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A five lemma for free products of groups with amalgamations

J. L. Dyer

This paper explores a five-lemma situation in the context of a free product of a family of groups with amalgamated subgroups (that is, a colimit of an appropriate diagram in the category of groups). In particular, for two families $\{A_\alpha\}$, $\{B_\alpha\}$ of groups with amalgamated subgroups $\{A_{\alpha\beta}\}$, $\{B_{\alpha\beta}\}$ and free products A, B we assume the existence of homomorphisms $A_\alpha \to B_\alpha$ whose restrictions $A_{\alpha\beta} \to B_{\alpha\beta}$ are isomorphisms and which induce an isomorphism $A \to B$ between the products. We show that the usual five-lemma conclusion is false, in that the morphisms $A_\alpha \to B_\alpha$ are in general neither monic nor epic. However, if all $B_\alpha \to B$ are monic, $A_\alpha \to B_\alpha$ is always epic; and if $A_\alpha \to A$ is monic, for all α, then $A_\alpha \to B_\alpha$ is an isomorphism.

A frequent situation in homological algebra and its applications is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

of exact sequences in which it is assumed that both f' and f'' are isomorphisms and concluded that f is also. We consider in this paper an analogous situation for products of groups with amalgamations. The

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conclusions are generalizations of similar ones of E.M. Brown [2, p. 485] and are of interest in topological situations involving applications of suitable generalizations of the classical van Kampen theorem (cf. R. Brown [3, Theorem 8.4.2, p. 282]).

As the results herein are purely group theoretic but seem most smoothly expressed in the language of categories, we begin by reviewing briefly certain definitions and results of category theory. We then summarize our results, and proceed to establish them in the ensuing sections.

1. Notation and statement of results

We assume that the notions of category, functor and natural transformation are familiar. The reader is referred to Mitchell ([6, Chapter II], especially §§ 1, 2 and 12) for more detail.

The category of all groups and homomorphisms will be denoted by G.

A *full* subcategory B of a category A is a subcategory such that for every pair of objects $A, B \in B$, every A-morphism from A to B is in B.

An *amalgam scheme* $\Sigma_* = \Sigma_*(I)$ is a category of the following type: I is a set. The objects of Σ_* are: all one-element subsets of I, all two-element subsets of I, and an object $*$. The morphisms of Σ_* are the identities and the following:

$$\{\alpha, \beta\} \to \{\alpha\} \ , \quad \{\alpha\} \to * \ ,$$
$$\{\alpha, \beta\} \to * = \{\alpha, \beta\} \to \{\alpha\} \to * = \{\alpha, \beta\} \to \{\beta\} \to * \ ; \quad \alpha, \beta \in I \ .$$

We denote by Σ the full subcategory generated by all objects except $*$, and by Σ_0 the full subcategory generated by $*$ and all $\{\alpha, \beta\} \in \Sigma_*$.

An *amalgam* is a covariant functor $F : \Sigma \to G$.

A *colimit for* F is a functor $F_* : \Sigma_* \to G$ which extends F and which satisfies the universal mapping property: if H is any extension of F to Σ_* there exists a unique natural transformation $\eta : F_* \to H$ such that

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$$\begin{array}{ccc} F_* & \xrightarrow{\eta} & H \\ \uparrow & \text{iden} & \uparrow \\ F & \longrightarrow & F \end{array}$$

commutes, where the vertical arrows are inclusions.

PROPOSITION 1. *If* $\eta : F \to G$ *is a natural transformation of amalgams, there is a unique extension of* η *to a natural transformation* $\eta_* : F_* \to G_*$.

THEOREM 2 (see also [7]). *Every amalgam has a colimit, unique up to natural equivalence.*

We shall write $F_{\alpha\beta} = F(\{\alpha, \beta\})$, $F_\alpha = F(\{\alpha\})$ for all $\{\alpha\}$, $\{\alpha, \beta\} \in \Sigma$; and $*F = F_*(*)$ where F_* is a colimit of F . The group $*F$ may be constructed as the largest quotient group of the free product $\coprod_I F_\alpha$ of the groups F_α , such that

$$F_{\alpha\beta} \to F_\alpha \to \coprod_I F_\alpha \to *F = F_{\alpha\beta} \to F_\beta \to \coprod_I F_\alpha \to *F .$$

Let $\eta : F \to G$ be a natural transformation of amalgams $F, G : \Sigma(I) \to G$. Denote by F_0, G_0 the restrictions of F, G to Σ_0 and let $\eta_0 : F_0 \to G_0$ be the induced natural transformation.

The situation with which we deal is the following: suppose η_0 is a natural equivalence. What can be said about η ? Restated in group-theoretic terms, the question reads thusly:

$$\begin{array}{ccccc} F_{\alpha\beta} & \longrightarrow & F_\alpha & \longrightarrow & *F \\ \downarrow{\eta_{\alpha\beta}} & & \downarrow{\eta_\alpha} & & \downarrow{*\eta} \\ G_{\alpha\beta} & \longrightarrow & G_\alpha & \longrightarrow & *G \end{array}$$

Suppose $*\eta$ and all of the $\eta_{\alpha\beta}$ are isomorphisms. What can be said about the η_α ? We note that $*F$ is the smallest subgroup of $*F$ which contains $\operatorname{Im}(F_\alpha \to *F)$ for all $\alpha \in I$.

We shall henceforth assume that $F_{\alpha\beta} \to F_\alpha$ *is monic.* Observe that in

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this case our terminology agrees with that of Hanna Neumann [9] and B.H. Neumann [7]. The colimit $*F$ is in [7] the group $P*$, called there the canonic group (of the amalgam) and the generalized free product of the F_α with amalgamated subgroups $F_{\alpha\beta}$ is $*F$, provided $*F$ satisfies some additional conditions (see §4 below).

We shall show that in general η_α is neither monic (§2) nor epic (§3). If $G_\alpha \to *G$ is monic for all $\alpha \in I$, η_α is epic (§4). If $F_\alpha \to *F$ is monic for all $\alpha \in I$, then $G_\alpha \to *G$ is also monic and η_α is an isomorphism (§4). Finally (§5) we restate a theorem of Neumann ([7] or [9]) which guarantees that $F_\alpha \to *F$ is monic, deduce the theorem of E.M. Brown as a corollary and then prove a monicity theorem of our own.

2.

EXAMPLE. η_α *need not be monic.*

In the light of existing examples of amalgams $F : \Sigma(I) \to G$ in which $\mathrm{card}(I) = 3$ and $F_\alpha \to *F$ is not monic (*cf.* [7] or [8]) it is not surprising that η_α should fail to be monic. We shall present a particularly economical example of this behaviour (*cf.* §5). It utilizes a 2-generator, 1-relator group found by Baumslag and Solitar [1] which is non-hopfian, in other words, is isomorphic to a proper factor group of itself.

We use $\langle g : r \rangle$ to denote the group generated by the set of elements g subject to the relations r (*cf.* Magnus *et al.* [5]), and $[x, y] = x^{-1}y^{-1}xy$, the commutator of x and y.

Let $I = \{1, 2, 3\}$, and define the amalgams $F, G : \Sigma \to G$ as follows:

$$F_1 = \left\langle a,\ t : a = [a^2, t] \right\rangle , \qquad G_1 = \left\langle b,\ u : b = [b^2, u] \right\rangle ,$$
$$F_2 = \langle t_2 \rangle , \qquad G_2 = \langle u_2 \rangle ,$$
$$F_3 = \langle t_3 \rangle , \qquad G_3 = \langle u_3 \rangle ,$$
$$F_{ij} = F_{ji} = \left\langle t_{ij} \right\rangle , \qquad G_{ij} = G_{ji} = \left\langle u_{ij} \right\rangle$$

for $i, j \in I$, $i \neq j$. We note that t and $ta^{-1}[a, t]^2$ are of infinite order modulo the commutator subgroup of G_1 and therefore are of infinite order in G_1 . The monomorphisms $F_{ij} \to F_i$ and $G_{ij} \to G_i$ are defined by

$$t_{ij} \to \begin{cases} t_i & i = 2, 3 , \\ t & i = 1 , \ j = 2 , \\ ta^{-1}[a, t]^2 & i = 1 , \ j = 3 , \end{cases} \qquad u_{ij} \to \{u_i \qquad i = 1, 2, 3 ,$$

and so the colimits $F_*, G_* : \Sigma_* \to G$ may be defined as follows:

$$*F = \left\langle a,\ t : a = [a^2, t],\ a = [a, t]^2 \right\rangle = \left\langle a,\ c,\ t : c = [a, t],\ a = [a^2, t],\ a = c^2 \right\rangle$$
$$= \left\langle c,\ t : c = [c^2, t],\ c^2 = [c^4, t] \right\rangle = \left\langle c,\ t : c = [c^2, t] \right\rangle ,$$
$$*G = \left\langle b,\ u : b = [b^2, u] \right\rangle .$$

The morphisms are determined by the functoriality of F_* and G_* together with the maps $F_1 \to *F$, $G_1 \to *G$ respectively:

$$\begin{cases} a \to c^2 \\ t \to t \end{cases} \qquad \begin{cases} b \to b \\ u \to u . \end{cases}$$

Define the transformation η by

$$\eta_{ij}(t_{ij}) = u_{ij} \qquad \eta_i(t_i) = u_i$$
$$\eta_1(a) = b^2 \qquad \eta_1(t) = u .$$

We note that η_1 is epic, for $b = [b^2, u] \in \operatorname{Im}\eta_1$, and that $\eta_1\left(a^{-1}[a, t]^2\right) = 1$ while $a^{-1}[a, t]^2 \neq 1$ (*cf.* [5, pp. 260-261] or [1]). Finally, the induced map $*\eta : *F \to *G$ is an isomorphism as $*\eta(t) = u$, and $\eta_1([a, t]) = [b^2, u] = b$ implies $*\eta(c) = *\eta([a, t]) = b$.

3.

EXAMPLE. η_α *need not be epic.*

Once again $I = \{1, 2, 3\}$. Let $A \subset B$ be two simple groups (for example let B be the alternating group on 6 symbols and A the subgroup fixing the first symbol). Define the amalgams $F, G : \Sigma \to G$ as follows:

$$\begin{aligned} F_1 &= \langle t \rangle \times A , & G_1 &= \langle u \rangle \times B , \\ F_2 &= \langle t_2 \rangle , & G_2 &= \langle u_2 \rangle , \\ F_3 &= \langle t_3 \rangle , & G_3 &= \langle u_3 \rangle , \\ F_{ij} &= F_{ji} = \left\langle t_{ij} \right\rangle , & G_{ij} &= G_{ji} = \left\langle u_{ij} \right\rangle \end{aligned}$$

where $i, j \in I$, $i \neq j$. The homomorphisms $F_{ij} \to F_i$ and $G_{ij} \to G_i$ are defined by the following:

$$t_{ij} \to \begin{cases} t_i & i = 2, 3 , \\ t \times 1 & i = 1 , \quad j = 2 , \\ t \times a & i = 1 , \quad j = 3 , \end{cases} \qquad u_{ij} \to \begin{cases} u_i & i = 2, 3 , \\ u \times 1 & i = 1 , \quad j = 2 , \\ u \times a & i = 1 , \quad j = 3 , \end{cases}$$

where $1 \neq a \in A$. The colimit groups are easily seen to be

$$*F = \langle t \rangle , \quad *G = \langle u \rangle ,$$

and the homomorphisms are the obvious ones. Let $\theta : \langle t \rangle \to \langle u \rangle$ be defined by $\theta(t) = u$, and let $I : A \to B$ be the inclusion. The transformation η defined by $\eta_1 = \theta \times I$ and isomorphisms elsewhere induces the isomorphism $*\eta : *F \to *G$ sending t to u , but $\operatorname{Im}\eta_1$ is clearly the proper subgroup $\langle u \rangle \times A$ of $G_1 = \langle u \rangle \times B$.

4. The main theorems

We shall prove

THEOREM 3. *If* $G_\alpha \to *G$ *is monic for each* $\alpha \in I$, *then* $\eta_\alpha : F_\alpha \to G_\alpha$ *is epic.*

We first define the generalized free product of the F_α with

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amalgamated subgroups $F_{\alpha\beta} = F_{\beta\alpha}$ in the sense of B.H. Neumann [7] or Hanna Neumann [9] which we denote by g.f.p.$\langle F_\alpha : F_{\alpha\beta}\rangle$ or g.f.p.F. Our first two lemmas reduce the theorem to the case in which g.f.p.F exists. We then use a theorem of Hanna Neumann [9], stated below, to conclude that g.f.p.G exists. Finally we exploit the properties of a normal form for elements of such g.f.p.'s to conclude that η_α is epic.

Homomorphisms will henceforth be labelled in accordance with the diagram below.

$$\begin{tikzcd} F_{\alpha\beta} \arrow[r, "i_{\alpha\beta}"] \arrow[d, "\eta_{\alpha\beta}"] & F_\alpha \arrow[r, "i_\alpha"] \arrow[d, "\eta_\alpha"] & {}^*F \arrow[d, "\eta"] \\ G_{\alpha\beta} \arrow[r, "j_{\alpha\beta}"'] & G_\alpha \arrow[r, "j_\alpha"'] & {}^*G \end{tikzcd} .$$

DEFINITION. ${}^*F$ is g.f.p.F if the following conditions are satisfied:

$$\forall\, \alpha \in I\,, \qquad i_\alpha \text{ is monic}$$
$$\forall\, \alpha,\, \beta \in I\,, \qquad \mathrm{Im}\big(i_\alpha\big) \cap \mathrm{Im}\big(i_\beta\big) = \mathrm{Im}\big(i_\alpha i_{\alpha\beta}\big)\,.$$

We note that these conditions are independent (*cf.* [8]).

For the following two lemmas, it is assumed that j_α is monic, $\alpha \in I$.

LEMMA 4. *We may assume* η_α *and* i_α *monic.*

Proof. Let $K_\alpha = \mathrm{Ker}\,\eta_\alpha$. Observe that $\mathrm{Im}\big(i_{\alpha\beta}\big) \cap K_\alpha = 1$ for the composition $j_{\alpha\beta}\eta_{\alpha\beta}$ is monic, and that $K_\alpha \subseteq \mathrm{Ker}\big(i_\alpha\big)$. Thus we have a factorization

$$\begin{tikzcd}
F_{\alpha\beta} \arrow[r, "i_{\alpha\beta}"] \arrow[d, "\mathrm{id}"] & F_\alpha \arrow[r, "i_\alpha"] \arrow[d] & {*F} \arrow[d, "\mathrm{id}"] \\
F_{\alpha\beta} \arrow[r] \arrow[d, "\eta"] & F_\alpha/K_\alpha \arrow[r] \arrow[d] & {*F} \arrow[d, "\eta"] \\
G_{\alpha\beta} \arrow[r, "j_{\alpha\beta}"'] & G_\alpha \arrow[r, "j_\alpha"'] & {*G}
\end{tikzcd}$$

such that $F_\alpha \to F_\alpha/K_\alpha$ is epic. Therefore η_α is epic if and only if $F_\alpha/K_\alpha \to G_\alpha$ is epic. Also $F_\alpha/K_\alpha \to G_\alpha$ is monic and the lower right square commutes so $F_\alpha/K_\alpha \to *F$ is monic.

LEMMA 5. *We may assume further that* $*F = \text{g.f.p.}F$.

Proof. By Lemma 4, we need ensure the intersection property. Set

$$F'_{\alpha\beta} = \mathrm{Im}(i_\alpha) \cap \mathrm{Im}(i_\beta)$$
$$G'_{\alpha\beta} = \eta(F'_{\alpha\beta})$$

and define $F'_{\alpha\beta} \to F_\alpha$ and $G'_{\alpha\beta} \to G_\alpha$ as the inverses of i_α restricted to $F'_{\alpha\beta}$ and j_α restricted to $G'_{\alpha\beta}$ respectively. These definitions are sensible as i_α, j_α are monic, $F'_{\alpha\beta} \subseteq \mathrm{Im}(i_\alpha)$ by construction and $G'_{\alpha\beta} = \eta\left[\mathrm{Im}(i_\alpha) \cap \mathrm{Im}(i_\beta)\right] \subseteq \mathrm{Im}(j_\alpha) \cap \mathrm{Im}(j_\beta)$.

THEOREM 6 (Hanna Neumann [9]). *Set* $U_\alpha = \left\langle \cup_\beta \mathrm{Im}(i_{\alpha\beta}) \right\rangle \subseteq F_\alpha$. *Then* $\text{g.f.p.}\langle U_\alpha : F_{\alpha\beta}\rangle$ *exists if and only if* $\text{g.f.p.}\langle F_\alpha : F_{\alpha\beta}\rangle$ *exists.*

PROPOSITION 7. *If* $\text{g.f.p.}F$ *exists, then so does* $\text{g.f.p.}G$.

Proof. The group $V_\alpha = \left\langle \cup_\beta \mathrm{Im}(j_{\alpha\beta}) \right\rangle$ is isomorphic to U_α , with isomorphism the restriction of η_α to U_α . Therefore $\text{g.f.p.}\langle V_\alpha : G_{\alpha\beta}\rangle$ exists, which implies that $\text{g.f.p.}G$ exists.

COROLLARY 8. *If* $F_\alpha \to *F$ *is monic, for all* $\alpha \in I$, *then so is* $G_\alpha \to *G$.

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NORMAL FORM THEOREM 9 (Hanna Neumann [*10*]). *Suppose* $*F = \text{g.f.p.}F$, *and let* $U = \left\langle \cup_{\alpha,\beta} \operatorname{Im}\left(i_\alpha i_{\alpha\beta}\right) \right\rangle \subseteq *F$. *For each* $\alpha \in I$ *choose a system* S^α *of right coset representatives of* $i_\alpha(U_\alpha)$ *in* $\operatorname{Im}(i_\alpha)$ *such that* $1 \in S^\alpha$ $\left[\right.$*that is,* $\operatorname{Im}(i_\alpha)$ *is the disjoint union of* $i_\alpha(U_\alpha) \cdot s$, $s \in S^\alpha\left.\right]$. *For each* $\alpha \in I$ *choose a system* T^α *of right coset representatives of* $i_\alpha(U_\alpha)$ *in* U *such that* $1 \in T^\alpha$. *Then each element of* $*F$ *may be written as*

$$u_0 t_1 s_1 \ldots t_r s_r$$

where $u_0 \in U$, $t_i \in T^{\alpha(i)}$, $s_i \in S^{\alpha(i)}$, $t_i \neq 1$, *and* $s_i = 1$ *implies* $\alpha(i) \neq \alpha(i+1)$. *Distinct symbols represent distinct elements.*

Proof of Theorem 3. We may assume $*F = \text{g.f.p.}F$, $*G = \text{g.f.p.}G$. Set $V = \left\langle \cup_{\alpha,\beta} \operatorname{Im}\left(j_\alpha j_{\alpha\beta}\right) \right\rangle \subseteq *G$. Choose right coset representative S^α and T^α in $*F$ as described in Theorem 9 above. We have that $\eta(U) = V$, $\eta i_\alpha(U_\alpha) = j_\alpha(V_\alpha)$ so $\eta(T^\alpha)$ is a system of right coset representatives of $j_\alpha(V_\alpha)$ in V . Since $\eta\left[\operatorname{Im}(i_\alpha)\right] \subseteq \operatorname{Im}(j_\alpha)$ we may obtain a system of right coset representatives of $j_\alpha(V_\alpha)$ in $\operatorname{Im}(j_\alpha)$ of the form $\eta(S^\alpha) \cup X^\alpha$. Assume that $x \in X^\alpha$, and let $y = \eta^{-1}(x) \in *F$. We may write

$$y = u_0 t_1 s_1 \ldots t_r s_r$$

as in the normal form theorem. Then $\eta(y) = x = \eta(u_0)\eta(t_1) \ldots \eta(s_r)$ and we have obtained two distinct normal form symbols for the same element of $*G$. Therefore $X^\alpha = \emptyset$ and so

$$\operatorname{Im}(j_\alpha) = \left\langle \eta(S^\alpha) \cup V^\alpha \right\rangle = \operatorname{Im}(\eta i_\alpha) = \operatorname{Im}(j_\alpha \eta_\alpha)$$

or η_α is epic.

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THEOREM 10. *Assume that* $i_\alpha : F_\alpha \to {}^*F$ *is monic for all* $\alpha \in I$. *Then* η_α *is an isomorphism.*

Proof. Since $j_\alpha \eta_\alpha = \eta i_\alpha$, it is clear that η_α is monic. By Corollary 8, j_α is monic for all $\alpha \in I$, hence η_α is also epic and so is an isomorphism.

5. Existence of the g.f.p.

It seems clear that one now wishes necessary and sufficient conditions that the homomorphisms $i_\alpha : F_\alpha \to {}^*F$ be monic. A summary of results concerning the existence of the g.f.p. may be found in Neumann [8]. In particular we have

THEOREM 11 (*cf.* Neumann [7]). *If* $\operatorname{card}(I) \leq 2$, *then the* g.f.p. *of any amalgam* $F : \Sigma(I) \to G$ *exists.*

This theorem and our Theorem 10 imply the result of E.M. Brown [2, Proposition 12.5, p. 485] cited in the introduction.

We now require a definition.

DEFINITION. Let $F : \Sigma(I) \to G$ be an amalgam. The *graph of* F consists of vertices α for each $\alpha \in I$, vertices $\{\alpha, \beta\}$ for each $\{\alpha, \beta\} \subset I$ such that $F_{\alpha\beta} \neq \{1\}$, and arrows beginning at $\{\alpha, \beta\}$ and terminating at α , for each $\{\alpha, \beta\}$ in the graph.

THEOREM 12. *If the amalgam* $F : \Sigma \to G$ *has a simply connected graph, then* $F_\alpha \to {}^*F$ *is monic.*

We shall omit the proof of this theorem, remarking only that one may first reduce to the case of an amalgam with connected, simply connected graph and finitely many vertices. There remains an amalgam whose graph is a finite tree, and one proceeds by pulling off branches.

I have learned that this result was announced by A. Karrass and D. Solitar and will appear in the *Transactions of the American Mathematical Society* [4]. (The colimit ${}^*F$ of an amalgam with simply connected graph is in [4] called a tree product.)

We remark finally that Corollary 13 below follows easily from Theorem 12 (by Corollary 8) and that Theorem 14 below may be deduced rather naturally from Theorem 11 via Theorem 12.

COROLLARY 13. *Given the amalgam* $F : \Sigma(I) \to G$, *let* I' *be some collection of 2-element subsets of* I . *Define the functor* $F' : \Sigma(I) \to G$ *by:* $F' = F$ *on the full subcategory of* $\Sigma(I)$ *generated by the objects of* $\Sigma(I)$ *not in* I' , *and* $F'_{\alpha\beta} = \{1\}$, $\{\alpha, \beta\} \in I'$. *Assume* $*F' = *F$ *and that the graph of* F' *is simply connected. Then* $F_\alpha \to *F$ *is monic for all* $\alpha \in I$.

THEOREM 14 (B.H. Neumann [7] or Hanna Neumann [9]). *Assume* $F : \Sigma(I) \to G$ *is an amalgam for which there exists* $H \in G$ *and isomorphisms* $H \to F_{\alpha\beta}$ *for all* $\alpha, \beta \in I$ *such that*

$$H \to F_{\alpha\beta} \to F_\alpha = H \to F_{\alpha\gamma} \to F_\alpha$$

for all $\alpha, \beta, \gamma \in I$. *Then* g.f.p.F *exists.*

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Herbert H. Lehman College,
City University of New York,

and

Institute for Advanced Study,
Princeton, New Jersey,
USA.