BULL. AUSTRAL. MATH. SOC. 40G05, 42B99
VOL. 31 (1985), 199-214.

# LEBESGUE CONSTANTS FOR DOUBLE HAUSDORFF MEANS

F. USTINA

As is well known, the divergence of the set of constants known as the Lebesgue constants corresponding to a particular method of summability implies the existence of a continuous, periodic function whose Fourier series, summed by the method, diverges at a point, and of another such function the sums of whose Fourier series converge everywhere but not uniformly in the neighborhood of some point.

In 1961, Lorch and Newman established that if $L(n; g)$ is the $n$th Lebesgue constant for the Hausdorff summability method corresponding to the weight function $g(u)$, then

$$L(n; g) = C(g) \log n + o(\log n) , \quad n \to \infty ,$$

where

$$C(g) = \left(2/\pi^2\right)|g(1)-g(1^-)| + (1/\pi)M\left\{\left|\sum \left\{g\left(\varepsilon_k^+\right)-g\left(\varepsilon_k^-\right)\right\} \sin \varepsilon_k u\right|\right\} ,$$

where the summation is taken over the jump discontinuities $\{\varepsilon_k\}$ of $g(u)$ and $M\{f(u)\}$ denotes the mean value of the almost periodic function $f(u)$ .

In this paper, a partial extension of this result to the two dimensional analogue is obtained. This extension is summarized in Theorem 1.3.

Received 5 September 1984.

## 1. Introduction

The Lebesgue constants for double Fourier series are defined by

$$L(m, n) = 4/\pi^2 \int_{0,0}^{\pi,\pi} |D_{m,n}(s, t)|dsdt$$

where

$$D_{m,n}(s, t) = \frac{\sin(m+\frac{1}{2})s}{2\sin\frac{1}{2}s} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}$$

is the Dirichlet kernel. As is well known, the divergence of these constants implies the existence of a continuous, periodic function of two variables whose Fourier series diverges at a point (the du Bois-Reymond singularity), and another such function whose Fourier series converges everywhere, but not uniformly in the neighborhood of some point (the Lebesgue singularity).

If the sequence $\{D_{k,l}(s, t)\}$ is transformed by a summability method $S$, and we denote the $mn$th transform of this sequence by $K_{m,n}(s, t)$, then the sequence of constants defined by

$$L(m, n; S) = 4/\pi^2 \int_{0,0}^{\pi,\pi} |K_{m,n}(s, t)|dsdt \tag{1.1}$$

are the Lebesgue constants for the summability method $S$. The divergence of these constants implies the du Bois-Reymond singularity and the Lebesgue singularity for the summability method $S$.

Lorch and Newman [4] investigated the Lebesgue constants for the regular Hausdorff means in the one dimensional case and established the following:

**THEOREM 1.2.** *Let $L(m; g)$ denote the mth Lebesgue constant for the regular Hausdorff method with weight function $g(u)$. Then*

$$L(m; g) = c(g) \log m + o(\log m) , \quad m \to \infty ,$$

*where, with $u_k$ the kth discontinuity of $g(u)$ and $M\{f(x)\}$ the mean value of the almost periodic function $f(x)$,*

$$c(g) = \frac{2}{\pi^2} |g(1)-g(1^-)| + \frac{1}{\pi} M\left\{\left|\sum_k \{g(u_k^+)-g(u_k^-)\} \sin u_k x\right|\right\} .$$

*Furthermore,*

$$0 \le c(g) \le (4/\pi^2)V(g)\ , \quad 0 \le u \le 1\ ,$$

*and* $c(g) = 0$ *if and only if* $g(u)$ *is continuous. If the method is totally regular so that* $V(g) = 1$ *, then* $c(g) = 4/\pi^2$ *if and only if the method is ordinary convergence. If* $g(1) = g(1^-)$ *, then* $c(g) \le 2/\pi^2$ *, and* $c(g) = 2/\pi^2$ *if and only if the method is of Euler type.*

The object of this paper is to get a partial extension of the above result to the two dimensional case. A full extension will follow in a subsequent paper. More specifically, we prove

**THEOREM 1.3.** *If* $L(m, n; g)$ *is the* $mn$*th Lebesgue constant for the Hausdorff means corresponding to the regular weight function* $g(u, v)$ *, then*

$$\begin{aligned}(1.4)\quad L(m, n; g) = {} & 4/\pi^2 \int_{1,1}^{n^{\frac12},m^{\frac12}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s}\,\frac{\sin tv}{t}\,dg(u, v)\right| dsdt \\ & + 4/\pi^3 \log n \int_1^{m^{\frac12}} \left|\int_0^1 \frac{\sin su}{s}\,\{dg(u, 1)-dg(u, 1^-)\}\right| ds \\ & + 4/\pi^3 \log m \int_1^{n^{\frac12}} \left|\int_0^1 \frac{\sin tv}{t}\,\{dg(1, v)-dg(1^-, v)\}\right| dt \\ & + 4/\pi^4 |g(1, 1; 1^-, 1^-)| \log m \log n + o(\log m \log n)\ ,\end{aligned}$$

*as* $m, n \to \infty$ *, where*

$$g(1, 1; 1^-, 1^-) = g(1, 1) - g(1, 1^-) - g(1^-, 1) + g(1^-, 1^-)\ .$$

*If* $g(u, 1) = g(u, 1^-)$ *and* $g(1, v) = g(1^-, v)$ *,* $0 \le u, v < 1$ *, then*

$$\begin{aligned}(1.5)\quad L(m, n; g) = {} & 4/\pi^2 \int_{1,1}^{n^{\frac12},m^{\frac12}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s}\,\frac{\sin tv}{t}\,dg(u, v)\right| dsdt \\ & + 4/\pi^4 |g(1, 1; 1^-, 1^-)| \log m \log n + o(\log m \log n)\ ,\end{aligned}$$

*as* $m, n \to \infty$ *. If, in addition,* $g(u, v)$ *is continuous in its domain at* $(1, 1)$ *, or if only* $g(1, 1; 1^-, 1^-) = 0$ *, then*

$$(1.6)\quad L(m,\,n;\,g) = 4/\pi^2 \int_{1,1}^{n^{\frac12},m^{\frac12}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s}\,\frac{\sin tv}{t}\,dg(u,\,v)\right| ds dt + o(\log m \log n)\ ,\quad m,\,n \to \infty\ .$$

NOTE 1.7. In the sequel, all $o$- and $O$- terms will be taken as $m$ and $n$ tend to $\infty$ without further mention.

## 2. A preliminary result

If $K_{m,n}(s,\,t)$ is the $mn$th Hausdorff transform of the sequence $\{D_{k,l}(s,\,t)\}$ , corresponding to the regular weight function $g(u,\,v)$ , then

$$(2.1)\quad K_{m,n}(s,\,t)$$

$$= \sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} D_{k,l}(s,\,t) \int_{0,0}^{1,1} u^{i}(1-u)^{m-k}v^{l}(1-v)^{n-l}dg(u,\,v)$$

$$= \tfrac14 \int_{0,0}^{1,1} \frac{\operatorname{Im}\left\{\left(1-u+ue^{is}\right)^m e^{i(s/2)}\right\}}{\sin\frac12 s}\,\frac{\operatorname{Im}\left\{\left(1-v+ve^{it}\right)^n e^{i(t/2)}\right\}}{\sin\frac12 t}\,dg(u,\,v)$$

$$= \tfrac14 \int_{0,0}^{1,1} \rho_1^m\rho_2^n \sin m\alpha \sin n\beta \cot\frac{s}{2}\cot\frac{t}{2}\,dg(u,\,v)$$

$$+ \tfrac14 \int_{0,0}^{1,1} \rho_1^m\rho_2^n \sin m\alpha \cos n\beta \cot\frac{s}{2}\,dg(u,\,v)$$

$$+ \tfrac14 \int_{0,0}^{1,1} \rho_1^m\rho_2^n \sin n\beta \cos m\alpha \cot\frac{t}{2}\,dg(u,\,v)$$

$$+ \tfrac14 \int_{0,0}^{1,1} \rho_1^m\rho_2^n \cos m\alpha \cos n\beta dg(u,\,v)$$

where $\rho_1$, $\rho_2$, $\alpha$ and $\beta$ are defined as in [6] by

$$(2.2)\quad \rho_1 \sin\alpha = u \sin s\ ,\quad \rho_1 \cos\alpha = 1 - u + u\cos s\ ,$$

$$\rho_2 \sin\beta = v \sin t\ ,\quad \rho_2 \cos\beta = 1 - v + v\cos t\ ,$$

or

$$(2.3)\quad \rho_1 e^{i\alpha} = 1 - u + ue^{is}\ ,\quad \rho_2 e^{i\beta} = 1 - v + ve^{it}\ .$$

It follows that

https://doi.org/10.1017/S000497270000469X Published online by Cambridge University Press

(2.4) $\rho_1^2 = 1 - 2u(1-u)(1 - \cos s)$ , $\rho_2^2 = 1 - 2v(1-v)(1 - \cos t)$ .

Hence $0 \le \rho_1^2, \rho_2^2 \le 1$ in the region of integration.

Substituting $\cot x/2 = \{\cot x/2 - 2/x\} + 2/x$ in (2.1) and simplifying the results it follows that

$$(2.5)\quad K_{m,n}(s, t) = \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin m\alpha}{s} \frac{\sin n\beta}{t}\, dg(u, v)$$
$$+ \int_{0,0}^{1,1} \rho_1^m \frac{\sin m\alpha}{s} \phi(n; t, v) dg(u, v)$$
$$+ \int_{0,0}^{1,1} \rho_2^n \frac{\sin n\beta}{t} \phi(m; s, u) dg(u, v)$$
$$+ \int_{0,0}^{1,1} \phi(m; s, u)\phi(n; t, v) dg(u, v)$$

where

$$\phi(m; s, u) = \rho_1^m \left\{\sin m\alpha \left\{\cot \frac{s}{2} - \frac{2}{s}\right\} + \cos m\alpha\right\}$$

with $\phi(n; t, v)$ defined in an analogous manner.

Note that by a lemma due to Livingston [3],

$$(2.6)\quad \rho_1^2 < e^{-2u(1-u)s^2/\pi^2} .$$

Hence $\phi(m; s, u) = o(1)$ , $m \to \infty$ and $\phi(n; t, v) = o(1)$ , $n \to \infty$ , $0 < u, v < 1$ and $0 < s, t < \pi$ .

Next, by a lemma due to Szász [5], $\alpha = \alpha(s, u) = su + O\left(u(1-u)s^3\right)$ . Hence

$$\left(\rho_1^m/s\right)(\sin m\alpha - \sin msu) = \left(2\rho_1^m/s\right)\left(\cos \frac{m}{2} (\alpha+su) \sin \frac{m}{2} (\alpha-su)\right)$$
$$= \left(2\rho_1^m/s\right) \sin O\left(mu(1-u)s^3\right)$$
$$< 2e^{-mu(1-u)s^2/\pi^2} O\left(mu(1-u)s^2\right)$$
$$= o(1) ,$$

uniformly in $s$ and $u$ . Hence we may write

https://doi.org/10.1017/S000497270000469X Published online by Cambridge University Press

$$(2.7)\qquad \rho_1^m \frac{\sin m\alpha}{s} = \rho_1^m \frac{\sin msu}{s} + \psi(m;\ s,\ u)\ ,$$

$$\rho_2^n \frac{\sin n\beta}{t} = \rho_2^n \frac{\sin ntv}{t} + \psi(n;\ t,\ v)\ ,$$

where the $\phi$-function tends to zero. Substituting these in (2.5) and simplifying, it follows that

$$(2.8)\quad K_{m,n}(s,\ t) = \int_{0,0}^{1,1} \rho_1^m\rho_2^n \frac{\sin msu}{s}\frac{\sin ntv}{t}\,dg(u,\ v) + \int_{0,0}^{1,1} \rho_1^m \frac{\sin msu}{s}\gamma(n;\ t,\ v)dg(u,\ v) + \int_{0,0}^{1,1} \rho_2^n \frac{\sin ntv}{t}\gamma(n;\ t,\ v)dg(u,\ v) + \int_{0,0}^{1,1} \gamma(m;\ s,\ u)\gamma(n;\ t,\ v)dg(u,\ v)$$

where

$$\gamma(m;\ s,\ u) = \psi(m;\ s,\ u) + \psi(m;\ s,\ u)\ ,$$

$$\gamma(n;\ t,\ v) = \psi(n;\ t,\ v) + \psi(n;\ t,\ v)\ .$$

Thus

$$(2.9)\quad L(m,\ n;\ g) = 4/\pi^2 \int_{0,0}^{\pi,\pi} \left|\int_{0,0}^{1,1} \rho_1^m\rho_2^n \frac{\sin msu}{s}\frac{\sin ntv}{t}\,dg(u,\ v)\right| dsdt + O\left(\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1}\rho_1^m\frac{\sin msu}{2}\gamma(n;\ t,\ v)dg(u,\ v)\right|dsdt\right) + O\left(\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1}\rho_2^n\frac{\sin ntv}{2}\gamma(m;\ s,\ u)dg(u,\ v)\right|dsdt\right) + O\left(\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1}\gamma(m;\ s,\ u)\gamma(n;\ t,\ v)dg(u,\ v)\right|dsdt\right)\ .$$

The last integral in (2.9) is clearly $O(1)$ or $o(\log m \log n)$. That the second and third integrals are also $o(\log m \log n)$ follows from the following.

**LEMMA 2.10.** *If $h(m,\ n;\ s,\ t,\ u,\ v)$ is uniformly bounded in each argument, and if $g(u,\ v)$ is of bounded variation in the sense of Hardy* *and Krause, then*

$$\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1} h(m,\ n;\ s,\ t,\ u,\ s)\ \frac{\sin msu}{s}\ dg(u,\ v)\right| dsdt = o(\log m \log n)\ .$$

Proof.

$$\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1} h(m,\ n;\ s,\ t,\ u,\ v)\ \frac{\sin msu}{s}\ dg(u,\ v)\right| dsdt$$

$$\leq \int_{0,0}^{\pi,\pi}\int_{0,0}^{1,1} |h(m,\ n;\ s,\ t,\ u,\ v)|\left|\frac{\sin msu}{s}\right||dg(u,\ v)|dsdt$$

$$= O(1)\int_{0}^{\pi}\int_{0,0}^{1,1}\left|\frac{\sin msu}{s}\right||dg(u,\ v)|ds$$

$$= O(1)\left\{\int_{0}^{1} + \int_{1}^{\pi m}\right\}\int_{0,0}^{1,1}\left|\frac{\sin su}{s}\right||dg(u,\ v)|ds$$

$$= O(1)\{1 + \log \pi + \log m\}V(g)$$

$$= O(\log m) = o(\log m \log n)\ .$$

Thus the second and third integrals in (2.9) are also $o(\log m \log n)$ . We state the result in a theorem.

THEOREM 2.11. *If* $L(m,\ n;\ g)$ *denotes the* $mn$*th Lebesgue constant for the Hausdorff means corresponding to the regular weight function* $g(u,\ v)$ *, then*

$$(2.12)\quad L(m,\ n;\ g) = \frac{4}{\pi^2}\int_{0,0}^{\pi,\pi}\left|\int_{0,0}^{1,1} \rho_1^m\rho_2^n\ \frac{\sin msu}{s}\ \frac{\sin ntv}{t}\ dg(u,\ v)\right| dsdt$$
$$+ o(\log m \log n)\ ,\quad m,\ n \to \infty\ ,$$

*where* $\rho_1^2 = 1 + 2u(1-u)(1 - \cos s)$ *and* $\rho_2^2 = 1 - 2v(1-v)(1 - \cos t)$ .

## 3. Proof of Theorem 1.3

For $a < c$ and $b < d$ , let $[c,\ d;\ a,\ b]$ denote the rectangular cell with vertices at $(a,\ b)$, $(a,\ d)$, $(c,\ d)$ and $(c,\ b)$ , and let $g(c,\ d;\ a,\ b) = g(c,\ d) - g(a,\ d) - g(c,\ b) + g(a,\ b)$ . Thus the integral in Theorem 2.11 is over the cell $[\pi,\ \pi;\ 0,\ 0]$ with respect to $s$ and $t$ . Divide this cell into four subcells, $[m^{-\frac{1}{2}},\ n^{-\frac{1}{2}};\ 0,\ 0]$ , $[m^{-\frac{1}{2}},\ \pi;\ 0,\ n^{-\frac{1}{2}}]$ , $[\pi,\ n^{-\frac{1}{2}};\ m^{-\frac{1}{2}},\ 0]$ and $[\pi,\ \pi;\ m^{-\frac{1}{2}},\ n^{-\frac{1}{2}}]$ , and denote

https://doi.org/10.1017/S000497270000469X Published online by Cambridge University Press

this integral over the respective subcells by $I_1$, $I_2$, $I_3$ and $I_4$ .

**LEMMA 3.1.**

$$I_1 = \int_{1,1}^{n^{\frac{1}{2}},m^{\frac{1}{2}}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} dg(u, v)\right| dsdt + o(\log m \log n) .$$

Proof. By (2.4),

$$\rho_1^2 = 1 - 2u(1-u)(1 - \cos s)$$
$$= 1 - 4u(1-u)\sin^2(s/2)$$
$$\geq \cos^2(s/2) .$$

Thus in the cell $[m^{-\frac{1}{2}}, n^{-\frac{1}{2}}; 0, 0]$ , $1 - m(s/2)^2 \leq \cos^m(s/2) \leq \rho_1^m \leq 1$ . Set $\rho_1^m = 1 + \phi_m(s, u)$ where $|\phi_m(s, u)| \leq m(s/2)^2$ . Similarly, $\rho_2^n = 1 + \phi_n(t, v)$ where $|\phi_n(t, v)| \leq n(t/2)^2$ . Replacing $\rho_1^m$ and $\rho_2^n$ in the integral in (2.12) by these equivalents, it follows that

(3.2)

$$I_1 = \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v)\right| dsdt + O(1) \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}}$$
$$\left|\int_{0,0}^{1,1} \{\phi_m(s, u)+\phi_n(t, v)+\phi_m(s, u)\phi_n(t, v)\} \frac{\sin msu}{s} \frac{\sin ntv}{t} dt(u, v)\right| dsdt .$$

Now $|\phi_m(s, u)(\sin msu)/s| \leq |\phi_m(s, u)/s| \leq ms/4$ . Hence

https://doi.org/10.1017/S000497270000469X Published online by Cambridge University Press

$$(3.3)\quad \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \phi_m(s, u) \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v)\right| dsdt$$

$$\le \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}} (ms/4) \int_{0,0}^{1,1} \left|\frac{\sin ntv}{t}\right| |dg(u, v)| dsdt$$

$$= (1/8)\left\{\int_0^1 + \int_1^{n^{\frac{1}{2}}}\right\} \int_{0,0}^{1,1} \left|\frac{\sin tv}{t}\right| |dg(u, v)| dt$$

$$< (1/8)\left\{1 + \int_1^{n^{\frac{1}{2}}} \frac{dt}{t}\right\} \int_{0,0}^{1,1} |dg(u, v)|$$

$$= (1/8)\{1 + \tfrac{1}{2} \log n\} V(g)$$

$$= o(\log m \log n) ,$$

since $|(\sin tv)/t| \le |(\sin t)/t| \le 1$ in $(0, 1]$ and $|(\sin tv)/t| \le (1/t)$ , $1 \le t$ . Similarly,

$$(3.4)\quad \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \phi_n(t, v) \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v)\right| dsdt = o(\log m \log n)$$

and

$$(3.5)\quad \int_{0,0}^{n^{-\frac{1}{2}},m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \phi_m(s, u)\phi_n(t, v) \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v)\right| dsdt$$

$$= o(1) = 1(\log m \log n) .$$

Thus by (3.3), (3.4) and (3.5), the last integral on the right in (3.2) is $o(\log m \log n)$ .

Next, replace $ms$ by $s$ and $nt$ by $t$ in the remaining integral in (3.2) and obtain

$$(3.6)\quad I_1 = \int_{0,0}^{n^{\frac{1}{2}},m^{\frac{1}{2}}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} dg(u, v)\right| dsdt + o(\log m \log n) .$$

Finally, shifting the lower limit on the outer integral in (3.6) from $(0, 0)$ to $(1, 1)$ introduces an error of

$$\left\{\int_{0,0}^{1,1} + \int_{0,1}^{1,m^{\frac{1}{2}}} + \int_{1,0}^{n^{\frac{1}{2}},1}\right\} \left|\int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} dg(u, v)\right| dsdt .$$

The first integral is clearly $O(1)$ since $(\sin su)/s$ and $(\sin tv)/t$ are bounded in the region of integration. The second and third integrals are $O(\log m)$ and $O(\log n)$ respectively, and their sum is $o(\log m \log n)$. Collecting the results, it follows that

$$I_1 = \int_{1,1}^{m^{\frac{1}{2}},m^{\frac{1}{2}}} \left|\int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t}\, dg(u, v)\right| ds dt + o(\log m \log n)$$

and the lemma is proved.

**LEMMA 3.7.**

$$I_2 = (1/\pi) \log n \int_1^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds + o(\log m \log n) ,$$

$$I_3 = (1/\pi) \log m \int_1^{n^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin tv}{t} \{dg(1, v)-dg(1^-, v)\}\right| dt + o(\log m \log n) .$$

Proof. We will prove the first part of the lemma. The second part follows in a similar manner.

Proceeding as in the proof of Lemma 3.1 with $\rho_1^m = 1 + \phi_m(s, u)$ and $|\phi_m(s, u)/s| < ms/4$ ,

$$(3.8) \quad I_2 = \int_{n^{-\frac{1}{2}},0}^{\pi,m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t}\, dg(u, v)\right| ds dt$$

$$= \int_{n^{-\frac{1}{2}},0}^{\pi,m^{-\frac{1}{2}}} \left|\int_{0,0}^{1,1} \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t}\, dg(u, v)\right| ds dt + o(\log m \log n)$$

since

$$\int_{n^{-\frac12},0}^{\pi,m^{-\frac12}} \left|\int_{0,0}^{1,1} \rho_2^n \phi_m(s, u) \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v)\right| dsdt$$

$$< \int_{n^{-\frac12},0}^{\pi,m^{-\frac12}} (ms)/(4t) \int_{0,0}^{1,1} |dg(u, v)| dsdt$$

$$= (1/8)V(g) \log \pi n^{\frac12}$$

$$= o(\log m \log n) .$$

Next set

$$\int_{0,0}^{1,1} \rho_2^n \sin msu \sin ntvdg(u, v)$$

$$= \left\{\int_{0,0}^{0^+,1} + \int_{0^+,0}^{1^-,1} + \int_{1^-,0}^{1,1}\right\} \rho_2^n \sin msu \sin ntvdg(u, v)$$

$$= i_1 + i_2 + i_3 .$$

Since $g(u, v)$ is a regular weight function, $g(u, 0) = g(u, 0^+)$ , and so

$$|i_1| \le \int_{0,0}^{0^+,1} |dg(u, v)| = \int_0^1 |dg(u, 0^+)-dg(u, 0)| = 0 .$$

To estimate $i_2$ , note that here $0 < t \le \pi$ and $0 < v < 1$ so that, by (2.6),

$$\rho_2^2 < e^{-2v(1-v)t^2/\pi^2} .$$

Hence

$$\rho_2^n < e^{-nv(1-v)t^2/\pi^2}$$

and so $\rho_2^n$ tends to zero boundedly as $n \to \infty$ . Hence

$$i_2 = \int_{0^+,0}^{1^-,1} \rho_2^n \sin msu \sin ntvdg(u, v)$$

$$= o(1) \int_{0^+,0}^{1^-,1} \sin msu \sin ntvdg(u, v) .$$

Finally,

$$i_3 = \int_{1^-,0}^{1,1} \rho_2^n \sin msu \sin ntv dg(u, v)$$
$$= \int_0^1 \sin msu \sin nt\{dg(u, 1)-d(u, 1^-)\}$$

since $\rho_2^n = 1$ when $v = 1$. Collecting these estimates and replacing $ms$ and $nt$ by $s$ and $t$ respectively, it follows that

$$(3.9)\quad I_2 = \int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} |i_1+i_2+i_3| \frac{ds}{s}\frac{dt}{t}$$
$$= \int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s}\frac{\sin t}{t}\{dg(u, 1)-dg(u, 1^-)\}\right| dsdt$$
$$+ \int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} \left\{O(1)\int_{0^+,0}^{1^-,1} \frac{\sin su}{s}\frac{\sin tv}{t} dg(u, v)\right\} dsdt$$
$$= \int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s}\frac{\sin t}{t}\{dg(u, 1)-dg(u, 1^-)\}\right| dsdt$$
$$+ O(\log m \log n)$$

since

$$\int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} \left\{O(1)\int_{0^+,0}^{1^-,1} \frac{\sin su}{s}\frac{\sin tv}{t} dg(u, v)\right\} dsdt$$
$$\le O(1)\int_{n^{\frac{1}{2}}}^{\pi n} \frac{dt}{t}\left\{\int_0^1 ds + \int_1^{m^{\frac{1}{2}}} \frac{ds}{s}\right\}\int_{0^+,0}^{1^-,1} |dg(u, v)|$$
$$= O(1) \log \pi n^{\frac{1}{2}}(1 + \tfrac{1}{2}\log m)V(g)$$
$$= O(\log m \log n) ,$$

where we have used the observation that for $0 < s \le 1$, $|(\sin su)/s| \le 1$ and for $1 \le s, t$, $|(\sin su)/s| < 1/s$ and $|(\sin tv)/t| < 1/t$.

Returning to the last integral in (3.9), note that

$$\int_1^\infty \left\{|\sin t| - \frac{2}{\pi}\right\} \frac{dt}{t}$$

converges, and so

(3.10)

$$\int_{n^{\frac{1}{2}},0}^{\pi n, m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \frac{\sin t}{t} \{dg(u, 1)-dg(u, 1^-)\}\right| dsdt$$

$$= \int_{n^{\frac{1}{2}}}^{\pi n} \{\{|\sin t|-(2/\pi)\}+(2/\pi)\} \frac{dt}{t} \int_0^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds$$

$$= \{o(1)+(2/\pi) \log \pi+(1/\pi) \log n\} \int_0^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds$$

$$= (1/\pi) \log n \int_0^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds + o(\log m \log n)$$

since

$$\int_0^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds = O(\log m) = o(\log m \log n) .$$

Finally, shifting the lower limit of integration with respect to $s$ from 0 to 1 in the last integral in (3.10) introduces an error of

$$(1/\pi) \log n \int_0^1 \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds$$

$$< (1/\pi) \log n \int_0^1 ds \int_0^1 |dg(u, 1)-dg(u, 1^-)|$$

$$\leq (1/\pi)V(g) \log n$$

$$= o(\log m \log n) .$$

Hence

$$I_2 = (1/\pi) \log n \int_1^{m^{\frac{1}{2}}} \left|\int_0^1 \frac{\sin su}{s} \{dg(u, 1)-dg(u, 1^-)\}\right| ds + o(\log m \log n)$$

and the lemma is proved.

**LEMMA 3.11.**

$$I_4 = (1/\pi^2)\,|g(1,\ 1;\ 1^-,\ 1^-)|\ \log m \log n + o(\log m \log n)\ .$$

*Proof.* We have

$$I_4 = \int_{n^{-\frac{1}{2}},m^{-\frac{1}{2}}}^{\pi,\pi} \left|\int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t}\, dg(u,\ v)\right| dsdt\ .$$

As in the proof of Lemma 3.7, let

$$\int_{0,0}^{1,1} \rho_1^m \rho_2^n \sin msu \sin ntv dg(u,\ v)$$

$$= \left\{\int_{0,0}^{0^+,1} + \int_{0^+,0}^{1^-,1} + \int_{1^-,0}^{1,0^+} + \int_{1^-,1^-}^{1,1}\right\} \rho_1^m \rho_2^n \sin msu \sin ntv dg(u,\ v)$$

$$= i_1 + i_2 + i_3 + i_4 + i_5\ .$$

Since $g(u,\ 0^+) = g(u,\ 0)$ ,

$$|i_1| \le \int_{0,0}^{0^+,1} |dg(u,\ v)| = \int_0^1 |dg(u,\ 0^+) - dg(u,\ 1)| = 0\ .$$

By the same argument, $i_3 = 0$ .

To estimate $i_2$ , for $t > 0$ and $0 < v < 1$ , $\rho_2^n \to 0$ , and so $\rho_1^m \rho_2^n \sin msu \sin ntv \to 0$ boundedly as $n \to \infty$ . Hence

$$i_2 = \int_{0^+,0}^{1^-,1} \rho_1^m \rho_2^n \sin msu \sin ntv dg(u,\ v)$$

$$= o(1)\ .$$

By a similar argument, $i_4 = o(1)$ . Finally,

$$i_5 = \int_{1^-,1^-}^{1,1} \rho_1^m \rho_2^n \sin msu \sin ntv dg(u,\ v)$$

$$= g(1,\ 1;\ 1^-,\ 1^-) \sin ms \sin nt\ .$$

Collecting the results,

$$(3.12)\quad I_4 = \int_{n^{-\frac12},m^{-\frac12}}^{\pi,\pi} |i_1+i_2+i_3+i_4+i_5|\,\frac{ds}{s}\,\frac{dt}{t}$$

$$= |g(1,\,1;\,1^-,\,1^-)|\int_{n^{-\frac12},m^{-\frac12}}^{\pi,\pi}\left|\frac{\sin ms}{s}\,\frac{\sin nt}{t}\right|dsdt + \int_{n^{-\frac12},m^{-\frac12}}^{\pi,\pi} o(1)\,\frac{ds}{s}\,\frac{dt}{t}$$

$$= |g(1,\,1;\,1^-,\,1^-)|\int_{n^{\frac12},m^{\frac12}}^{\pi n,\pi m}\left|\frac{\sin s}{s}\,\frac{\sin t}{t}\right|dsdt + o(\log m\log n)\ .$$

Next, as in (3.10),

$$\int_{n^{\frac12}}^{\pi n}\left|\frac{\sin t}{t}\right|dt = O(1) + (1/\pi)\log n$$

and similarly

$$\int_{m^{\frac12}}^{\pi m}\left|\frac{\sin s}{s}\right|ds = O(1) + (1/\pi)\log m\ .$$

Applying these estimates to (3.12), we get

$$I_4 = |g(1,\,1;\,1^-,\,1^-)|\{O(1)+(1/\pi)\log m\}\{O(1)+(1/\pi)\log n\} + o(\log m\log n)$$

$$= \left(1/\pi^2\right)|g(1,\,1;\,1^-,\,1^-)|\log m\log n + O(1) + O(\log m) + O(\log n) + o(\log m\log n)$$

$$= \left(1/\pi^2\right)|g(1,\,1;\,1^-,\,1^-)|\log m\log n + o(\log m\log n)$$

and the lemma is proved.

To complete the proof of Theorem 1.3, we collect the results of Lemmas 3.1, 3.7 and 3.11, and get, by Theorem 2.11,

https://doi.org/10.1017/S000497270000469X Published online by Cambridge University Press

$$(\pi^2/4)L(m, n; g) = \int_{0,0}^{\pi,\pi} \left| \int_{0,0}^{1,1} \rho_1^m \rho_2^n \frac{\sin msu}{s} \frac{\sin ntv}{t} dg(u, v) \right| dsdt$$

$$= \int_{1,1}^{n\ ,m} \left| \int_{0,0}^{1,1} \frac{\sin su}{s} \frac{\sin tv}{t} dg(u, v) \right| dsdt$$

$$+ (1/\pi) \log m \int_1^n \left| \int_0^1 \frac{\sin tv}{t} \{dg(1, v) - dg(1^-, v)\} \right| dt$$

$$+ (1/\pi) \log n \int_1^m \left| \int_0^1 \frac{\sin su}{s} \{dg(u, 1) - dg(u, 1^-)\} \right| ds$$

$$+ \left(1/\pi^2\right) |g(1, 1; 1^-, 1^-)| \log m \log n + o(\log m \log n) .$$

Multiplying both sides by $\left(4/\pi^2\right)$ completes the proof of the main part of the theorem. The rest is obvious.

## References

[1] C.R. Adams, "Hausdorff transformations for double sequences", *Bull. Amer. Math. Soc.* **39** (1933), 303-312.

[2] C.R. Adams, "On definitions of bounded variation for functions of two variables", *Trans. Amer. Math. Soc.* **35** (1933), 824-854.

[3] A.E. Livingston, "The Lebesgue constants for Euler $(E, p)$ summation of Fourier series", *Duke Math. J.* **21** (1954), 309-314.

[4] L. Lorch and D.J. Newman, "The Lebesgue constants for regular Hausdorff methods", *Canad. J. Math.* **13** (1961), 283-298.

[5] O. Szász, "Gibbs phenomenon for Hausdorff means", *Trans. Amer. Math. Soc.* **28** (1926), 50-73.

[6] F. Ustina, "Gibbs phenomenon for the Hausdorff means of double sequences", *J. Austral. Math. Soc.* **11** (1970), 169-185.

Department of Mathematics,
University of Alberta,
Edmonton,
Canada T6G 2G1.