AN INEQUALITY IMPLICIT FUNCTION THEOREM

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Abstract

Let \( f \) be a continuous function, and \( u \) a continuous linear function, from a Banach space into an ordered Banach space, such that \( f - u \) satisfies a Lipschitz condition and \( u \) satisfies an inequality implicit-function condition. Then \( f \) also satisfies an inequality implicit-function condition. This extends some results of Flett, Craven and S. M. Robinson.


Following Rockafellar [13], by a convex process is meant a map \( T \) of points in a Banach space \( X \) into the subsets of another Banach space \( Y \) such that \( 0 \in T0 \), \( T(\lambda x) = \lambda Tx \) and \( Tx_1 + Tx_2 \subseteq T(x_1 + x_2) \) for all \( \lambda > 0 \), \( x_1 \), \( x_2 \) and \( x \) in \( X \). This is the case if and only if the graph \( G(T) \) of \( T \) is a convex cone in \( X \times Y \). \( T \) is a closed convex process if \( G(T) \) is a closed convex cone. If \( T \) is also onto \( Y \) (in the sense that for each \( y \in Y \) there exists \( x \in X \) such that \( y \in Tx \)) then it is an open mapping (see [10, Theorem 2] and also [5, page 182], [8, Theorem 1]), that is, there exists a constant \( k > 0 \) with the following property: for each \( y \in Y \) there is \( x \in X \) with \( ||x|| \leq k||y|| \) such that \( y \in Tx \). (In this case we say that \( T \) is \( k \)-open.)

Suppose \( K \) is a closed convex cone in \( Y \). Then, for any continuous linear map \( u \) from \( X \) into \( Y \), we can associate a closed convex process \( U \) by putting

\[
U(x) = u(x) + K \quad (x \in X).
\]

Thus, if \( U \) is onto \( Y \), then \( U \) is \( k \)-open for some \( k > 0 \). The following Theorems 1 and 2 were proved by Flett [4, Lemmas 1 and 3] in the special case that \( K = \{0\} \) (see also Craven [2], and [3, page 147]).
THEOREM 1. Let $U$ be $k$-open for some $k > 0$. Let $f$ be a continuous (not necessarily linear) map from a subset of $D$ of $X$ containing $0$ into $Y$ such that $f(0) = 0$ and

$$||\{f(x_1) - u(x_1)\} - \{f(x_2) - u(x_2)\}|| \leq (\eta/k)||x_1 - x_2||$$

for some $\eta \in (0,1)$ and all $x_1, x_2 \in D$. If $z \in X$ and $D$ contains the ball $B$ with centre $z$ and radius $R$ with $R > (\eta/(1-\eta)||z||$, then there exists $x \in B$ such that $u(z) \in f(x) + K$.

The proof is based on the following contraction lemma, essentially due to Robinson [11] who considered Hausdorff distance $\rho_H$ instead of unbalanced $d$ (our proof is also simpler then that given in [11]). See also [7]. For subsets $A, B$ of a metric space $(\Omega, \rho)$ and $x \in \Omega$, we define $d(x, B) := \inf\{\rho(x, b): b \in B\}$, $d(A, B) := \sup\{d(a, B): a \in A\}$, and $\rho_H(A, B) := \max\{d(A, B), d(B, A)\}$.

LEMMA 1. Let $(\Omega, \rho)$ be a complete metric (or semi-metric) space, and let $T: \Omega \to 2^\Omega$ satisfy

$$d(Tx_1, Tx_2) \leq \eta\rho(x_1, x_2)$$

for some $\eta \in (0,1)$ and all $x_1, x_2$ in a subset $D$ of $\Omega$. Suppose $D$ contains a ball $B$ with centre $x_0$ and radius $R > d(x_0, Tx_0)/(1 - \eta)$. Then there exists $x \in B$ with $x \in Tx$. 

PROOF. Take $\varepsilon > 0$ such that $R > d(x_0, Tx_0)/(1 - \eta) + \varepsilon$, and let $\sigma = d(x_0, Tx_0) + \varepsilon(1 - \eta)$. Since $d(x_0, Tx_0) < \sigma$, there exists $x_1 \in Tx_0$ such that $\rho(x_0, x_1) < \sigma$. By (2),

$$d(x_1, Tx_1) \leq d(Tx_0, Tx_1) \leq \eta\rho(x_0, x_1) < \eta\sigma,$$

so there is $x_2 \in Tx_1$ such that $\rho(x_1, x_2) < \eta\sigma$. Suppose that $x_1, \ldots, x_n$ from $B$ have been selected respectively from $Tx_0, \ldots, Tx_{n-1}$ such that $\rho(x_{k-1}, x_k) < \eta^{k-1}\sigma$ for all $k \leq n$. Then, since

$$d(x_n, Tx_n) \leq d(Tx_{n-1}, Tx_n) \leq \eta\rho(x_{n-1}, x_n) < \eta^n\sigma,$$

one can select $x_{n+1} \in Tx_n$ such that $\rho(x_n, x_{n+1}) < \eta^n\sigma$. Note that $\rho(x_0, x_{n+1}) < \sigma(1 + \eta + \cdots + \eta^n) < \sigma/(1 - \eta) = d(x_0, Tx_0)/(1 - \eta) + \varepsilon$; in particular $x_{n+1} \in B$. In this way, we have a Cauchy sequence, which converges, say to $v$. Then $d(x_0, v) \leq d(x_0, Tx_0)/(1 - \eta) + \varepsilon$ so $v \in B$. The proof that $v \in Tv$ is similar to [10]: take $\gamma > 0$ and a positive integer $n$. Then there is $y \in Tv$ such that $\rho(x_n, y) < d(x_n, Tv) + \gamma$ so

$$\rho(x_n, y) < d(Tx_{n-1}, Tv) + \gamma \leq \eta\rho(x_{n-1}, v) + \gamma.$$
and
\[ d(v, T v) \leq \rho(v, y) \leq \rho(v, x_n) + \rho(x_n, y) \leq \rho(v, x_n) + \eta \rho(x_{n-1}, v) + \gamma. \]
Letting \( n \to \infty \) and \( \gamma \to 0 \), we see that \( v \in T v \).

We now turn to the proof of Theorem 1. We shall apply Lemma 1 to \( \Omega = X \) with \( \rho \) the usual metric induced by the norm. The inverse \( U^{-1} \) of the multivalued function \( U \) is defined by
\[ U^{-1}y = \{ x \in X : y \in Ux \} \quad (y \in Y). \]

By assumption each \( U^{-1}y \) is non-empty. We will show that
\[ d(U^{-1}y_1, U^{-1}y_2) \leq k ||y_1 - y_2|| \quad (y_1, y_2 \in Y). \]
In fact, let \( x_1 \in U^{-1}y_1 \). Since \( U \) is \( k \)-open, there is \( x \in X \) with \( ||x|| \leq k ||y_2 - y_1|| \) such that \( y_2 - y_1 \in Ux \). Then
\[ y_2 = (y_2 - y_1) + y_1 \in u(x) + K + u(x_1) + K = u(x + x_1) + K = U(x + x_1) \]
because \( K \) is a convex cone. Therefore \( x + x_1 \in U^{-1}y_2 \), and
\[ d(x_1, U^{-1}y_2) \leq \rho(x_1, x + x_1) = ||x|| \leq k ||y_2 - y_1||. \]

Since \( x_1 \) is arbitrary in \( U^{-1}y_1 \), (3) is proved.

Now define \( T \) on \( D \) by \( Tw = U^{-1}(g(w)) \) where \( g(w) := u(z) - f(w) + u(w) \).
By (1), we have, for all \( w_1, w_2 \in D \), that
\[ ||g(w_1) - g(w_2)|| = ||\{ f(w_2) - u(w_2) \} - \{ f(w_1) - u(w_1) \}|| \leq \eta / k ||w_1 - w_2||; \]
it follows from (3) that \( d(Tw_1, Tw_2) \leq \eta ||w_1 - w_2|| \). Moreover, since \( g(0) = u(z) \),
\[ z \in U^{-1}(u(z)) = T0, \]
we have
\[ d(z, Tz) \leq d(T0, Tz) \leq \eta ||z - 0|| = \eta ||z||. \]

By the Contraction Lemma, there exists \( x \in B \) such that \( x \in T x \). Take a sequence \( \{ x_n \} \) in \( Tx \) convergent to \( x \). Then \( g(x) \in U(x_n) = u(x_n) + K \), that is,
\[ u(z) - f(x) + u(x) \in u(x_n) + K. \]
Since \( K \) is closed it follows that \( u(z) \in f(x) + K \).

**Theorem 2.** Let \( C \) be a closed convex cone in \( Y \), and \( Q \) a subset of \( Y \) such that \( Q + C \subseteq Q \) and \( \lambda Q \subseteq Q \) for all \( \lambda \in [0, 1] \). Let \( f \) be a \( C^1 \)-function at 0 from an open set in \( X \) containing 0 into \( Y \), with \( f(0) = 0 \) and \( f'(0) = u \). Define \( U \) by \( U(x) = u(x) - C \) for all \( x \in X \). If \( U \) is onto \( Y \), then \( U^{-1}(Q) \) is contained in the tangent cone of \( f^{-1}(Q) \) at 0.

**Proof.** It is known that \( U \) is \( k \)-open for some \( k > 0 \) as noted before. Let \( h \in U^{-1}(q) \) with \( ||h|| = 1 \) and \( q \in Q \). Then \( q \in U(h) = u(h) - C \) so \( u(h) \in C + Q \subseteq Q \).
and consequently \( u(\lambda h) \in Q \) for all \( \lambda \in [0,1] \). Take \( \eta \in (0,1) \); then there exists \( \xi > 0 \) such that \( ||f'(x) - u|| \leq \eta/k \) for all \( x \) in \( \xi B_X \) the \( \xi \)-ball with centre 0 in \( X \). By the Mean Value Theorem, \( (1) \) of Theorem 1 holds with \( D := \xi B_X \). Take \( \lambda > 0 \), small enough that \( D \) contains the open ball with centre \( \lambda h \) and radius \( 2\eta\lambda/(1-\eta) \). Applying Theorem 1 there is \( x \in X \) with \( ||x-\lambda h|| \leq 2\eta||\lambda h||/(1-\eta) \) such that \( u(\lambda h) \in f(x) - C \), that is, \( f(x) \in u(\lambda h) + C \subseteq Q + C \subseteq Q \).

Do the above for all \( \eta = 1/n \) with integers \( n > 3 \) and choose \( \lambda = \lambda_n > 0 \) such that \( \lambda_n \to 0 \) as \( n \to \infty \); we write \( x_n \) for \( x \) accordingly constructed above. Note that \( x_n \neq 0, x_n \in f^{-1}(Q) \), \( x_n \to 0 \) and

\[
||x_n||x_n||^{-1} - \lambda_n h||\lambda_n h||^{-1}|| \leq 2||x_n - \lambda_n h||||\lambda_n h||^{-1}
\leq 4\eta/(1-\eta) \to 0 \text{ as } n \to \infty,
\]

where we have used the elementary inequality \( ||a||a||^{-1} - b||b||^{-1}|| \leq 2||a - b|| ||b||^{-1} \) for non-zero elements in a normed space, which is true because

\[
||(a||b|| - a||a|| - b||a|| + a||a||)||a|| ||b||^{-1}|| \leq 2||a|| ||b - a||(||a|| ||b||)^{-1}.
\]

Therefore \( h \) is in the tangent cone of \( f^{-1}(Q) \) at 0.

**Remark.** A related result has been given by Robinson [12, Corollary 2] where he considered the case \( Q = C \). Applications of results of this type to Optimization Theory, have been given in [1], [2], [3], [4], [6], [9], [12] and [14].

**References**


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