Bull. Aust. Math. Soc. **89** (2014), 149–158
doi:10.1017/S0004972713000439

THE p-HARMONIC BOUNDARY AND D_p-MASSIVE SUBSETS OF A GRAPH OF BOUNDED DEGREE

MICHAEL J. PULS

(Received 3 December 2012; accepted 19 March 2013; first published online 12 June 2013)

Abstract

Let p be a real number greater than one and let Γ be a graph of bounded degree. We investigate links between the p-harmonic boundary of Γ and the D_p-massive subsets of Γ. In particular, if there are n pairwise disjoint D_p-massive subsets of Γ, then the p-harmonic boundary of Γ consists of at least n elements. We show that the converse of this statement is also true.

2010 *Mathematics subject classification*: primary 31C20; secondary 05C38, 31C45, 60J50.

Keywords and phrases: p-harmonic boundary, D_p-massive set, p-harmonic function, asymptotically constant functions, extreme points of a path.

1. Introduction

Throughout this paper p will always denote a real number greater than one. A graph is said to have the *p-Liouville property* if every bounded p-harmonic function on the graph is constant. Similarly, a graph is said to have the *D_p-Liouville property* if every bounded p-harmonic function on the graph with finite p-Dirichlet sum is constant. When a graph has the p-Liouville property (D_p-Liouville property), the set of bounded p-harmonic functions (with finite p-Dirichlet sum) can be identified with $\mathbb{R}$, the real numbers. Now let G be a finitely generated group. Our main motivation for studying the p-harmonic boundary of a graph arose from the problem of determining the first reduced ℓ^p-cohomology space of G. A locally finite graph with bounded degree, called the Cayley graph of G, can be associated with G. Thus it makes sense to define the p-harmonic boundary for G, and to say that G has the p-Liouville property (D_p-Liouville property). It turns out that the first reduced ℓ^p-cohomology space of G vanishes if and only if G has the D_p-Liouville property if and only if the p-harmonic boundary of G consists of one point or is empty. A more complete discussion about this characterisation can be found in [11] and the references therein. Another reason for studying locally finite graphs with bounded degree is their intimate connection via discrete approximation to complete Riemannian manifolds with bounded geometry.

The research for this paper was partially supported by PSC-CUNY grant 63873-00 41.

The papers [2, 4, 5] contain a wealth of information concerning this link between graphs and manifolds.

Recently, a generalised version of the D_p-Liouville property for graphs has been studied in [8, 9]: specifically, the conditions on a graph under which the bounded p-harmonic functions with finite p-Dirichlet sum can be identified with $\mathbb{R}^n$, $n \in \mathbb{N}$. When $n \geq 2$, this also means that there are nonconstant p-harmonic functions on the graph. Holopainen and Soardi proved in [2, Lemma 5.7] that there is a nonconstant bounded p-harmonic function with finite p-Dirichlet sum on a graph of bounded degree if and only if there exist two disjoint D_p-massive subsets of vertices of the graph.

The purpose of this paper is to bring into sharper focus this connection between D_p-massive subsets and nonconstant p-harmonic functions on a graph. As a consequence, we are able to determine exactly when the set of bounded p-harmonic functions on a graph with finite p-Dirichlet sum can be identified with $\mathbb{R}^n$. The main tool we use to obtain our results is the p-harmonic boundary of a graph.

The p-harmonic boundary is a subset of the p-Royden boundary. When $p = 2$ these sets are respectively known as the harmonic boundary and the Royden boundary. In [15, Ch. 6] the Royden and harmonic boundaries were studied for locally finite graphs of bounded degree. Many of the results in [15, Ch. 6] were translated from corresponding results on complete Riemannian surfaces. See [13, Ch. 3] for information about the Royden and harmonic boundaries in the setting of complete Riemannian surfaces. However, there are some major differences between these two cases. In [15, Example 6.27] it was shown that the Royden boundary and the harmonic boundary coincide for a locally finite graph of bounded degree that satisfies a strong isoperimetric inequality. This is in stark contrast with the complete Riemannian surface case. More precisely, if the harmonic boundary is removed from the Royden boundary of a complete Riemannian surface, then the resulting set is dense in the Royden boundary! See [13, page 157] for the details of this fact. Furthermore, if the graph is a k-regular tree, $k \geq 3$, then there are no isolated points in the harmonic boundary of the tree [15, page 145].

The problem of explicitly computing the p-harmonic boundary of a locally finite graph of bounded degree appears to be quite difficult. The only result we can find in this direction is in the paper [16] where it is shown that the Royden boundary of a 2-regular tree, which can be considered as a Cayley graph for the integers, is a quotient space of $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$. In [11, Ch. 7] the author gave some examples of finitely generated groups whose p-harmonic boundary is empty or contains exactly one point by using the fact that the first reduced ℓ^p-cohomology of those particular groups is zero.

In Section 2 we define the main concepts used in this paper. We also state our main result. Section 3 is devoted to the proof of the main result. We explain in Section 4 how our result extends the main result of [9].

2. Definitions and statement of main result

Let Γ be a graph with vertex set V_Γ and edge set E_Γ. We will write V for V_Γ and E for E_Γ. For $x \in V$, N_x will be the set of neighbours of x and $\deg(x)$ will denote the number of neighbours of x. We shall say that Γ is of *bounded degree* if there exists a positive integer k for which $\deg(x) \le k$ for every $x \in V$. A path γ in Γ is a sequence of vertices $x_1, x_2, \ldots, x_n, \ldots$ where $x_{i+1} \in N_{x_i}$ for $1 \le i \le n-1$ and $x_i \ne x_j$ if $i \ne j$. Note that all paths considered in this paper have no self-intersections. A graph is *connected* if any two distinct vertices of the graph are joined by a path. All graphs considered in this paper will be connected, of bounded degree with no self-loops and have countably infinite number of vertices. By assigning length one to each edge of Γ, V becomes a metric space with respect to the shortest path metric. We will denote this metric by $d(x, y)$, where $x, y \in V$. Thus $d(x, y)$ gives the length of the shortest path joining the vertices x and y. For $S \subseteq V$, the outer boundary ∂S of S is the set of vertices in $V \setminus S$ with at least one neighbour in S, and $|S|$ will denote the cardinality of S. We use 1_V to represent the function that takes the value 1 on all elements of V. Finally, if $x \in V$ and $n \in \mathbb{N}$, the natural numbers, then $B_n(x)$ will denote the metric ball that contains all elements of V that have distance less than n from x.

We now proceed to define some function spaces that will be used in this paper. Let $S \subseteq V$ and let f be a real-valued function on $S \cup \partial S$. We define the pth power of the *gradient*, the *p-Dirichlet sum*, and the *p-Laplacian* of $x \in S$ by

$$|Df(x)|^p = \sum_{y \in N_x} |f(y) - f(x)|^p,$$

$$I_p(f, S) = \sum_{x \in S} |Df(x)|^p,$$

$$\Delta_p f(x) = \sum_{y \in N_x} |f(y) - f(x)|^{p-2}(f(y) - f(x)).$$

In the case $1 < p < 2$, we make the convention that $|f(y) - f(x)|^{p-2}(f(y) - f(x)) = 0$ if $f(y) = f(x)$. A function f is said to be *p-harmonic* on S if $\Delta_p f(x) = 0$ for all $x \in S$. Observe that a function which is p-harmonic on S is also defined on ∂S. We now give an alternative definition that is commonly used for a function to be p-harmonic on S when S is a finite (compact) subset of V. We begin by setting

$$\Xi(f, S) = \frac{1}{2}\Big(I_p(f, S) + \sum_{x \in \partial S} \sum_{y \in N_x \cap S} |f(x) - f(y)|^p\Big).$$

A function f is said to be p-harmonic on S if it is the minimiser of Ξ among the functions in $S \cup \partial S$ with the same value in ∂S as f, that is , if

$$\Xi(f, S) \le \Xi(u, S)$$

for every function u in $S \cup \partial S$ with $f = u$ in ∂S. The interested reader can find more information about p-harmonic functions and harmonic functions on graphs in the papers [1–3, 6, 8, 9, 11, 14, 15, 17] and the references therein.

We shall say that f is *p-Dirichlet finite* if $I_p(f, V) < \infty$. The set of all p-Dirichlet finite functions on Γ will be denoted by $D_p(\Gamma)$. This is a reflexive Banach space with respect to the norm

$$\|f\|_{D_p} = (I_p(f, V) + |f(o)|^p)^{1/p},$$

where o is a fixed vertex of Γ and $f \in D_p(\Gamma)$. We use $HD_p(\Gamma)$ to represent the set of p-harmonic functions on V that are contained in $D_p(\Gamma)$. Note that the constant functions are members of $HD_p(\Gamma)$. Let $\ell^\infty(\Gamma)$ denote the set of bounded functions on V and let $\|f\|_\infty = \sup_V |f|$ for $f \in \ell^\infty(\Gamma)$. Set $BD_p(\Gamma) = D_p(\Gamma) \cap \ell^\infty(\Gamma)$. The set $BD_p(\Gamma)$ is a Banach space under the norm

$$\|f\|_{BD_p} = (I_p(f, V))^{1/p} + \|f\|_\infty,$$

where $f \in BD_p(\Gamma)$. Let $BHD_p(\Gamma)$ be the set of bounded p-harmonic functions contained in $D_p(\Gamma)$. The space $BD_p(\Gamma)$ is closed under the usual operations of scalar multiplication, addition and pointwise multiplication. Furthermore, for $f, g \in BD_p(\Gamma)$ we have that $\|fg\|_{BD_p} \le \|f\|_{BD_p}\|g\|_{BD_p}$. Thus $BD_p(\Gamma)$ is a commutative Banach algebra. Let $C_c(\Gamma)$ be the set of functions on V with finite support. Denote the closure of $C_c(\Gamma)$ in $D_p(\Gamma)$ by $\overline{C_c(\Gamma)}_{D_p}$. Set $B(\overline{C_c(\Gamma)}_{D_p}) = \overline{C_c(\Gamma)}_{D_p} \cap \ell^\infty(\Gamma)$. Using the fact that the inequality $(a + b)^{1/p} \le a^{1/p} + b^{1/p}$ is true when $a, b \ge 0$ and $1 < p \in \mathbb{R}$, we see immediately that $\|f\|_{D_p} \le \|f\|_{BD_p}$. Consequently, $B(\overline{C_c(\Gamma)}_{D_p})$ is closed in $BD_p(\Gamma)$.

2.1. The *p*-harmonic boundary. In this subsection we construct the p-harmonic boundary of a graph Γ. For a more detailed discussion about this construction, see [11, Section 2.1]. Let $Sp(BD_p(\Gamma))$ denote the set of complex-valued characters on $BD_p(\Gamma)$, that is, the nonzero ring homomorphisms from $BD_p(\Gamma)$ to $\mathbb{C}$. We will implicitly use the following property of elements in $Sp(BD_p(\Gamma))$ throughout the paper.

LEMMA 2.1. *Let $\chi \in Sp(BD_p(\Gamma))$. If $f \in BD_p(\Gamma)$, then $\chi(f)$ is a real number.*

PROOF. Suppose that there exists an $f \in BD_p(\Gamma)$ for which $\chi(f) = a + bi$, where $b \neq 0$. Set $F = (f - a)/b$ and observe that $\chi(F) = i$. Since $BD_p(\Gamma)$ is a Banach algebra, F, F^2 and $F^2 + 1_V$ all belong to $BD_p(\Gamma)$. Also, $\chi(F^2 + 1_V) = 0$. For $x \in V$ and $y \in N_x$,

$$\left|\frac{1}{F^2(y) + 1_V} - \frac{1}{F^2(x) + 1_V}\right|^p \le |F^2(x) - F^2(y)|^p,$$

because $F^2 + 1_V \ge 1$ on V. It now follows that $(F^2 + 1_V)^{-1} \in BD_p(\Gamma)$, and so $F^2 + 1_V$ has a multiplicative inverse in $BD_p(\Gamma)$. Hence, $\chi(F^2 + 1_V) \neq 0$, a contradiction. Therefore, $\chi(f)$ is a real number. □

With respect to the weak $*$-topology, $Sp(BD_p(\Gamma))$ is a compact Hausdorff space. If $A \subseteq Sp(BD_p(\Gamma))$, $\overline{A}$ will indicate the closure of A in $Sp(BD_p(\Gamma))$. Given a topological space X, let $C(X)$ denote the ring of continuous functions on X endowed with the sup-norm. The Gelfand transform defined by $\hat{f}(\chi) = \chi(f)$ yields a monomorphism of Banach algebras from $BD_p(\Gamma)$ into $C(Sp(BD_p(\Gamma)))$ with dense image. Furthermore, the map $i : V \to Sp(BD_p(\Gamma))$ given by $(i(x))(f) = f(x)$ is an injection, and $i(V)$ is an open dense subset of $Sp(BD_p(\Gamma))$. For the rest of this paper we shall write f for $\hat{f}$, where $f \in BD_p(\Gamma)$. The *p-Royden boundary* of Γ, which we shall denote by $R_p(\Gamma)$, is the compact set $Sp(BD_p(\Gamma)) \setminus i(V)$. The *p-harmonic boundary* of Γ is the following subset of $R_p(\Gamma)$:

$$\partial_p(\Gamma) := \{\chi \in R_p(\Gamma) \mid \hat{f}(\chi) = 0 \text{ for all } f \in B(\overline{C_c(\Gamma)}_{D_p})\}.$$

We shall write $|\partial_p(\Gamma)|$ to indicate the cardinality of $\partial_p(\Gamma)$.

2.2. D_p-massive sets. We now define the concept of a D_p-massive subset of a graph. An infinite connected subset U of V with $\partial U \neq \emptyset$ is called a D_p-massive subset of V if there exists a nonnegative function $u \in BD_p(\Gamma)$ with the following properties:

(1) $\Delta_p u(x) = 0$ for $x \in U$;
(2) $u(x) = 0$ for $x \in \partial U$; and
(3) $\sup_{x \in U} u(x) = 1$.

We call any u that satisfies these conditions an *inner potential* of the D_p-massive subset U. The next result is [11, Proposition 4.11] and will be needed later.

Proposition 2.2. *If U is a D_p-massive subset of V, then $\overline{i(U)}$ contains at least one point of $\partial_p(\Gamma)$.*

2.3. Statement of the main result. We now give the main result of this paper.

Theorem 2.3. *Let $1 < p \in \mathbb{R}$ and let Γ be a graph of bounded degree. Suppose that $n \in \mathbb{N}$. Then there exist n pairwise disjoint D_p-massive subsets $D_1, D_2, \ldots, D_n$ of V if and only if $|\partial_p(\Gamma)| \geq n$.*

By combining this theorem with [11, Corollary 2.7] we obtain the following corollary.

Corollary 2.4. *Let $1 < p \in \mathbb{R}$, $n \in \mathbb{N}$ and let Γ be a graph of bounded degree. If there exist n pairwise disjoint D_p-massive subsets of V, but there do not exist $n + 1$ disjoint D_p-massive subsets of V, then $BHD_p(\Gamma)$ can be identified with $\mathbb{R}^n$.*

3. Proof of Theorem 2.3

The following lemma will be needed for the proof of Theorem 2.3. For a proof of the lemma see the first part of the proof of [2, Lemma 5.7]

Lemma 3.1. *Let h be a nonconstant function in $BHD_p(\Gamma)$, and let U be an infinite connected subset of V. Let a and b be real numbers such that*

$$\inf_{x \in U} h < a < b < \sup_{x \in U} h.$$

Then each component of the set $\{x \in U \mid h(x) > b\}$ and each component of $\{x \in U \mid h(x) < a\}$ is D_p-massive.

We are now ready to prove Theorem 2.3. Let $D_1, D_2, \ldots, D_n$ be a collection of pairwise disjoint D_p-massive subsets of V. For each k, with $1 \le k \le n$, let u_k be an inner potential for D_k. We may and do assume that $u_k = 0$ on $V \setminus D_k$. Also, $\overline{D_k} \cap \partial_p(\Gamma) \neq \emptyset$ by Proposition 2.2. For each k we will produce an element $\chi_k \in \overline{D_k} \cap \partial_p(\Gamma)$ for which $\chi_k(u_k) \neq 0$ and $\chi_k(u_j) = 0$ if $j \neq k$. This will establish $|\partial_p(\Gamma)| \ge n$. Extend u_k to a continuous function on $Sp(BD_p(\Gamma))$. By [11, Theorem 2.6] there exists a p-harmonic function h_k on V such that $h_k = u_k$ on $\partial_p(\Gamma)$. The maximum principle [11, Theorem 4.7] says that $0 < h_k < 1$ on V. Let $B_k = \{x \in D_k \mid h_k(x) > 1 - \epsilon\}$, where $0 < \epsilon < \frac{1}{4}$. Since $\sup u_k = 1$ on D_k, $B_k \neq \emptyset$. Let C_k be a component of B_k. By Lemma 3.1, C_k is D_p-massive. Thus $\overline{C_k} \cap \partial_p(\Gamma) \neq \emptyset$. Select $\chi_k \in \overline{C_k} \cap \partial_p(\Gamma)$. Because $h_j = u_j$ on $\partial_p(\Gamma)$, $\chi_k(h_j) = \chi_k(u_j)$. Consequently, $\chi_k(u_k) = 1$ and $\chi_k(u_j) = 0$ if $k \neq j$. Hence, $|\partial_p(\Gamma)| \ge n$ if there exist n pairwise disjoint D_p-massive subsets of V.

Conversely, let $\chi_1, \chi_2, \ldots, \chi_n$ be distinct elements from $\partial_p(\Gamma)$. By Urysohn's lemma there exists a continuous function $f_1 : Sp(BD_p(\Gamma)) \to [0, 1]$ with $f_1(\chi_1) = 1$ and $f_1(\chi_k) = 0$ if $k \neq 1$. Let $M_1 = f_1^{-1}(1)$. For each integer k with $2 \le k \le n$ we can inductively define a continuous function $f_k : Sp(BD_p(\Gamma)) \to [0, 1]$ with the following properties:

$$f_k(x) = \begin{cases} 1, & x = \chi_k, \\ 0, & x = \chi_i, i \neq k, \\ 0, & x \in \bigcup_{i=1}^{k-1} M_i, \end{cases}$$

where $M_k = f_k^{-1}(1)$.

By the density of $BD_p(\Gamma)$ in $C(Sp(BD_p(\Gamma)))$, we can assume that $f_k \in BD_p(\Gamma)$ for each k. Using [11, Theorems 4.6 and 4.8], we obtain a unique $h_k \in BHD_p(\Gamma)$ with $h_k = f_k$ on $\partial_p(\Gamma)$ for each k. Also, $0 < h_k < 1$ on V. Observe that if $h_k(\chi) = 1 = h_j(\chi)$ for some $\chi \in \partial_p(\Gamma)$, then $k = j$. Let $\epsilon > 0$ and consider the set $A_{k,\epsilon} = \{x \in V \mid h_k(x) > 1 - \epsilon\}$. For each k let $D_{k,\epsilon}$ be a component of $A_{k,\epsilon}$. Furthermore, choose the $D_{k,\epsilon}$ so that $D_{k,\epsilon_1} \subseteq D_{k,\epsilon_2}$ if $0 < \epsilon_1 < \epsilon_2$. Lemma 3.1 yields that $D_{k,\epsilon}$ is D_p-massive. The proof will be complete if there exists an $\epsilon > 0$ such that $D_{k,\epsilon} \cap D_{j,\epsilon} = \emptyset$ if $k \neq j$. Assume for the purposes of contradiction that this condition is not true. Then there exist j, k with $D_{k,\epsilon} \cap D_{j,\epsilon} \neq \emptyset$ for all $\epsilon > 0$. Let $i \in \mathbb{N}$. Denote by C_i a component of $D_{k,2^{-i}} \cap D_{j,2^{-i}}$. By the comparison principle [2, Theorem 3.14] C_i is infinite. Using Lemma 3.1, we can produce a D_p-massive subset of C_i. An appeal to Proposition 2.2 produces a $\psi_i \in \overline{C_i} \cap \partial_p(\Gamma)$. Clearly $\psi_i(h_j) > 1 - 2^{-i}$ and $\psi_i(h_k) > 1 - 2^{-i}$. The sequence (ψ_i) in $\partial_p(\Gamma)$ has a convergent subsequence that converges to some ψ in $\partial_p(\Gamma)$. Consequently, $\psi(h_k) = 1 = \psi(h_j)$. This contradicts our earlier observation that if $h_k(\chi) = 1 = h_j(\chi)$ for some $\chi \in \partial_p(\Gamma)$, then $k = j$. Therefore, there exists an $\epsilon > 0$ for which $D_{k,\epsilon} \cap D_{j,\epsilon} = \emptyset$ for each j, k with $1 \le j, k \le n$. The proof of the theorem is now complete.

4. A result of Kim and Lee

In this section we elaborate on how Theorem 2.3 improves the main result of [9]. We start by giving some needed definitions.

Recall that E represents the edge set of a graph Γ. Denote by $\mathcal{F}(E)$ the set of all real-valued functions on E and let $\mathcal{F}^+(E)$ be the subset of $\mathcal{F}(E)$ that consists of all nonnegative functions. For $f \in \mathcal{F}(E)$ set

$$\xi_p(f) = \sum_{e\in E} |f(e)|^p.$$

The edge set of a path γ in Γ will be denoted by $Ed(\gamma)$. Let Q be a set of paths with no self-intersections in Γ. Denote by $\mathcal{A}(Q)$ the set of all $f \in \mathcal{F}^+(E)$ that satisfy $\xi_p(f) < \infty$ and $\sum_{e\in Ed(\gamma)} f(e) \geq 1$ for all $\gamma \in Q$. The *extremal length* of order p for Q is defined by

$$\lambda_p(Q)^{-1} = \inf\{\xi_p(f) \mid f \in \mathcal{A}(Q)\}.$$

The number $\lambda_p(Q)^{-1}$ is commonly known as the *p-modulus* of the path family Q. We shall say that a property holds for *p-almost every path* in a collection of paths if the set of paths for which the property does not hold has infinite extremal length (or p-modulus zero).

Let $A \subseteq V$, and write Γ_A for the largest subgraph of Γ that has vertex set A. Let γ be a one-sided infinite path in Γ. For a real-valued function f on V, set $f(\gamma) = \lim_{n\to\infty} f(x_n)$ as $n \to \infty$ along the vertices of γ. Let P_A be the set of all one-sided infinite paths with no self-intersections contained in Γ_A. We define a real-valued function f to be *asymptotically constant* on A if there exists a constant c such that

$$f(\gamma) = c \text{ for } p\text{-almost every path } \gamma \in P_A.$$

We shall say that an infinite connected set U has *property AC* if each function in $BHD_p(\Gamma)$ is asymptotically constant on U.

An infinite connected subset S of V is said to be *p-hyperbolic* if there exists a nonempty finite subset A of V for which

$$\operatorname{Cap}_p(A, \infty, S) = \inf_u I_p(u, S) > 0,$$

where the infimum is taken over all finitely supported functions u on $S \cup \partial S$ such that $u = 1$ on A. If S is not p-hyperbolic, then it is said to be *p-parabolic*. The quantity $\operatorname{Cap}_p(A, \infty, S)$ is known as the *p-capacity* of S.

Motivated by [18, Theorem 3.1], Kim and Lee prove the following result in [9, Theorem 1.1].

Theorem 4.1. *Let $n \in \mathbb{N}$ and let Γ be a graph with n p-hyperbolic ends. Suppose that each p-hyperbolic end has property AC. Then given any real numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$, there exists a unique $h \in BHD_p(\Gamma)$ such that*

$$h(\gamma) = a_i \text{ for } p\text{-almost every path } \gamma \in P_{F_i}$$

for each $i = 1, 2, \ldots, n$, where $F_1, F_2, \ldots, F_n$ are the p-hyperbolic ends of Γ.

We see immediately that if a graph Γ satisfies the hypothesis of this theorem, then $BHD_p(\Gamma)$ can be identified with $\mathbb{R}^n$, which is the same conclusion as Corollary 2.4. However, the hypotheses of Theorem 4.1 are quite strong. The number of ends of a graph Γ is independent of p, and the AC property is also very restrictive. For example, let G denote a co-compact lattice in the real rank-one simple Lie groups $Sp(n, 1)$, $n \geq 2$. The Cayley graph of the group G has one end, but there are nonconstant p-harmonic functions with finite p-Dirichlet sum on G precisely when $p > 4n + 2$. See [10, Section 4] for the details.

When the cardinality of $\partial_p(\Gamma)$ is finite, Theorem 2.3 completely characterises the number of elements in $\partial_p(\Gamma)$ in terms of pairwise disjoint D_p-massive sets. It is the case that D_p-massive sets are also p-hyperbolic. The reason why we are able to drop the property AC assumption from Theorem 4.1 in our Theorem 2.3 is given in Proposition 4.3 below. Before we prove the proposition we need the following lemma.

Lemma 4.2. *Let Γ be a graph with bounded degree and let $1 < p \in \mathbb{R}$. Suppose that F is an infinite connected subset of V with property AC. For $h \in BHD_p(\Gamma)$, denote by c_h the constant for which $h(\gamma) = c_h$ for p-almost every path in P_F. If $\chi \in \overline{F} \cap \partial_p(\Gamma)$, then $\chi(h) = c_h$.*

Proof. Let $h \in BHD_p(\Gamma)$. Suppose that $c_h < \chi(h)$. Let $\epsilon > 0$ such that $c_h < \chi(h) - \epsilon$. Define $A = \{x \in F \mid h(x) > \chi(h) - \epsilon\}$ and let C be a component of A. Observe that $\lambda_p(P_C) = \infty$ due to $h(\gamma) > c_h$ for each $\gamma \in P_C$. By Lemma 3.1, C is D_p-massive. Now [12, Proposition 5.3] yields the contradiction $\lambda_p(P_C) < \infty$. A similar argument shows that it is also not the case that $\chi(h) < c_h$. Therefore $\chi(h) = c_h$. □

Denote by $V(\gamma)$ the vertex set of an infinite path γ in Γ. Write $\overline{V}(\gamma)$ for the closure of $i(V(\gamma))$ in $Sp(BD_p(\Gamma))$. The set of extreme points of γ is given by

$$Ex(\gamma) = \overline{V}(\gamma) \setminus i(V(\gamma)).$$

Proposition 4.3. *Let $1 < p \in \mathbb{R}$ and let Γ be a graph of bounded degree. Let F be a p-hyperbolic subset of V. Then F has property AC if and only if $|\overline{F} \cap \partial_p(\Gamma)| = 1$.*

Proof. Because F is p-hyperbolic, it is the case that $\lambda_p(P_F) < \infty$. From [12, Lemma 5.2] we get that $\overline{F} \cap \partial_p(\Gamma) \neq \emptyset$. Now suppose that χ_1 and χ_2 are distinct elements from $\overline{F} \cap \partial_p(\Gamma)$. Since $BD_p(\Gamma)$ separates points in $Sp(BD_p(\Gamma))$, there exists an $f \in BD_p(\Gamma)$ for which $\chi_1(f) \neq \chi_2(f)$. Combining [11, Theorems 4.6 and 4.8], we obtain an $h \in BHD_p(\Gamma)$ with the property $f = h$ on $\partial_p(\Gamma)$. Thus $\chi_1(h) \neq \chi_2(h)$, contradicting Lemma 4.2. Hence, $|\overline{F} \cap \partial_p(\Gamma)| = 1$.

Now assume that $|\overline{F} \cap \partial_p(\Gamma)| = 1$ and let χ be the unique element in $\overline{F} \cap \partial_p(\Gamma)$. Select an $h \in BHD_p(\Gamma)$ and let $c_h = \chi(h)$. We will now show that $h(\gamma) = c_h$ for p-almost every path in P_F. Denote by P_∞ the set of all $\gamma \in P_F$ for which $h(\gamma)$ does not exist. Let $\gamma = x_0 x_1 \ldots x_n \ldots \in P_\infty$. The identity $h(x_n) = h(x_0) - \sum_{k=1}^{n}(h(x_{k-1}) - h(x_k))$ implies that $\sum_{k=1}^{\infty} |h(x_{k-1}) - h(x_k)| = \infty$. It now follows [7, Lemma 2.3] that $\lambda_p(P_\infty) = \infty$.

For each $n \in \mathbb{N}$, set

$$P_{1/n} = \{\gamma \in P_F \setminus P_\infty \mid |h(\gamma) - c_h| > 1/n\}.$$

Now suppose that $\lambda_p(P_{1/n}) < \infty$ for some $n \in \mathbb{N}$. By [12, Lemma 5.2],

$$\left(\overline{\bigcup_{\gamma}\{Ex(\gamma) \mid \gamma \in P_{1/n}\}}\right) \cap \partial_p(\Gamma) \neq \emptyset.$$

Let ψ be an element in this intersection. The definition of $P_{1/n}$ implies that $\psi(h) \neq c_h$. Combining the fact that $P_{1/n} \subseteq P_F$ with the hypothesis $|\overline{F} \cap \partial_p(\Gamma)| = 1$ yields $\psi = \chi$, contradicting the fact that $\chi(h) = c_h$. Hence $\lambda_p(P_{1/n}) = \infty$ for all $n \in \mathbb{N}$. Let $P_U = \bigcup_{n=1}^{\infty} P_{1/n}$. According to [7, Lemma 2.2], $\lambda_p(P_U) = \infty$, and $\lambda_p(P_U \cup P_\infty) = \infty$. Let $P_h = \{\gamma \in P_F \mid h(\gamma) = c_h\}$. Then $P_F = P_h \cup P_U \cup P_\infty$. Another appeal to [7, Lemma 2.2] shows that $\lambda_p(P_h) < \infty$ since $\lambda_p(P_F) < \infty$. Thus $h(\gamma) = c_h$ for p-almost every path in P_F. Therefore, h is asymptotically constant on F. □

It follows immediately from this proposition that if a graph Γ satisfies the assumptions of Theorem 4.1, then $|\partial_p(\Gamma)| = n$.

Acknowledgement

I would like to thank the referee for several excellent suggestions that improved the exposition of this paper.

References

[1] A. Cantón, J. L. Fernández, D. Pestana and J. M. Rodríguez, 'On harmonic functions on trees', *Potential Anal.* **15**(3) (2001), 199–244.

[2] I. Holopainen and P. M. Soardi, 'p-harmonic functions on graphs and manifolds', *Manuscripta Math.* **94**(1) (1997), 95–110.

[3] I. Holopainen and P. M. Soardi, 'A strong Liouville theorem for p-harmonic functions on graphs', *Ann. Acad. Sci. Fenn. Math.* **22**(1) (1997), 205–226.

[4] M. Kanai, 'Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds', *J. Math. Soc. Japan* **37**(3) (1985), 391–413.

[5] M. Kanai, 'Rough isometries and the parabolicity of Riemannian manifolds', *J. Math. Soc. Japan* **38**(2) (1986), 227–238.

[6] R. Kaufman, J. G. Llorente and J.-M. Wu, 'Nonlinear harmonic measures on trees', *Ann. Acad. Sci. Fenn. Math.* **28**(2) (2003), 279–302.

[7] T. Kayano and M. Yamasaki, 'Boundary limit of discrete Dirichlet potentials', *Hiroshima Math. J.* **14**(2) (1984), 401–406.

[8] S. W. Kim and Y. H. Lee, 'Positive p-harmonic functions on graphs', *Bull. Korean Math. Soc.* **42**(2) (2005), 421–432.

[9] S. W. Kim and Y. H. Lee, 'Energy finite p-harmonic functions on graphs and rough isometries', *Commun. Korean Math. Soc.* **22**(2) (2007), 277–287.

[10] M. J. Puls, 'The first L^p-cohomology of some finitely generated groups and p-harmonic functions', *J. Funct. Anal.* **237**(2) (2006), 391–401.

[11] M. J. Puls, 'Graphs of bounded degree and the p-harmonic boundary', *Pacific J. Math.* **248**(2) (2010), 429–452.

[12] M. J. Puls, ‘Some results concerning the p-Royden and p-harmonic boundaries of a graph of bounded degree’, *Ann. Acad. Sci. Fenn. Math.* **37** (2012), 81–90.

[13] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Die Grundlehren der mathematischen Wissenschaften, 164 (Springer, New York, 1970).

[14] N. Shanmugalingam, ‘Some convergence results for p-harmonic functions on metric measure spaces’, *Proc. Lond. Math. Soc.* (3) **87**(1) (2003), 226–246.

[15] P. M. Soardi, *Potential theory on infinite networks*, Lecture Notes in Mathematics, 1590 (Springer, Berlin, 1994).

[16] J. Wysoczański, ‘Royden compactification of integers’, *Hiroshima Math. J.* **26**(3) (1996), 515–529.

[17] M. Yamasaki, ‘Parabolic and hyperbolic infinite networks’, *Hiroshima Math. J.* **7**(1) (1977), 135–146.

[18] M. Yamasaki, ‘Ideal boundary limit of discrete Dirichlet functions’, *Hiroshima Math. J.* **16**(2) (1986), 353–360.

MICHAEL J. PULS, Department of Mathematics, John Jay College-CUNY,
524 West 59th Street, New York, NY 10019, USA
e-mail: mpuls@jjay.cuny.edu