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A CHARACTERIZATION OF THE MINKOWSKI NORMS

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ABSTRACT. If $n > 2$ and $M(m_1, \ldots, x_n)$ is a symmetric norm of the form $m(x_1, m(x_2, m(\ldots)\ldots)$, where m is a symmetric norm on $\mathbb{R}^2$, then $m(x,y) = (|x|^p + |y|^p)^{1/p}$ for some $p \geq 1$ or else $m(x,y) = \max\{|x|, |y|\}$.

1. **Introduction.** Mizel and Sundaresan [1] generalized the Minkowski (ℓ^p) norms by defining a sequential norm iteratively, starting from an essentially arbitrary symmetric norm on $\mathbb{R}^2$. Then it was shown in [2] that the Mizel-Sundaresan norms are topologically equivalent in the infinite dimensional case to Orlicz norms and that therefore no new BK spaces arise from their construction. If the only use for sequential norms was to identify new BK space, this result would have put an end to all questions about the Mizel-Sundaresan construction.

There are however other uses for norms. For example, Anderson [3] recently showed that an ℓ^p norm with an integer value of p can be used to generate a sample of size p from the gamma probability distribution with shape parameter $1/p$. Statistical estimation methods based on minimum distance, as described in [4], for example, often use norms, and the statistical error distributions may depend strongly on the particular norm used.

The Mizel-Sundaresan norms have several important properties which make them attractive for statistical purposes. In the first place, they are very simple to compute, much simpler, for example, than any version of Orlicz norms. Moreover, Mizel-Sundaresan norms have the important theoretical property that the conjugate of the Mizel-Sundaresan (iterative) norm generated from a two-dimensional norm m is just the iterative norm generated from the conjugate of m. The situation surrounding Orlicz norms, as described in [5], for example, is much more complicated.

One property that Mizel-Sundaresan norms do not generally possess is the property of symmetry (K-symmetry). Thus the iterative norm of a sequence is generally not left invariant by permutations of the coordinates. In fact, the following question has been outstanding for a number of years: can a Mizel-Sundaresan norm be K-symmetric except in the classical ℓ^p case? The purpose of this note is simply to answer this question in the negative.

2. **Main results.** A Mizel-Sundaresan (iterative) norm I_m is determined by a norm m on $\mathbb{R}^2$ under the conditions

$$\max\{|x_1|, |x_2|\} \leq m(x_1, x_2) = m(|x_2|, |x_1|) \leq |x_1| + |x_2|$$

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in addition to the norm conditions. Norms on $\mathbb{R}^3, \mathbb{R}^4, \ldots$ are defined iteratively by

$$m_3(x_1, x_2, x_3) = m(m(x_1, x_2), x_3),$$
$$m_4(x_1, x_2, x_3, x_4) = m(m_3(x_1, x_2, x_3), x_4),$$

and so on. An extended norm I_m on the space of infinite sequences is then defined by letting $I_m(x_1, x_2, \ldots)$ be the limit of $m_n(x_1, \ldots, x_n)$, which is monotonic in n, as $n \longrightarrow \infty$. If we set $g(\alpha) = m(1, \alpha) - 1$, then I_m is topologically equivalent to the Orlicz norm O_g defined by

$$O_g(x) = \inf \left\{ \lambda > 0 \mid \sum_i g(|x_i| / \lambda) \leq 1 \right\}$$

as was shown, essentially, in [2].

The only obvious case in which O_g is exactly the same as I_m is the case in which m is one of the Minkowski ℓ^p norms m_p, defined by

$$m_p(x_1, x_2) = \begin{cases} (|x_1|^p + |x_2|^p)^{1/p} & \text{for } p \in [1, \infty) \\ \max\{|x_1|, |x_2|\} & \text{for } p = \infty. \end{cases}$$

In general, O_g is symmetric but I_m is not. Of course, the two-dimensional norm m is assumed to be symmetric and it is easy to see that if m_3 is symmetric then so are $m_4, m_5, \ldots$ and also I_m. This is also the case in which I_m can be described as in the abstract, i.e., as the limit of $m_n(x_n, \ldots, x_1)$.

Note also that a proper sequential norm is symmetric if and only if its conjugate is symmetric. Hence, a number of questions are settled by the following.

THEOREM. *If I_m is symmetric, then m is m_p for some $p \in [1, \infty]$.*

Note. In terms of the function $G(t) = m(1, |t|)$, the theorem defines the only solutions G for the functional equation

$$G(s)G(t/s) = G(t)G(s/t), \text{ for } s, t > 0$$

which satisfy the additional conditions implied above, *viz.* that $G(s) = sG(1/s)$ is convex and even, achieving its minimum at $G(0) = 1$, with $|G'(s)| \leq 1$.

PROOF. Define $H_n = m_n(J_n)$, where J_n is the point in $\mathbb{R}^n$ which has each of its coordinates equal to 1. If $H_2 = 1$, then $m = m_\infty$ and there is nothing more to prove. Hence we assume $H_2 > 1$ and define $p \in [1, \infty)$ by $H_2 = 2^p$. Notice that $H_{i+j} = m(H_i, H_j)$ and hence H has the multiplicative property $H_{ij} = H_i H_j$. Thus, for any positive rational number $r = m/n$, one can consistently define $H(r) = H_m/H_n$, noting that $H(r) = H_r$ when r is a positive integer. Note also that for $m = 1, 2, \ldots$,

$$0 \leq H_{m+1} - H_m = H_m g\left(\frac{1}{H_m}\right) \leq g(1).$$

It follows that for any positive real number x, the infimum of $H(r)$ for rational $r > x$ is the same as the supremum for $r < x$ and therefore H has a unique continuous extension

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to the positive reals. Let this extension be made. Then clearly, $H(x) = x^p$ for all $x > 0$ and, in particular, $H_n = n^p$ for $n = 1, 2, \ldots$.

Now we just have to show that the norm m is characterized by the sequence $(H_1, H_2, \ldots)$. In fact, for any positive real number x, we can approximate x by numbers of the form $H(m/n)$ with m and n integral. Then as $m/n \longrightarrow x$ we have

$$H_{m+n}/H_m = m(1, H_n/H_m) \longrightarrow m(1, x).$$

This concludes the proof.

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