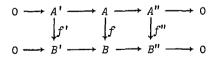
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# A five lemma for free products of groups with amalgamations

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This paper explores a five-lemma situation in the context of a free product of a family of groups with amalgamated subgroups (that is, a colimit of an appropriate diagram in the category of groups). In particular, for two families  $\{A_{\alpha}\}, \{B_{\alpha}\}$  of groups with amalgamated subgroups  $\{A_{\alpha\beta}\}, \{B_{\alpha\beta}\}$  and free products A, B we assume the existence of homomorphisms  $A_{\alpha} \rightarrow B_{\alpha}$  whose restrictions  $A_{\alpha\beta} \rightarrow B_{\alpha\beta}$  are isomorphisms and which induce an isomorphism  $A \rightarrow B$  between the products. We show that the usual five-lemma conclusion is false, in that the morphisms  $A_{\alpha} \rightarrow B_{\alpha}$  are in general neither monic nor epic. However, if all  $B_{\alpha} \rightarrow B$  are monic,  $A_{\alpha} \rightarrow B_{\alpha}$  is always epic; and if  $A_{\alpha} \rightarrow A$  is monic, for all  $\alpha$ , then  $A_{\alpha} \rightarrow B_{\alpha}$  is an isomorphism.

A frequent situation in homological algebra and its applications is a commutative diagram



of exact sequences in which it is assumed that both f' and f'' are isomorphisms and concluded that f is also. We consider in this paper an analogous situation for products of groups with amalgamations. The

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conclusions are generalizations of similar ones of E.M. Brown [2, p. 485] and are of interest in topological situations involving applications of suitable generalizations of the classical van Kampen theorem (cf. R. Brown [3, Theorem 8.4.2, p. 282]).

As the results herein are purely group theoretic but seem most smoothly expressed in the language of categories, we begin by reviewing briefly certain definitions and results of category theory. We then summarize our results, and proceed to establish them in the ensuing sections.

### 1. Notation and statement of results

We assume that the notions of category, functor and natural transformation are familiar. The reader is referred to Mitchell ([6, Chapter II], especially §§ 1, 2 and 12) for more detail.

The category of all groups and homomorphisms will be denoted by G .

A full subcategory B of a category A is a subcategory such that for every pair of objects  $A, B \in B$ , every A-morphism from A to B is in B.

An amalgam scheme  $\Sigma_* = \Sigma_*(I)$  is a category of the following type: I is a set. The objects of  $\Sigma_*$  are: all one-element subsets of I, all two-element subsets of I, and an object \*. The morphisms of  $\Sigma_*$ are the identities and the following:

 $\{\alpha, \beta\} \rightarrow \{\alpha\} , \ \{\alpha\} \rightarrow * ,$  $\{\alpha, \beta\} \rightarrow * = \{\alpha, \beta\} \rightarrow \{\alpha\} \rightarrow * = \{\alpha, \beta\} \rightarrow \{\beta\} \rightarrow * ; \ \alpha, \beta \in I .$ 

We denote by  $\Sigma$  the full subcategory generated by all objects except \*, and by  $\Sigma_{O}$  the full subcategory generated by \* and all  $\{\alpha, \beta\} \in \Sigma_{\star}$ .

An amalgam is a covariant functor  $F: \Sigma \rightarrow G$ .

A colimit for F is a functor  $F_*: \Sigma_* \to G$  which extends F and which satisfies the universal mapping property: if H is any extension of F to  $\Sigma_*$  there exists a unique natural transformation  $\eta: F_* \to H$  such that



commutes, where the vertical arrows are inclusions.

**PROPOSITION 1.** If  $\eta : F \neq G$  is a natural transformation of amalgams, there is a unique extension of  $\eta$  to a natural transformation  $\eta_* : F_* \neq G_*$ .

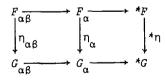
THEOREM 2 (see also [7]). Every amalgam has a colimit, unique up to natural equivalence.

We shall write  $F_{\alpha\beta} = F(\{\alpha, \beta\})$ ,  $F_{\alpha} = F(\{\alpha\})$  for all  $\{\alpha\}$ ,  $\{\alpha, \beta\} \in \Sigma$ ; and  $*F = F_{*}(*)$  where  $F_{*}$  is a colimit of F. The group \*F may be constructed as the largest quotient group of the free product  $\coprod_{T} F_{\alpha}$  of the groups  $F_{\alpha}$ , such that

$$F_{\alpha\beta} \rightarrow F_{\alpha} \rightarrow \coprod_{I} F_{\alpha} \rightarrow *F = F_{\alpha\beta} \rightarrow F_{\beta} \rightarrow \coprod_{I} F_{\alpha} \rightarrow *F \ .$$

Let  $\eta : F \neq G$  be a natural transformation of amalgams  $F, G : \Sigma(I) \neq G$ . Denote by  $F_0, G_0$  the restrictions of F, G to  $\Sigma_0$ and let  $\eta_0 : F_0 \neq G_0$  be the induced natural transformation.

The situation with which we deal is the following: suppose  $\eta_0$  is a natural equivalence. What can be said about  $\eta$ ? Restated in group-theoretic terms, the question reads thusly:



Suppose \*n and all of the  $n_{\alpha\beta}$  are isomorphisms. What can be said about the  $n_{\alpha}$ ? We note that \*F is the smallest subgroup of \*F which contains  $\operatorname{Im}(F_{\alpha} \rightarrow *F)$  for all  $\alpha \in I$ .

We shall henceforth assume that  $F_{\alpha\beta} + F_{\alpha}$  is monic. Observe that in

this case our terminology agrees with that of Hanna Neumann [9] and B.H. Neumann [7]. The colimit \*F is in [7] the group  $P^*$ , called there the canonic group (of the amalgam) and the generalized free product of the  $F_{\alpha}$ with amalgamated subgroups  $F_{\alpha\beta}$  is \*F, provided \*F satisfies some additional conditions (see §4 below).

We shall show that in general  $\eta_{\alpha}$  is neither monic (§2) nor epic (§3). If  $G_{\alpha} \rightarrow {}^{*}G$  is monic for all  $\alpha \in I$ ,  $\eta_{\alpha}$  is epic (§4). If  $F_{\alpha} \rightarrow {}^{*}F$  is monic for all  $\alpha \in I$ , then  $G_{\alpha} \rightarrow {}^{*}G$  is also monic and  $\eta_{\alpha}$ is an isomorphism (§4). Finally (§5) we restate a theorem of Neumann ([7] or [9]) which guarantees that  $F_{\alpha} \rightarrow {}^{*}F$  is monic, deduce the theorem of E.M. Brown as a corollary and then prove a monicity theorem of our own.

2.

**EXAMPLE.**  $n_{a}$  need not be monic.

In the light of existing examples of amalgams  $F : \Sigma(I) \rightarrow G$  in which  $\operatorname{card}(I) = 3$  and  $F_{\alpha} \rightarrow {}^*F$  is not monic (cf. [7] or [8]) it is not surprising that  $\eta_{\alpha}$  should fail to be monic. We shall present a particularly economical example of this behaviour (cf. \$5). It utilizes a 2-generator, 1-relator group found by Baumslag and Solitar [1] which is non-hopfian, in other words, is isomorphic to a proper factor group of itself.

We use  $\langle g : r \rangle$  to denote the group generated by the set of elements g subject to the relations r (*cf.* Magnus *et al.* [5]), and  $[x, y] = x^{-1}y^{-1}xy$ , the commutator of x and y.

Let  $I = \{1, 2, 3\}$ , and define the amalgams  $F, G : \Sigma \rightarrow G$  as follows:

#### A five lemma

$$F_{1} = \left\langle a, t : a = [a^{2}, t] \right\rangle, \quad G_{1} = \left\langle b, u : b = [b^{2}, u] \right\rangle$$

$$F_{2} = \left\langle t_{2} \right\rangle, \quad G_{2} = \left\langle u_{2} \right\rangle,$$

$$F_{3} = \left\langle t_{3} \right\rangle, \quad G_{3} = \left\langle u_{3} \right\rangle,$$

$$F_{ij} = F_{ji} = \left\langle t_{ij} \right\rangle, \quad G_{ij} = G_{ji} = \left\langle u_{ij} \right\rangle$$

for  $i, j \in I$ ,  $i \neq j$ . We note that t and  $ta^{-1}[a, t]^2$  are of infinite order modulo the commutator subgroup of  $G_1$  and therefore are of infinite order in  $G_1$ . The monomorphisms  $F_{ij} \neq F_i$  and  $G_{ij} \neq G_i$  are defined by

$$t_{ij} \neq \begin{cases} t_i & i = 2, 3, \\ t & i = 1, j = 2, u_{ij} \neq \{u_i & i = 1, 2, 3, \\ ta^{-1}[a, t]^2 & i = 1, j = 3, \end{cases}$$

and so the colimits  $F_*, G_* : \Sigma_* \to G$  may be defined as follows:

$$F = \left\langle a, t : a = [a^{2}, t], a = [a, t]^{2} \right\rangle = \left\langle a, c, t : c = [a, t], a = [a^{2}, t], a = c^{2} \right\rangle$$
$$= \left\langle c, t : c = [c^{2}, t], c^{2} = [c^{4}, t] \right\rangle = \left\langle c, t : c = [c^{2}, t] \right\rangle,$$
$$FG = \left\langle b, u : b = [b^{2}, u] \right\rangle.$$

The morphisms are determined by the functoriality of  $F_*$  and  $G_*$  together with the maps  $F_1 \rightarrow {}^*F$ ,  $G_1 \rightarrow {}^*G$  respectively:

$$\begin{cases} a \div c^2 \\ t \div t \end{cases} \qquad \begin{cases} b \div b \\ u \div u \end{cases}.$$

Define the transformation  $\,\eta\,$  by

$$\begin{split} & \mathsf{n}_{ij}(t_{ij}) = u_{ij} \qquad \mathsf{n}_i(t_i) = u_i \\ & \mathsf{n}_1(a) = b^2 \qquad \mathsf{n}_1(t) = u \ . \end{split}$$

We note that  $\eta_1$  is epic, for  $b = [b^2, u] \in \operatorname{Im} \eta_1$ , and that  $\eta_1(a^{-1}[a, t]^2) = 1$  while  $a^{-1}[a, t]^2 \neq 1$  (cf. [5, pp. 260-261] or [1]). Finally, the induced map  $*\eta : *F \neq *G$  is an isomorphism as  $*\eta(t) = u$ , and  $\eta_1([a, t]) = [b^2, u] = b$  implies  $*\eta(c) = *\eta([a, t]) = b$ .

3.

EXAMPLE.  $\eta_{n}$  need not be epic.

Once again  $I = \{1, 2, 3\}$ . Let  $A \subset B$  be two simple groups (for example let B be the alternating group on 6 symbols and A the subgroup fixing the first symbol). Define the amalgams  $F, G : \Sigma \rightarrow G$  as follows:

$$\begin{array}{l} F_1 = \langle t \rangle \times A , & G_1 = \langle u \rangle \times B , \\ F_2 = \langle t_2 \rangle , & G_2 = \langle u_2 \rangle , \\ F_3 = \langle t_3 \rangle , & G_3 = \langle u_3 \rangle , \\ F_{ij} = F_{ji} = \left\langle t_{ij} \right\rangle , & G_{ij} = G_{ji} = \left\langle u_{ij} \right\rangle \end{array}$$

where  $i, j \in I$ ,  $i \neq j$ . The homomorphisms  $F_{ij} \neq F_i$  and  $G_{ij} \neq G_i$ are defined by the following:

$$t_{ij} \neq \begin{cases} t_i & i = 2, 3, \\ t \times 1 & i = 1, j = 2, u_{ij} \neq \\ t \times a & i = 1, j = 3, \end{cases} \qquad u_{ij} \neq \begin{cases} u_i & i = 2, 3, \\ u \times 1 & i = 1, j = 2, \\ u \times a & i = 1, j = 3, \end{cases}$$

where  $1 \neq a \in A$ . The colimit groups are easily seen to be

$$*F = \langle t \rangle$$
,  $*G = \langle u \rangle$ 

,

and the homomorphisms are the obvious ones. Let  $\theta : \langle t \rangle \rightarrow \langle u \rangle$  be defined by  $\theta(t) = u$ , and let  $I : A \rightarrow B$  be the inclusion. The transformation  $\eta$  defined by  $\eta_1 = \theta \times I$  and isomorphisms elsewhere induces the isomorphism  $*\eta : *F \rightarrow *G$  sending t to u, but  $\text{Im}\eta_1$  is clearly the proper subgroup  $\langle u \rangle \times A$  of  $G_1 = \langle u \rangle \times B$ .

#### 4. The main theorems

We shall prove

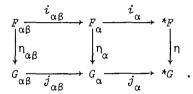
THEOREM 3. If  $G_{\alpha} + *G$  is monic for each  $\alpha \in I$ , then  $n_{\alpha} : F_{\alpha} \neq G_{\alpha}$  is epic.

We first define the generalized free product of the  $F_{\alpha}$  with

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amalgamated subgroups  $F_{\alpha\beta} = F_{\beta\alpha}$  in the sense of B.H. Neumann [7] or Hanna Neumann [9] which we denote by g.f.p. $\langle F_{\alpha} : F_{\alpha\beta} \rangle$  or g.f.p.F. Our first two lemmas reduce the theorem to the case in which g.f.p.F exists. We then use a theorem of Hanna Neumann [9], stated below, to conclude that g.f.p.G exists. Finally we exploit the properties of a normal form for elements of such g.f.p.'s to conclude that  $\eta_{\alpha}$  is epic.

Homomorphisms will henceforth be labelled in accordance with the diagram below.



DEFINITION. \*F is g.f.p.F if the following conditions are satisfied:

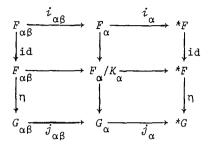
$$\begin{array}{ll} \forall \ \alpha \in I \ , & i_{\alpha} \ \ \, \text{is monic} \\ \forall \ \alpha, \ \beta \in I \ , & \operatorname{Im}(i_{\alpha}) \ \cap \ \, \operatorname{Im}(i_{\beta}) \ = \ \, \operatorname{Im}(i_{\alpha}i_{\alpha\beta}) \ . \end{array}$$

We note that these conditions are independent (cf. [8]).

For the following two lemmas, it is assumed that  $\,j_{\,\alpha}\,$  is monic,  $\,\alpha\,\in\,I$  .

LEMMA 4. We may assume  $n_{\alpha}$  and  $i_{\alpha}$  monic.

Proof. Let  $K_{\alpha} = \operatorname{Kern}_{\alpha}$ . Observe that  $\operatorname{Im}(i_{\alpha\beta}) \cap K_{\alpha} = 1$  for the composition  $j_{\alpha\beta}n_{\alpha\beta}$  is monic, and that  $K_{\alpha} \subseteq \operatorname{Ker}(i_{\alpha})$ . Thus we have a factorization



such that  $F_{\alpha} \neq F_{\alpha}/K_{\alpha}$  is epic. Therefore  $\eta_{\alpha}$  is epic if and only if  $F_{\alpha}/K_{\alpha} \neq G_{\alpha}$  is epic. Also  $F_{\alpha}/K_{\alpha} \neq G_{\alpha}$  is monic and the lower right square commutes so  $F_{\alpha}/K_{\alpha} \neq *F$  is monic.

LEMMA 5. We may assume further that \*F = g.f.p.F.

Proof. By Lemma 4, we need ensure the intersection property. Set

$$F'_{\alpha\beta} = \operatorname{Im}(i_{\alpha}) \cap \operatorname{Im}(i_{\beta})$$
$$G'_{\alpha\beta} = \operatorname{n}(F'_{\alpha\beta})$$

and define  $F'_{\alpha\beta} \rightarrow F_{\alpha}$  and  $G'_{\alpha\beta} \rightarrow G_{\alpha}$  as the inverses of  $i_{\alpha}$  restricted to  $F'_{\alpha\beta}$  and  $j_{\alpha}$  restricted to  $G'_{\alpha\beta}$  respectively. These definitions are sensible as  $i_{\alpha}$ ,  $j_{\alpha}$  are monic,  $F'_{\alpha\beta} \subseteq \operatorname{Im}(i_{\alpha})$  by construction and  $G'_{\alpha\beta} = \eta \left( \operatorname{Im}(i_{\alpha}) \cap \operatorname{Im}(i_{\beta}) \right) \subseteq \operatorname{Im}(j_{\alpha}) \cap \operatorname{Im}(j_{\beta})$ .

THEOREM 6 (Hanna Neumann [9]). Set  $U_{\alpha} = \left\langle \bigcup_{\beta} \operatorname{Im}(i_{\alpha\beta}) \right\rangle \subseteq F_{\alpha}$ . Then g.f.p. $\left\langle U_{\alpha} : F_{\alpha\beta} \right\rangle$  exists if and only if g.f.p. $\left\langle F_{\alpha} : F_{\alpha\beta} \right\rangle$  exists. PROPOSITION 7. If g.f.p.F exists, then so does g.f.p.G.

Proof. The group  $V_{\alpha} = \left\langle \bigcup_{\beta} \operatorname{Im}(j_{\alpha\beta}) \right\rangle$  is isomorphic to  $U_{\alpha}$ , with isomorphism the restriction of  $\eta_{\alpha}$  to  $U_{\alpha}$ . Therefore g.f.p. $\left\langle V_{\alpha} : G_{\alpha\beta} \right\rangle$ exists, which implies that g.f.p.*G* exists.

COROLLARY 8. If  $F_{\alpha} \to {}^{*}F$  is monic, for all  $\alpha \in I$ , then so is  $G_{\alpha} \to {}^{*}G$ .

NORMAL FORM THEOREM 9 (Hanna Neumann [10]). Suppose \*F = g.f.p.F, and let  $U = \left\langle U_{\alpha,\beta} \operatorname{Im}(i_{\alpha}i_{\alpha\beta}) \right\rangle \subseteq *F$ . For each  $\alpha \in I$  choose a system  $S^{\alpha}$ of right coset representatives of  $i_{\alpha}(U_{\alpha})$  in  $\operatorname{Im}(i_{\alpha})$  such that  $1 \in S^{\alpha}$ (that is,  $\operatorname{Im}(i_{\alpha})$  is the disjoint union of  $i_{\alpha}(U_{\alpha}) \cdot s$ ,  $s \in S^{\alpha}$ ). For each  $\alpha \in I$  choose a system  $T^{\alpha}$  of right coset representatives of  $i_{\alpha}(U_{\alpha})$  in U such that  $1 \in T^{\alpha}$ . Then each element of \*F may be written as

$$u_0 t_1 s_1 \dots t_p s_p$$

where  $u_0 \in U$ ,  $t_i \in T^{\alpha(i)}$ ,  $s_i \in S^{\alpha(i)}$ ,  $t_i \neq 1$ , and  $s_i = 1$  implies  $\alpha(i) \neq \alpha(i+1)$ . Distinct symbols represent distinct elements.

Proof of Theorem 3. We may assume  ${}^*F = g.f.p.F$ ,  ${}^*G = g.f.p.G$ . Set  $V = \left\langle \bigcup_{\alpha,\beta} \operatorname{Im}(j_{\alpha}j_{\alpha\beta}) \right\rangle \subseteq {}^*G$ . Choose right coset representative  $S^{\alpha}$ and  $T^{\alpha}$  in  ${}^*F$  as described in Theorem 9 above. We have that  $\eta(U) = V$ ,  $\eta i_{\alpha}(U_{\alpha}) = j_{\alpha}(V_{\alpha})$  so  $\eta(T^{\alpha})$  is a system of right coset representatives of  $j_{\alpha}(V_{\alpha})$  in V. Since  $\eta\left(\operatorname{Im}(i_{\alpha})\right) \subseteq \operatorname{Im}(j_{\alpha})$  we may obtain a system of right coset representatives of  $j_{\alpha}(V_{\alpha})$  in  $\operatorname{Im}(j_{\alpha})$  of the form  $\eta(S^{\alpha}) \cup X^{\alpha}$ . Assume that  $x \in X^{\alpha}$ , and let  $y = \eta^{-1}(x) \in {}^*F$ . We may write

$$y = u_0 t_1 s_1 \dots t_p s_p$$

as in the normal form theorem. Then  $\eta(y) = x = \eta(u_0)\eta(t_1) \dots \eta(s_p)$  and we have obtained two distinct normal form symbols for the same element of \*G. Therefore  $X^{\alpha} = \phi$  and so

$$\operatorname{Im}(j_{\alpha}) = \langle \eta(S^{\alpha}) \cup V^{\alpha} \rangle = \operatorname{Im}(\eta i_{\alpha}) = \operatorname{Im}(j_{\alpha}\eta_{\alpha})$$

or  $\eta_{\alpha}$  is epic.

THEOREM 10. Assume that  $i_{\alpha}: F_{\alpha} \rightarrow {}^{*}F$  is monic for all  $\alpha \in I$ . Then  $n_{\alpha}$  is an isomorphism.

Proof. Since  $j_{\alpha}\eta_{\alpha} = \eta i_{\alpha}$ , it is clear that  $\eta_{\alpha}$  is monic. By Corollary 8,  $j_{\alpha}$  is monic for all  $\alpha \in I$ , hence  $\eta_{\alpha}$  is also epic and so is an isomorphism.

## 5. Existence of the g.f.p.

It seems clear that one now wishes necessary and sufficient conditions that the homomorphisms  $i_{\alpha}: F_{\alpha} \to {}^*F$  be monic. A summary of results concerning the existence of the g.f.p. may be found in Neumann [8]. In particular we have

THEOREM 11 (cf. Neumann [7]). If  $card(I) \le 2$ , then the g.f.p. of any amalgam  $F : \Sigma(I) \rightarrow G$  exists.

This theorem and our Theorem 10 imply the result of E.M. Brown [2, Proposition 12.5, p. 485] cited in the introduction.

We now require a definition.

DEFINITION. Let  $F : \Sigma(I) \rightarrow G$  be an amalgam. The graph of F consists of vertices  $\alpha$  for each  $\alpha \in I$ , vertices  $\{\alpha, \beta\}$  for each  $\{\alpha, \beta\} \subset I$  such that  $F_{\alpha\beta} \neq \{1\}$ , and arrows beginning at  $\{\alpha, \beta\}$  and terminating at  $\alpha$ , for each  $\{\alpha, \beta\}$  in the graph.

THEOREM 12. If the amalgam  $F : \Sigma \to G$  has a simply connected graph, then  $F_{\alpha} \to *F$  is monic.

We shall omit the proof of this theorem, remarking only that one may first reduce to the case of an amalgam with connected, simply connected graph and finitely many vertices. There remains an amalgam whose graph is a finite tree, and one proceeds by pulling off branches.

I have learned that this result was announced by A. Karrass and D. Solitar and will appear in the *Transactions of the American* Mathematical Society [4]. (The colimit \*F of an amalgam with simply connected graph is in [4] called a tree product.)

We remark finally that Corollary 13 below follows easily from Theorem 12 (by Corollary 8) and that Theorem 14 below may be deduced rather naturally from Theorem 11 via Theorem 12.

COROLLARY 13. Given the amalgam  $F : \Sigma(I) \rightarrow G$ , let I' be some collection of 2-element subsets of I. Define the functor  $F' : \Sigma(I) \rightarrow G$  by: F' = F on the full subcategory of  $\Sigma(I)$  generated by the objects of  $\Sigma(I)$  not in I', and  $F'_{\alpha\beta} = \{1\}$ ,  $\{\alpha, \beta\} \in I'$ . Assume \*F' = \*F and that the graph of F' is simply connected. Then  $F_{\alpha} \rightarrow *F$  is monic for all  $\alpha \in I$ .

THEOREM 14 (B.H. Neumann [7] or Hanna Neumann [9]). Assume  $F: \Sigma(I) \rightarrow G$  is an amalgam for which there exists  $H \in G$  and isomorphisms  $H \rightarrow F_{\alpha\beta}$  for all  $\alpha, \beta \in I$  such that

$$H \rightarrow F_{\alpha\beta} \rightarrow F_{\alpha} = H \rightarrow F_{\alpha\gamma} \rightarrow F_{\alpha}$$

for all  $\alpha$ ,  $\beta$ ,  $\gamma \in I$ . Then g.f.p.F exists.

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