mathbb{R}^2$ in the usual way.

The irreducible unitary representations of E are of two types. The first kind are the one dimensional representations that arise from the representations of $\mathbb{T}$ by passing to the quotient of E by the translations. Explicitly, for each integer n we have a representation χ_n given by $\chi_n(t(a)r(\alpha)) = e^{in\alpha}$. These representations are inequivalent for distinct integers. The second kind are infinite dimensional, and may be given explicitly as follows. For each complex number $a \neq 0$ we define a unitary representation U^a of E on $L_2(\mathbb{T})$ by $[U^a(g)F](s) = e^{i(z,sa)}F(r(\alpha)^{-1}s)$, where $s \in \mathbb{T}$, $(z, sa) = Re(z\overline{sa})$, $F \in L_2(\mathbb{T})$, and $g = t(z)r(\alpha)$. It is shown in [12] that U^a and U^b are equivalent if and only if $|a| = |b|$, so that we need only consider the case of $a > 0$. With $a > 0$, we have in polar form $[U^a(g)F](\theta) = e^{ia\rho\cos(\varphi-\theta)}F(\theta-\alpha)$, where $g = t(\rho e^{i\varphi})r(\alpha)$.

Below we shall need a few additional facts which are easily deduced from the material in [14], in particular, the following theorem.

Theorem 2.1 *For all integers n, m and real numbers $a > 0$, we have $\chi_n \otimes \chi_m \cong \chi_{(n+m)}$ and $\chi_n \otimes U^a \cong U^a$.*

Proof The map from $\mathbb{C} \otimes \mathbb{C}$ to $\mathbb{C}$ given by $a \otimes b \mapsto ab$ gives a unitary equivalence between $\chi_n \otimes \chi_m$ and $\chi_{(n+m)}$. To see the second equivalence, first note that $\chi_n \otimes U^a$ is an infinite dimensional irreducible unitary representation of E, so for some $b > 0$, $\chi_n \otimes U^a \cong U^b$. To see that $b = a$, we examine the proof of the classification theorem for irreducible representations of E given as [14, Theorem 2.1, p. 165]. In that proof it is shown that the spectral measure for the restriction of U^a to $\mathbb{R}^2$, the translations, is concentrated on a circle of radius a centered at the origin. Thus two representations U^a and U^b are equivalent if and only if their restrictions to $\mathbb{R}^2$ are. If we consider the restriction of $\chi_n \otimes U^a$ to $\mathbb{R}^2$, we see that it is just $1 \otimes U^a|_{\mathbb{R}^2}$, which is unitarily equivalent to $U^a|_{\mathbb{R}^2}$ via the map from $\mathbb{C} \otimes L_2(\mathbb{T})$ to $L_2(\mathbb{T})$ given by $a \otimes v \mapsto av$. ■

3 Elementary C^*-Dynamical Systems

In this section we describe the classes of C^*-dynamical systems we shall be concerned with, beginning with the following definitions.

Definition 3.1 Given a locally compact group G, we define a *concrete elementary*

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C^*-dynamical system to be one of the form $(K(H), \mathrm{Ad}\, U)$, where U is a unitary representation of G on the Hilbert space H. A C^*-dynamical system (A, α) will be called an *elementary C^*-dynamical system* if and only if it is equivariantly isomorphic to some concrete elementary C^*-dynamical system. By a *finite atomic elementary C^*-dynamical system* we shall mean one for which the group is E and which is isomorphic to a concrete elementary C^*-dynamical system in which the representation is a finite direct sum of irreducible representations. Given two finite atomic elementary C^*-dynamical systems, (A, α) and (B, β), we shall follow established usage in calling an equivariant *-homomorphism $\varphi\colon A \to B$ *proper* if the hereditary subalgebra of B generated by $\varphi(A)$ is all of B.

We shall be concerned with C^*-dynamical systems of the following form: $(A, \alpha) = \varinjlim\{(A_n, \alpha_n), \varphi_{nm}\}$, where for each n, (A_n, α_n) is a finite atomic elementary C^*-dynamical system and the φ_{nm}s are all proper equivariant *-homomorphisms.

Our first step is to see what kind of proper inclusions of one finite atomic elementary C^*-dynamical system into another are possible. Let (A, α) and (B, β) be two such C^*-dynamical systems, and let $\varphi\colon A \to B$ be a proper equivariant *-homomorphism. The action β on B extends to an action, which we shall also call β, on $M(B)$, the multiplier algebra of B. Now $\varphi(A)' \cap M(B) \cong M_n$, where n is the multiplicity of the embedding φ. This copy of M_n is invariant under the extended action β. We then have $(B, \beta) \cong (A \otimes M_n, \alpha \otimes \beta|_{M_n})$ under an isomorphism that carries the map $\varphi\colon A \to B$ to the map $a \mapsto a \otimes 1$. It is easy to see that the possible actions of E on M_n are all of the form $Ad(\chi_{k_1} \oplus \cdots \oplus \chi_{k_n})$, for some set of one dimensional representations $\chi_{k_1}, \ldots, \chi_{k_n}$ of E. The list of representations appearing is uniquely determined up to tensoring the whole set with a single one dimensional representation. We can make the list unique by insisting that $k_i \geq 0$ for each i, and that the smallest $k_i = 0$ (recall Theorem 2.1). We summarise this in the following theorem.

Theorem 3.2 *Let (A, α) and (B, β) be two finite atomic elementary C^*-dynamical systems and let $\varphi\colon A \to B$ be a proper equivariant *-homomorphism. Then there exists a unique list of natural numbers $n_1, \ldots, n_k$ such that there is an isomorphism of (B, β) with $(A, \alpha) \otimes (M_{(k+1)}, \mathrm{Ad}\, W)$ that carries φ to the map $a \mapsto a \otimes 1$, where $W = \chi_0 \oplus \chi_{n_1} \oplus \cdots \oplus \chi_{n_k}$.*

Our next step is to consider the fixed point subalgebra in a finite atomic elementary C^*-dynamical system. Let (A, α) be such a system and let $a \in A^\alpha$. Assume $(A, \alpha) = (K(H), \mathrm{Ad}\, U)$, where $U = \chi_{n_1} \oplus \cdots \oplus \chi_{n_k} \oplus U^{b_1} \oplus \cdots \oplus U^{b_l}$. Let P denote the projection onto the subspace of H corresponding to the χ_{n_i}*s*. Then $P \in A^\alpha$. We shall show that $a = PaP$. Write $U = V \oplus U^{b_l}$, and let $q \in M(A)$ be the projection onto the subspace of H corresponding to V. Since U^{b_l} is both infinite dimensional and irreducible, there are no non-zero projections in $(1-q)A(1-q) \cap A^\alpha$. Consider now $(1-q)a(1-q)$. Then $(1-q) \in M(A)^\alpha$ and $a \in A^\alpha$, so $(1-q)a^*(1-q)a(1-q)$ is a self adjoint element of $(1-q)A(1-q) \cap A^\alpha$. The spectral theorem for compact operators combined with the absence of non-zero projections in this algebra shows that $(1-q)a^*(1-q)a(1-q) = 0$, and so $(1-q)a(1-q) = 0$. For $qa(1-q)$, we may apply a similar argument to $(1-q)a^*qa(1-q)$ and for $(1-q)aq$ we do the same thing with $(1-q)aqa^*(1-q)$. Now we get our result by repeating the above steps for successive U^{b_i}s.

From the above discussion we see that if (A,α) is a finite atomic elementary C^*-dynamical system, then A^α is a unital algebra with the projection P from above as its unit (we include $\{0\}$ as a unital C^*-algebra). Furthermore, from Theorem 3.2 we see that if (A,α) and (B,β) are two finite atomic elementary C^*-dynamical systems and $\varphi\colon A\to B$ is a proper equivariant *-homomorphism, then φ takes the unit of A^α to the unit of B^β. If we now consider $(A,\alpha)=\varinjlim\{(A_n,\alpha_n),\varphi_{mn}\}$, an inductive limit of finite atomic elementary C^*-dynamical systems with proper connecting maps, we see that A^α is unital.

Let $(A,\alpha)=\varinjlim\{(A_n,\alpha_n),\varphi_{mn}\}$ be an inductive limit C^*-dynamical system as above, and let $1_{A^E}\in A$ denote the unit of A^α. The action α restricts to an action of E on the hereditary subalgebra $1_{A^E}A1_{A^E}$. On this subalgebra the translations act trivially and we have an AF-type action of $\mathbb{T}$. The special case where $1_{A^E}A1_{A^E}=A$, *i.e.*, when all our elementary C^*-algebras are matrix algebras and the representations are finite dimensional, is discussed in Section 5 below. The special case where $1_{A^E}=0$, and all of our representations are infinite dimensional, is the subject of the next section.

4 Infinite Dimensional Representations

In this section, we consider the case in which the fixed point subalgebra is $\{0\}$ (only infinite dimensional representations). We begin by introducing some terminology.

Definition 4.1 Let INF denote the class of finite atomic elementary C^*-dynamical systems in which the fixed point subalgebra is $\{0\}$. Let $LIMINF$ denote the class of C^*-dynamical systems arising as inductive limits of elements of INF with proper connecting maps.

Suppose that $(A,\alpha)\in INF$, that (B,β) is a finite atomic elementary C^*-dynamical system, and that $\varphi\colon A\to B$ is a proper equivariant *-homomorphism. Then it is easy to see from Theorem 3.2 that $(B,\beta)\cong(A\otimes M_n,\alpha\otimes id)$ for some n via an isomorphism that carries φ to the map $a\mapsto a\otimes 1$. Thus if $\{(A_n,\alpha_n),\varphi_{nm}\}$ is an inductive system of elements of inf with proper connecting maps and (A,α) is the inductive limit, then $(A,\alpha)\cong(A_1\otimes M,\alpha_1\otimes id)$, where M is a UHF algebra. We shall prove the following theorem, showing that this decomposition is essentially unique.

Theorem 4.2 *Let $(A,\alpha)=\varinjlim\{(A_n,\alpha_n),\varphi_{mn}\}$, where, for each n, $(A_n,\alpha_n)\in INF$ and the connecting maps φ_{mn} are proper. For some UHF algebra M, we have $(A,\alpha)\cong(A_1,\alpha_1)\otimes(M,id)$. This decomposition is unique in the following sense. Suppose $(K(H),\operatorname{Ad}V)\otimes(M,id)\cong(K(H),\operatorname{Ad}W)\otimes(N,id)$, N and M are UHF algebras, $V=n_1U^{a_1}\oplus\cdots\oplus n_kU^{a_k}$, $W=m_1U^{b_1}\oplus\cdots\oplus m_lU^{b_l}$, with $a_i\neq a_j$ and $b_i\neq b_j$ for $i\neq j$, and that $\gcd(n_1,\dots n_k)=1=\gcd(m_1,\dots m_l)$. Then $N\cong M$ and $V\cong W$.*

Proof The existence of such a decomposition was observed above, so we proceed to show uniqueness. Recall the formula for the representation U^a on $L^2(\mathbb{T})$ given in

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Section 2 above. If we consider the restriction of U^a to just the rotations, and we let f_n denote the function $\theta \mapsto e^{in\theta}$ for $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we see that $\{f_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$ consisting of eigenvectors for this action, and that the eigenvalues are all distinct. We see that the fixed point subalgebra for the action restricted to the rotations, which we shall denote $A^{\mathbb{T}}$, is the copy of c_0 generated by the projections $\langle f_n \mid (\,\cdot\,)\rangle f_n$. Call the n-th such projection e_n.

Next, we consider the function $h(n,a)\colon E \to \mathbb{R}$ given by $g \mapsto \|e_n\alpha_g(e_n)\|$. We shall only need the case where $g > 0$, so that, in the notation of Section 2, $g = \rho$ and $\varphi = 0$. Then

$$\begin{aligned}
\|e_n\alpha_g(e_n)\| &= \|\langle f_n \mid (\alpha_g(e_n)(\,\cdot\,)\rangle f_n\| \\
&= \Big\| \Big\langle\, f_n \;\Big|\; \langle U^a(g)f_n \mid (\,\cdot\,)\rangle U^a(g)f_n \Big\rangle\, f_n \Big\| \\
&= \|\langle U^a(g)f_n \mid (\,\cdot\,)\rangle \langle f_n \mid U^a(g)f_n\rangle f_n\| \\
&= |\langle f_n \mid U^a(g)f_n\rangle|\, \|\langle U^a(g)f_n \mid (\,\cdot\,)\rangle f_n\| \\
&= |\langle f_n \mid U^a(g)f_n\rangle| \\
&= \left| \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}(e^{iag\cos\theta}e^{in\theta})\,d\theta \right| \\
&= \frac{1}{2\pi}\left| \int_0^{2\pi} e^{iag\cos\theta}\,d\theta \right|
\end{aligned}$$

From this we see that $h(n,a)$ is independent of n, and furthermore only depends the product ag. Let $h\colon \mathbb{R} \to \mathbb{R}$ denote the function such that $h(ax) = h(n,a)(x)$. It is easy to see that $h(0) = 1$, that $h(t) < 1$ for $t > 0$, and that h is continuous. It follows that if $a, b > 0$ and $a \neq b$, then for any integers n and m, $h(n,a) \neq h(m,b)$.

Now consider $(B,\beta) \cong (A,\alpha) \otimes (M, id)$ where $(A,\alpha) \cong (K(H), \operatorname{Ad} U^a)$ is an element of inf with no nontrivial invariant hereditary subalgebra, and M is a UHF algebra. If we let $B^{\mathbb{T}}$ denote the fixed point subalgebra of the restriction of β to the rotations, we see from the discussion above that $B^{\mathbb{T}} \cong c_0 \otimes M$. Furthermore, if we let p be a minimal central projection in $B^{\mathbb{T}}$, we have $\|p\beta_g(p)\| = h(ag)$ for $g > 0$. (In fact, we have $\|q\alpha_g(q)\| = h(ag)$ for $g > 0$ for any projection $q \leq p$, an observation we shall need below.) This shows that if $(B,\beta) \cong (C,\gamma) \otimes (N, id)$ for some other element (C,γ) of inf with no non-trivial invariant hereditary subalgebras, then $(C,\gamma) \cong (A,\alpha)$ and $N \cong M$.

Suppose $(B,\beta) \in \mathit{LIMINF}$, $A \subseteq B$, $(A, \beta|_A) \in \mathit{INF}$, and that the inclusion is proper. Suppose $(A, \beta|_A) \cong (K(H), \operatorname{Ad} V)$ where $V = n_1U^{a_1} \oplus \cdots \oplus n_kU^{a_k}$ with $a_i \neq a_j$ for $i \neq j$. The action $\beta|_A$ extends to an action on $B(H) \cong M(A)$, the multiplier algebra of A, via the same representation V. Then $V(E)' \cong \mathbb{C}^k$, so the fixed point subalgebra, $M(A)^\beta$, for the extension of the action to $M(A)$, is isomorphic to $\mathbb{C}^k$. Let $P_1, \ldots, P_k$ denote the minimal central projections for $M(A)^\beta$. Since the inclusion of A into B is proper, $M(A) \subseteq M(B)$, so we may view $P_1, \ldots, P_k$ as projections in $M(B)^\beta$.

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Now, in the same situation as in the paragraph above, assume further that $(B,\beta) \cong (A,\alpha) \otimes (M, id)$, where $\alpha = \beta|_A$ and M is a UHF algebra. We shall show that in this case $P_1,\ldots,P_k$ are central in $M(B)^\beta$. We have that the action β on B is given by a homomorphism $V\colon E \to U(B(H)) = U(M(A)) \subseteq U(M(B))$, where $U(C)$ denotes the unitary group of the unital C^*-algebra C, such that $\beta_g(x) = (\operatorname{Ad} V(g))(x)$. The extension of the action β to $M(B)$ is given by the same unitaries. Represent M faithfully and non-degenerately on a Hilbert space K. Then we have B represented faithfully and non-degenerately on $H \otimes K$, by a *-homomorphism, say π, and this representation extends to a faithful representation, which we shall also call π, of $M(B)$ on $H \otimes K$ such that $\pi(M(B))$ is the set of all operators in $B(H \otimes K)$ that multiply $\pi(B)$ into itself (*cf.* [12]). We then have $\pi(V(g)) = V(g) \otimes 1 \in B(H) \otimes 1 \subseteq B(H \otimes K)$, and similarly $\pi(P_i) = P_i \otimes 1$. Since the automorphisms in β are inner in $M(B)$, we have that $\pi(M(B)^\beta) \subseteq (\pi \circ V(E))'$, so we just have to see that for each i, P_i is central for $(\pi \circ V(E))'$. This follows from the fact that the representations $V(\cdot)P_i \otimes 1$ of E are pairwise disjoint (*cf.* [6]).

Assume the same notation as in the statement of the theorem. Let $B = K(H) \otimes M \cong K(H) \otimes N$, let $P_1,\ldots,P_k$ be the projections in $M(B)$ corresponding to the subspaces for $n_1Ua_1,\ldots,m_1U^{a_k}$ in V, and let $Q_1,\ldots,Q_l$ be those for $m_1U^{b_1},\ldots,m_lU^{b_l}$ in W. Then from above we have that $Q_iP_j = P_jQ_i$ for each i and j. Assume that for some fixed i and j we have $P_iQ_j \neq 0$. Let $D_{ij} = (P_iQ_j)B(Q_jP_i) = (Q_jBQ_j) \cap (P_iBP_i)$. Let $\{e_n\}$ denote the minimal central projections in $(Q_jBQ_j)^{\mathbb{T}} \cong c_0 \otimes M_{m_j} \otimes N$ and let $\{f_n\}$ denote the minimal central projections in $(P_iBP_i)^{\mathbb{T}} \cong c_0 \otimes M_{n_i} \otimes M$. Let a be a non-zero positive element in D_{ij}. Then the element $b = \int_{\mathbb{T}} \alpha_\theta(a)\,d\theta$ is a non-zero positive element in $(D_{ij})^{\mathbb{T}}$. For some k, the element $c = e_kbe_k$ is non-zero, and the hereditary subalgebra of B generated by this element is contained in e_kBe_k. The algebra e_kBe_k is a UHF algebra, and has real rank zero. It is known that in a C^*-algebra with real rank zero every hereditary subalgebra has an approximate unit consisting of projections (*cf.* [13]), so in particular we may choose a non-zero projection $p \in cBc$. Clearly $p \leq e_k$. Since D_{ij} is hereditary, $p \in (D_{ij})^{\mathbb{T}}$. For some n, we have that $f_npf_n \neq 0$, and $0 \leq f_npf_n \leq p$. The hereditary subalebra of B generated by f_npf_n is contained in f_nBf_n, which is a UHF algebra. As above, we may choose a non-zero projection q in the hereditary subalgebra generated by f_npf_n. Then $q \leq p \leq e_k$, and $q \leq f_n$. From the analysis of the case with just one representation, applied to Q_jBQ_j, and the first of these conditions, we see that $\|q\alpha_g(q)\| = h(b_jg)$ for $g > 0$. From the second condition we have $\|q\alpha_g(q)\| = h(a_ig)$ for $g > 0$. From this it follows that $b_j = a_i$. It is now easy to see that we must have $k = l$, and, possibly after reordering, $P_i = Q_i$, $a_i = b_i$, for $i = 1,\ldots,k$. Furthermore, the analysis of the case of one representation applied to each of P_iBP_i in turn shows that $M_{n_i} \otimes M \cong M_{m_i} \otimes N$ for each i.

Finally, it remains to check that the multiplicities are the same. With the notation as in the paragraph above, we identify N and M with the e_{11} corners in the respective decompositions of B and consider the K_0 classes $[1_N]$ and $[1_M]$ in $K_0(B) \subseteq \mathbb{Q}$. We have $m_i[1_M] = n_i[1_N]$ for $i = 1,\ldots,k$. It is easy to show that if p and q are positive non-zero rational numbers, $n_1,\ldots,n_k, m_1,\ldots,m_k$ are non-zero natural numbers, $\gcd(n_1,\ldots,n_k) = 1 = \gcd(m_1,\ldots,m_k)$, and $m_ip = n_iq$ for $i = 1,\ldots,k$, then $p = q$. Thus $[1_N] = [1_M]$, so $m_i = n_i$ for each i, and we have $V \cong W$ and $N \cong M$.

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This completes the proof of Theorem 4.2. ∎

If $V = n_1 U^{a_1} \oplus \cdots \oplus n_k U^{a_k}$ and $\gcd(n_1, \dots, n_k) = d$, then $(K(H), \operatorname{Ad} V) \cong (K(H), \operatorname{Ad} W) \otimes (M_d, id)$, where $W = (n_1/d)U^{a_1} \oplus \cdots \oplus (n_k/d)U^{a_k}$, so we see that any $(A, \alpha) \in LIMINF$ may be written in the form $(B, \beta) \otimes (M, id)$ with M a UHF algebra and $(B, \beta) \in \inf$ with the multiplicities in the corresponding representation having greatest common divisor 1 in a unique way. We call this tensor product expression the *canonical decomposition for* (A, α).

5 Finite Dimensional Representations

In this section, we review the classification of product-type actions of the circle on UHF algebras. To do this, one may use the results of any of [3, 4, 9, 10]. We shall follow [4]. In doing so we shall view the circle as $\mathbb{R}/2\pi\mathbb{Z}$ and regard circle actions as periodic actions of $\mathbb{R}$.

In [4, Definition 3.1] the ordered ring $\mathfrak{R}$ is defined via a Grothendieck construction. We begin with the set S of all unordered tuples of non-negative real numbers (finite sets of non-negative reals counted with multiplicity) and we define an addition $\oplus$ and a multiplication $\odot$ on S as follows. Writing $[x_1, \dots, x_n]$ for the element of S corresponding to the tuple $\langle x_1, \dots, x_m \rangle$, and $[\emptyset]$ for the element of S corresponding to the empty set,

$$[x_1, \dots, x_n] \oplus [y_1, \dots, y_m] = [x_1, \dots, x_n, y_1, \dots, y_m]$$
$$[\emptyset] \oplus X = X \oplus [\emptyset] = X \quad X \text{ any element of } S$$
$$[x_1, \dots, x_n] \odot [y_1, \dots, y_m] = [x_i + y_j, \quad 1 \le i \le n,\ 1 \le j \le m]$$
$$[\emptyset] \odot X = X \odot [\emptyset] = [\emptyset] \quad X \text{ any element of } S$$

With these definitions, $(S, \oplus)$ is an abelian semigroup with cancellation, and we can enlarge it into a group via the Grothendieck construction. The multiplication extends to make this group into a ring. With the original semigroup $(S, \oplus)$ as positive cone, the ring, which we denote $\mathfrak{R}$, becomes an ordered ring.

In [4, Definition 3.2] an invariant of an AF flow, called the coloured K_0 module, is defined as follows. Given an AF flow (A, α), we let A^α denote the fixed point subalgebra of A, and $D(A^\alpha)$ the dimension range of A^α. The coloured K_0 module, denoted $K\mathfrak{R}(A, \alpha)$, of the AF flow (A, α) is the universal right $\mathfrak{R}$ module generated by the set $D(A^\alpha)$ with the following relations:

(1) If p and q are projections in A^α and $p \perp q$, then $[p + q] = [p] + [q]$.
(2) If v is a partial isometry in A which is also an eigenoperator with eigenvalue a, then $[v^*v] = [vv^*][a]$, where $[a] \in \mathfrak{R}$.

We make the coloured K_0 module into an ordered module over the ordered ring $\mathfrak{R}$ by taking as positive cone the set of all positive $\mathfrak{R}$-linear combinations of elements of $D(A^\alpha)$. We refer to the copy of $D(A^\alpha)$ in $K\mathfrak{R}(A, \alpha)$ as the coloured scale, and denote it $\Sigma\mathfrak{R}(A, \alpha)$. If (A, α) and (B, β) are two AF flows and $\varphi \colon A \to B$ is an equivariant

*-homomorphism, then we define $K\mathfrak{R}(\varphi)\colon K\mathfrak{R}(A,\alpha) \to K\mathfrak{R}(B,\beta)$ by $K\mathfrak{R}(\varphi)[p] = [\varphi(p)]$ for any projection p in A^α and extending by linearity. This makes the coloured K_0 module into a functor from AF flows with equivariant *-homomorphisms to ordered $\mathfrak{R}$ modules with positive $\mathfrak{R}$ module maps respecting the scales.

The following existence result may easily be deduced from [4, Lemma 3.5].

Lemma 5.1 (Existence) *Let $(A,\alpha,\mathbb{R})$ and $(B,\beta,\mathbb{R})$ be two C^*-dynamical systems with A and B full matrix algebras, and let $\varphi\colon K\mathfrak{R}(A,\alpha) \to K\mathfrak{R}(B,\beta)$ be a positive $\mathfrak{R}$ module homomorphism mapping the class of the unit $[1_A] \in K\mathfrak{R}(A,\alpha)$ to $[1_B] \in K\mathfrak{R}(B,\beta)$. Then there exists a unital equivariant *-homomorphism $\tilde{\varphi}\colon A \to B$ such that $K\mathfrak{R}(\tilde{\varphi}) = \varphi$.*

Similarly, the following uniqueness result may be easily deduced from [4, Lemma 3.6].

Lemma 5.2 (Uniqueness) *Let $(A,\alpha,\mathbb{R})$ and $(B,\beta,\mathbb{R})$ be two C^*-dynamical systems with A and B full matrix algebras, and let ψ and φ be two unital equivariant *-homomorphisms from A to B. Suppose that $K\mathfrak{R}(\psi) = K\mathfrak{R}(\varphi)$. Then there exists a unitary U in the fixed point subalgebra of B such that $\psi = (\operatorname{Ad} U)\circ\varphi$.*

The final result we shall need from [4] is the following lemma.

Lemma 5.3 (Inductive limits) *Let $\{(A_n,\alpha_n,\mathbb{R}),\varphi_{nm}\}$ be an inductive system of C^*-dynamical systems where the A_ns are full matrix algebras, and the φ_{nm}s are unital equivariant *-homomorphisms, and let $(A,\alpha,\mathbb{R})$ denote the inductive limit of this system. Then $(K\mathfrak{R}(A,\alpha), K\mathfrak{R}^+(A,\alpha), [1_A])$ is the inductive limit, in the category of ordered $\mathfrak{R}$ modules with distinguished positive elements and positive $\mathfrak{R}$ module maps preserving the distinguished elements, of the inductive system*

$$\{(K\mathfrak{R}(A_n,\alpha_n), K^+\mathfrak{R}(A_n,\alpha_n), [1_{A_n}]), K\mathfrak{R}(\varphi_{nm})\}.$$

This may be deduced from [4, Remark 3.3 part 3; Lemma 3.7].

6 Classification

In this section we shall introduce an invariant that classifies C^*-dynamical systems arising as inductive limits of finite atomic elementary C^*-dynamical systems with proper connecting maps up to equivariant isomorphism (this is made precise in Theorem 6.2). We shall make free use of elementary results about K-theory. We refer the reader to [1] for this material.

Let $(A,\alpha) \cong (A_1,\alpha_1)\otimes(M,id)$ be the canonical decomposition for $(A,\alpha) \in LIMINF$. Suppose $(A_1,\alpha_1) \cong (K(H), \operatorname{Ad} V)$. Then $A^{\mathbb{T}} \cong c_0 \otimes M_d \otimes M$, where d is the sum of the multiplicities of the distinct irreducible representations appearing in V. The minimal central projections in $A^{\mathbb{T}}$ are all Murray–von Neumann equivalent in A, so they have the same class in $K_0(A)$. Call this K_0 class $g(A,\alpha)$. Write $\mu(A,\alpha)$ for the unitary equivalence class $[V]$ of the representation V of E appearing in the canonical decomposition of (A,α). We are now ready to define our invariant.

Definition 6.1 Let (A, α) be a C^*-dynamical system arising as an inductive limit of finite atomic elementary C^*-dynamical systems, and let $1_{A^E} \in A$ denote the unit of the fixed point subalgebra A^E. The invariant for (A, α), which we shall denote $\operatorname{Inv}(A, \alpha)$, consists of the following pieces of information:

(1) The coloured K_0 module $K\mathfrak{R}(1_{A^E}A1_{A^E}, \alpha)$ along with its positive cone

$$K\mathfrak{R}^+(1_{A^E}A1_{A^E}, \alpha),$$

and the class of the unit $[1_{A^E}]_{K\mathfrak{R}}$.

(2) The ordered K_0 group $(K_0(A), K_0(A)^+)$ along with the scale $D(A)$ and the distinguished elements $[1_{A^E}]$ and $g\big((1 - 1_{A^E})A(1 - 1_{A^E}), \alpha\big)$.

(3) The unitary equivalence class $\mu\big((1 - 1_{A^E})A(1 - 1_{A^E}), \alpha\big)$ of the representation appearing in the canonical decomposition of $\big((1 - 1_{A^E})A(1 - 1_{A^E}), \alpha\big)$.

By a morphism of invariants we shall mean a triple $(\varphi, \psi, =)$ where φ is a morphism of scaled, ordered, $\mathfrak{R}$-modules preserving the class of the unit, ψ is a morphism of scaled, ordered groups preserving the distinguished elements, and $=$ denotes identity of unitary equivalence classes of representations. The definition of composition of morphisms is the obvious one.

With these definitions, we may now state our main theorem.

Theorem 6.2 *Suppose (A, α) and (B, β) are two C^*-dynamical systems arising as inductive limits of finite atomic elementary C^*-dynamical systems with proper connecting maps and that $(\varphi, \psi, =)$ is an isomorphism of invariants from* $\operatorname{Inv}(A, \alpha)$ *to* $\operatorname{Inv}(B, \beta)$. *Then there exists an equivariant *-isomorphism $\gamma\colon A \to B$ such that $\varphi = K\mathfrak{R}(\gamma)$ and $\psi = K_0(\gamma)$.*

The proof of this theorem will follow the pattern of an Elliott intertwining argument, *cf.* [7]. We shall require the following lemma.

Lemma 6.3 *Suppose (A, α) and (B, β) are two finite atomic elementary C^*-dynamical systems with $1_{A^E} \neq 0$ and $\gamma = (\varphi, \psi, =)$ is a morphism of invariants from* $\operatorname{Inv}(A, \alpha)$ *to* $\operatorname{Inv}(B, \beta)$ *such that ψ is not the zero homomorphism. Then there exists a proper equivariant *-homomorphism $\tilde{\gamma}\colon A \to B$ such that $\varphi = K\mathfrak{R}(\tilde{\gamma})$ and $\psi = K_0(\tilde{\gamma})$. Furthermore, if $\delta\colon A \to B$ is another such proper equivariant *-homomorphism, then there exists a unitary U in the fixed point subalgebra of $M(B)$ such that $\delta = \operatorname{Ad} U \circ \tilde{\gamma}$.*

Proof of Lemma 6.3 We have that $(K_0(A), K_0^+(A)) \cong (\mathbb{Z}, \mathbb{Z}^+) \cong (K_0(B), K_0^+(B))$. We shall divide the proof into cases according to the values of $D(A)$ and $D(B)$. There are two combinations that may be ruled out. If either $D(A) = K_0^+(A)$ and $D(B) \neq K_0^+(B)$, or $D(A) \neq K_0^+(A)$ and $D(B) = K_0^+(B)$, then we cannot have

$$\mu\big((1 - 1_{A^E})A(1 - 1_{A^E}), \alpha\big) = \mu\big((1 - 1_{B^E})B(1 - 1_{B^E}), \beta\big).$$

We have then two cases to consider: the case where $D(A) \neq K_0^+(A)$ and $D(B) \neq K_0^+(B)$, and the case where $D(A) = K_0^+(A)$ and $D(B) = K_0^+(B)$. We deal with the first case first.

Suppose that, in the situation described in the statement of the lemma, we have further that $D(A) \neq K_0^+(A)$ and $D(B) \neq K_0^+(B)$, so that both A and B are full matrix algebras. In this case, both of the representations in the invariant are the zero representation, and $g\big((1-1_{A^E})A(1-1_{A^E}),\alpha\big)$ and $g\big((1-1_{B^E})B(1-1_{B^E}),\beta\big)$ are both zero too. From Lemma 5.1 we have that there exists a unital equivariant *-homomorphism $\tilde{\varphi}\colon A \to B$ such that $K\mathfrak{R}(\tilde{\varphi}) = \varphi$. Furthermore, from Lemma 5.2 we have that if γ is another unital equivariant *-homomorphism from A to B with $K\mathfrak{R}(\gamma) = \varphi$, then there exists a unitary U in the fixed point subalgebra of B such that $\gamma = (\mathrm{Ad}\ U)\circ\tilde{\varphi}$. All that remains to be checked is that $K_0(\tilde{\varphi}) = \psi$. This follows from the fact that $\tilde{\varphi}$ is a unital homomorphism, so that $K_0(\tilde{\varphi})$ takes the class of the unit of A to that of the unit of B, and that ψ does the same.

Next we consider the case where $D(A) = K_0^+(A)$ and $D(B) = K_0^+(B)$. In this case we have $A \cong B \cong K$. The assumption that ψ is not a zero homomorphism together with $1_{A^E} \neq 0$ implies that $1_{B^E} \neq 0$. We have that the coloured K_0 module for $(1_{A^E}A1_{A^E},\alpha)$ (resp., $(1_{B^E}B1_{B^E},\beta)$) is a singly generated free ordered right $\mathfrak{R}$ module with generator the class of a certain minimal projection in A (resp., B). It follows that there is an element $b = [0, b_1, \dots, b_n] \in \mathfrak{R}^+$ such that φ is the map $x \mapsto x \cdot b$. Since φ takes $[1_{A^E}]_{K\mathfrak{R}}$ to $[1_{B^E}]_{K\mathfrak{R}}$, we see that $b_1, \dots, b_n \in \mathbb{Z}^+$ and $(1_{B^E}B1_{B^E},\beta) \cong (1_{A^E}A1_{A^E},\alpha) \otimes (M_{(n+1)}, \mathrm{Ad}(\chi_0 \oplus \chi_{b_1} \oplus \cdots \oplus \chi_{b_n}))$. It follows that $[1_{B^E}]_{K_0} = (n+1)[1_{A^E}]_{K_0}$, identifying both groups with $\mathbb{Z}$, so the K_0 map ψ is multiplication by $n+1$. Since the same representation class appears in the canonical decompositions of

$$\big((1-1_{A^E})A(1-1_{A^E}),\alpha\big) \quad \text{and} \quad \big((1-1_{B^E})B(1-1_{B^E}),\beta\big)$$

and $(n+1)g\big((1-1_{A^E})A(1-1_{A^E}),\alpha\big) = g((1-1_{B^E})B(1-1_{B^E}),\beta\big)$, we see that if we write $(A,\alpha) \cong (K(H), \mathrm{Ad}\, W)$, where W is a finite direct sum of irreducible representations of E, and do the same for (B,β), then each infinite dimensional representation appears $n+1$ times as many times in the representation for (B,β) as in the one for (A,α). It follows that $(B,\beta) \cong (A,\alpha) \otimes (M_{(n+1)}, \mathrm{Ad}(\chi_0 \oplus \chi_{b_1} \oplus \cdots \oplus \chi_{b_n}))$, and the map $\tilde{\gamma}$ given by $a \mapsto a \otimes 1$ satisfies the existence part of the lemma.

Suppose that $\delta\colon A \to B$ is another proper equivariant *-homomorphism with $K\mathfrak{R}(\delta) = K\mathfrak{R}(\tilde{\gamma})$ and $K_0(\delta) = K_0(\tilde{\gamma})$. Since both δ and $\tilde{\gamma}$ are proper, they extend to unital equivariant *-homomorphisms, which we shall also call δ and $\tilde{\gamma}$ respectively, from $M(A)$ to $M(B)$. Consider the relative commutants $C = \tilde{\gamma}(M(A))' \cap M(B)$ and $D = \delta(M(A))' \cap M(B)$. Both C and D are invariant subalgebras of $M(B)$ isomorphic to $M_{(n+1)}$. We have $(C,\beta) \cong (M_{(n+1)}, \mathrm{Ad}(\chi_0 \oplus \chi_{b_1} \oplus \cdots \oplus \chi_{b_1}))$, and we may write $(D,\beta) \cong (M_{(n+1)}, \mathrm{Ad}\, W)$, where W is some other finite direct sum of χ_ks. It follows that $(B,\beta) \cong (A,\alpha) \otimes (M_{(n+1)}, \mathrm{Ad}\, W)$ via an equivariant isomorphism that carries δ to the map $a \mapsto a \otimes 1$. This implies that $(1_{B^E}B1_{B^E},\beta) \cong (1_{A^E}A1_{A^E},\alpha) \otimes (M_{(n+1)}, \mathrm{Ad}\, W)$, so we must have $(M_{(n+1)}, \mathrm{Ad}\, W) \cong (M_{(n+1)}, \mathrm{Ad}(\chi_0 \oplus \chi_{b_1} \oplus \cdots \oplus \chi_{b_1}))$. We thus have an equivariant automorphism of B that carries δ to $\tilde{\gamma}$, and as every equivariant automorphism of B is implemented by a unitary in the fixed point subalgebra of $M(B)$, this gives us our uniqueness result. This completes the proof of Lemma 6.3. ■

Remark 6.4 The assumption that $1_{A^E} \neq 0$ in Lemma 6.3 is necessary. Suppose that, in the situation of the lemma, we have that both 1_{A^E} and 1_{B^E} are zero. Let

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$V \in \mu(A,\alpha)$, and write V in terms of distinct irreducible representations as $V \cong n_1U^{a_1} \oplus \cdots \oplus n_kU^{a_k}$. For some m, l the canonical decompositions of (A,α) and (B,β) are $(K(H), \operatorname{Ad} V) \otimes (M_m, id)$ and $(K(H), \operatorname{Ad} V) \otimes (M_l, id)$, respectively. We have then $g(A,\alpha) = m(n_1 + \cdots + n_k)$ and $g(B,\beta) = l(n_1 + \cdots + n_k)$. It now follows from the condition $\psi(g(A,\alpha)) = g(B,\beta)$ that m divides l, $l = sm$ say, and that the map from A to B given by $\tilde{\gamma}\colon K(H)\otimes M_m \to K(H)\otimes M_l \cong K(H)\otimes M_m \otimes M_s; x\otimes y \mapsto x\otimes y\otimes 1_s$ meets the requirements of the existence part of the lemma. The uniqueness statement in the lemma, however, does not hold in this case. Let $(A,\alpha) \in \inf$ and let $(B,\beta) = (A,\alpha) \otimes (M_2, id)$. By Theorem 3.2 $(B,\beta) \cong (A,\alpha) \otimes (M_2, \operatorname{Ad}(id \oplus \chi_1))$. Let ψ denote the inclusion $a \mapsto a\otimes 1$ in the first case, and let φ denote the inclusion $a \mapsto a \otimes 1$ in the second tensor product decomposition. The invariants for these two equivariant inclusions are the same. The restrictions of β to the relative commutants $\psi(A)' \cap M(B)$ and $\varphi(A)' \cap M(B)$ however give (M_2, id) and $(M_2, \operatorname{Ad}(id \oplus \chi_1))$, which are not isomorphic, so there cannot be a unitary in the fixed point subalgebra of $M(B)$ taking ψ to φ.

Proof of Theorem 6.2 We deal first with the case where $1_{A^E} = 0$. Then, since the isomorphism ψ of $K_0(A)$ with $K_0(B)$ takes $[1_{A^E}]$ to $[1_{B^E}]$, we have $1_{B^E} = 0$ too, and (A,α), (B,β) are both elements of $\liminf$. Let $(K(H), \operatorname{Ad} V) \otimes (M, id)$ and $(K(H), \operatorname{Ad} V) \otimes (N, id)$ be the canonical decompositions of (A,α) and (B,β) respectively, and suppose $V = n_1U^{a_1} \oplus \cdots \oplus n_kU^{a_k}$ with $a_i \neq a_j$ for $i \neq j$. Then $g(A,\alpha) = (n_1 + \cdots + n_k)[e_{11} \otimes 1_M]$ and $g(B,\beta) = (n_1 + \cdots + n_k)[e_{11} \otimes 1_N]$, so $\psi(g(A,\alpha)) = g(B,\beta)$ implies that $\psi([e_{11} \otimes 1_M]) = [e_{11} \otimes 1_N]$. It follows from a fundamental result in K-theory (*cf.* [1]) that there exists an isomorphism $\tilde{\psi}\colon A \to B$ such that $\psi = K_0(\tilde{\psi})$, and we see that M and N are cutdowns of A and B, respectively by projections which are set Murray–von Neumann equivalent by this isomorphism, so we have $M \cong N$, under an isomorphism γ say. It follows that $1 \otimes \gamma\colon K(H) \otimes M \to K(H) \otimes N$ is an equivariant isomorphism satisfying the requirements of the theorem.

Suppose now that $\{(A_n, \alpha_n), i_{nm}\}$ and $\{(B_n, \beta_n), j_{nm}\}$ are two inductive systems of finite atomic elementary C^*-dynamical systems with proper connecting maps, that (A,α) and (B,β) are the inductive limits, that $1_{A^E} \neq 0$, and that γ is an isomorphism of $\operatorname{Inv}(A,\alpha)$ with $\operatorname{Inv}(B,\beta)$. It follows that $1_{B^E} \neq 0$, $1_{A^E} \in A_1$, and $1_{B^E} \in B_1$. Since K_0 is continuous with respect to inductive limits, and by Lemma 5.3 $K\mathfrak{R}$ is as well, we get a diagram,

$$\begin{array}{ccccccc}
\operatorname{Inv}(A_1,\alpha_1) & \longrightarrow & \operatorname{Inv}(A_2,\alpha_2) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Inv}(A,\alpha) \\
 & & & & & & \nu \uparrow \; \downarrow \gamma \\
\operatorname{Inv}(B_1,\beta_1 & \longrightarrow & \operatorname{Inv}(B_2,\beta_2) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Inv}(B,\beta)
\end{array}$$

in the category of invariants in which

$$\operatorname{Inv}(A,\alpha) = \varinjlim\{\operatorname{Inv}(A_n,\alpha_n), \big(K\mathfrak{R}(i_{nm}), K_0(i_{nm}), =\big) = \operatorname{Inv}(i_{nm})\},$$

and similarly for $\mathrm{Inv}(B,\beta)$.

Both $K_0(A_1)$ and $K\mathfrak{R}(1_{A^E}A_1 1_{A^E})$ are singly generated, so for some n we may find a morphism of invariants, γ_1 say, from $\mathrm{Inv}(A_1,\alpha_1)$ to $\mathrm{Inv}(B_n,\beta_n)$ such that $\gamma \circ \mathrm{Inv}(i_{1\infty}) = \mathrm{Inv}(j_{n\infty}) \circ \gamma_1$. Passing to a subsequence and renumbering, we may suppose that $n = 1$. Applying the same reasoning, we get a morphism of invariants, δ say, from $\mathrm{Inv}(B_1,\beta_1)$ to $\mathrm{Inv}(A_m,\alpha_m)$ for some m, such that $\mathrm{Inv}(i_{m\infty}) \circ \delta = \nu \circ \mathrm{Inv}(j_{1\infty})$. We then have that $\mathrm{Inv}(i_{1\infty}) = \mathrm{Inv}(i_{m\infty} \circ \delta \circ \gamma_1)$. Since both $K_0(A_1)$ and $K\mathfrak{R}(1_{A^E}A_1 1_{A^E})$ are singly generated, for some $k \geq m$ we have $\mathrm{Inv}(i_{1k}) = \mathrm{Inv}(i_{mk}) \circ \delta \circ \gamma_1$. Passing to a subsequence and renumbering, we may suppose $m = k = 2$. We write ν_1 for δ. Proceeding in this fashion, from left to right through the diagram above, we arrive at a commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Inv}(A_1,\alpha_1) & \longrightarrow & \mathrm{Inv}(A_2,\alpha_2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Inv}(A,\alpha) \\
\downarrow{\scriptstyle \gamma_1} & \nearrow{\scriptstyle \nu_1} & \downarrow{\scriptstyle \gamma_2} & \nearrow{\scriptstyle \nu_2} & & & {\scriptstyle \nu}\uparrow\downarrow{\scriptstyle \gamma} \\
\mathrm{Inv}(B_1,\beta_1 & \longrightarrow & \mathrm{Inv}(B_2,\beta_2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Inv}(B,\beta)
\end{array}$$

Next, we use the existence part of Lemma 6.3 to conclude that, for each n, there exist proper equivariant *-homomorphisms $\tilde{\gamma}_n \colon (A_n,\alpha_n) \to (B_n\beta_n)$ and $\tilde{\nu}_n \colon (B_n,\beta_n) \to (A_{n+1},\alpha_{n+1})$ such that $(K\mathfrak{R}(\tilde{\gamma}_n), K_0(\tilde{\gamma}_n), =) = \gamma_n$ and $(K\mathfrak{R}(\tilde{\nu}_n), K_0(\tilde{\nu}_n), =) = \nu_n$. This gives us a diagram

$$\begin{array}{ccccccc}
(A_1,\alpha_1) & \longrightarrow & (A_2,\alpha_2) & \longrightarrow & \cdots & \longrightarrow & (A,\alpha) \\
\downarrow{\scriptstyle \tilde{\gamma}_1} & \nearrow{\scriptstyle \tilde{\nu}_1} & \downarrow{\scriptstyle \tilde{\gamma}_2} & \nearrow{\scriptstyle \tilde{\nu}_2} & & & \\
(B_1,\beta_1 & \longrightarrow & (B_2,\beta_2) & \longrightarrow & \cdots & \longrightarrow & (B,\beta)
\end{array}$$

in which the triangles need not commute.

Finally, we use the uniqueness part of Lemma 6.3 to adjust the vertical maps in the above diagram to get a commuting one in which the maps have the same invariants as in the one above. We start with ν_1. Since i_{12} and $\nu_1 \circ \gamma_1$ have the same invariants, there exists a unitary U in the fixed point subalgebra of $M(A_2)$ such that $\mathrm{Ad}\, U \circ \nu_1 \circ \gamma_1 = i_{12}$. Since $U \in M(A_2)^{\alpha_2}$, $K_0(\mathrm{Ad}\, U \circ \nu_1) = K_0(\nu_1)$. Also, U commutes with 1_{A^E}, so $1_{A^E}U$ is a unitary in the fixed point subalgebra of $1_{A^E}A_1 1_{A^E}$ and $K\mathfrak{R}(\mathrm{Ad}\, U \circ \nu_1) = K\mathfrak{R}(\nu_1)$. We replace ν_1 with $\mathrm{Ad}\, U \circ \nu_1$ and proceed to γ_2. Continuing in this way gives us a diagram like the one above, and in which the maps have the same invariants, but in which each triangle, and hence the whole diagram, now commutes. It is well known, (*cf.* [7]) that such a commuting diagram gives rise to a pair of inverse isomorphisms, $\tilde{\gamma} \colon A \to B$ and $\tilde{\nu} \colon B \to A$ which make the whole diagram commute. In our case, these isomorphisms are also equivariant. That they have the right values on the invariants follows from commutativity of the diagram and functoriality of the invariant.

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