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Small doubling for discrete subsets of non-commutative groups and a theorem of Lagarias

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Abstract. Approximate lattices of Euclidean spaces, also known as Meyer sets, are aperiodic subsets with fascinating properties. In general, approximate lattices are defined as approximate subgroups of locally compact groups that are discrete and have finite co-volume. A theorem of Lagarias [Meyer's concept of quasicrystal and quasiregular sets. *Comm. Math. Phys.* **179**(2) (1996), 365–376] provides a criterion for discrete subsets of Euclidean spaces to be approximate lattices. It asserts that if a subset X of $\mathbb{R}^n$ is relatively dense and $X - X$ is uniformly discrete, then X is an approximate lattice. We prove two generalizations of Lagarias' theorem: when the ambient group is amenable and when it is a higher-rank simple algebraic group over a characteristic 0 local field. This is a natural counterpart to the recent structure results for approximate lattices in non-commutative locally compact groups. We also provide a reformulation in dynamical terms pertaining to return times of cross-sections. Our method relies on counting arguments involving the so-called periodization maps, ergodic theorems and a method of Tao regarding small doubling for finite subsets. In the case of simple algebraic groups over local fields, we moreover make use of deep superrigidity results due to Margulis and to Zimmer.

Key words: aperiodic order, approximate lattices, ergodic theory
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1. *Introduction*

A subset Λ of a locally compact group G is called an *approximate lattice* [**BH18**, **Hru20**] if it satisfies:

- (uniformly discrete) there is a neighbourhood W of the identity such that

$$\text{for every } \lambda_1 \neq \lambda_2 \in \Lambda, \quad \lambda_1 W \cap \lambda_2 W = \emptyset;$$

- (finite co-volume) there is $\mathcal{F} \subset G$ with finite Haar measure such that

$$\Lambda\mathcal{F} := \{\lambda f : \lambda \in \Lambda, f \in \mathcal{F}\} = G;$$

- (l-approximate subgroup) $e \in \Lambda$, $\Lambda = \Lambda^{-1}$ and there exists $F \subset G$ of size at most l such that

$$\Lambda^2 := \{\lambda_1\lambda_2 : \lambda_1, \lambda_2 \in \Lambda\} \subset F\Lambda := \{f\lambda : f \in F, \lambda \in \Lambda\}.$$

Ever since their introduction by Meyer in his seminal monograph [**Mey72**], approximate lattices of Euclidean spaces—and their more regular relatives the model sets—have been key objects of study in the theory of aperiodic order (see for instance [**Lag96**, **Mey72**, **Moo97**]). Their definition allowed to study in a common framework Penrose tilings (and their pentagrid description due de Bruijn [**dB81a**, **dB81b**]), Pisot–Vijayaraghavan numbers [**Mey72**], crystallographic subsets and certain mathematical models of quasi-crystals [**BG13**] or spectra of certain crystalline measures related to Poisson summation [**LO15**]. More recently, inspired by the breakthroughs in the study of non-commutative finite approximate subgroups (e.g. [**BGT12**, **Hel08**, **Hru12**]), the interest has grown concerning non-commutative approximate lattices. On the one hand, generalizations of Meyer's structure theorem [**Mey72**] have been extensively studied. This study was initiated by Björklund and Hartnick in [**BH18**] and generalizations to linear groups have been achieved through, notably, the breakthrough work of Hrushovski [**Hru20**] and the work of the author [**Mac20**, **Mac22**, **Mac23**]. On the other hand, approximate lattices have been used as instances of structured subsets with rich aperiodic behaviour in non-commutative frameworks (see for instance the monograph [**CHT24**] and recent works [**ER22**, **EV25**, **HK22**, **HW22**, **PRS22**]).

A result of Lagarias' [**Lag96**] fits naturally within that picture. It provides a geometrically significant condition for a discrete co-compact subset of a Euclidean space to be an approximate lattice (see [**LO15**, §5.2] for a short proof and a deep application). Lagarias' theorem asserts that if a subset $X \subset \mathbb{R}^n$ is such that $X + K = \mathbb{R}^n$ for some compact subset K and $X - X$ is uniformly discrete, then $X - X$ is an approximate lattice covered by finitely many translates of X. Our goal in this article is to provide generalizations of Lagarias' theorem that, in particular, hold in the non-commutative framework.

Our first theorem generalizes Lagarias' theorem to amenable groups.

THEOREM 1.1. *Let G be a unimodular amenable second countable locally compact group. Let $X \subset G$ be such that:*

(1) *$X^{-1}X$ is uniformly discrete that is $(X^{-1}X)^2 \cap V^{-1}V = \{e\}$ for some neighbourhood of the identity $V \subset G$;*

(2) *there is $\mathcal{F}$ Borel of finite Haar measure such that $\mathcal{F}X = G$.*

Then, there are an approximate lattice $\Lambda \subset XX^{-1}$ and a finite subset $F \subset G$ such that $X \subset \Lambda F$. In addition, F and Λ can be chosen so $|F| = O((\mu_G(\mathcal{F})/\mu_G(V))^3)$ and Λ is an $O(\mu_G(\mathcal{F})/\mu_G(V)^{12})$-approximate subgroup whose internal dimension (see §2.4) is bounded above by $O(\log(\mu_G(\mathcal{F})/\mu_G(V)))$.

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Qualitatively, Lagarias' theorem and Theorem 1.1 are instances of the following heuristic: 'small doubling' (that is, $X^{-1}X$ not much larger than X) implies 'approximate subgroup'. Here, if smallness is determined by how discrete a subset is, then Theorem 1.1 indeed tells us that a finite co-volume subset with small doubling is an approximate subgroup. Such heuristics are commonplace in additive combinatorics. The Plünnecke–Ruzsa theorem relates *quantitatively* small doubling for finite subsets—in terms of size—and approximate subgroups (see [**Pet12**, **Ruz99**], a non-commutative generalization due to Tao [**Tao08**], as well as the flattening lemma of Bourgain and Gamburd [**BG08**]). This analogy is anything but vain and, in fact, lies at the heart of our approach. Konieczny [**Kon23**] was the first to use this analogy to give a slick new proof, and indeed a quantitative improvement, of Lagarias' theorem in the case $G = \mathbb{R}^n$. However, the methods concerned with discrete subsets of Euclidean spaces do not generalize to amenable ambient groups, see §1.2 for a description of where it fails and how we circumvent this issue. Note finally that in Lagarias' original result, $X - X$ is already an approximate lattice. In contrast, one can check that in our results, the approximate lattice Λ can be taken equal to XX^{-1} if and only if the elements of X have open centralizers.

Our method shows furthermore that *finite co-volume* is better embodied by the following notion. Given $X \subset G$ closed, let Ω_X denote the closure—in the Chabauty space $\mathcal{C}(G)$ of closed subsets of G equipped with the Chabauty topology (§3.1)—of the orbit $G \cdot X$. We call Ω_X the *invariant hull* of X. Sometimes also called the tiling space or the continuous hull of X [**BG13**], Ω_X is a key object in the theory of aperiodic order that encapsulates the long-range structure of the (potentially) aperiodic subset X of G. We are able to prove the following generalization of Theorem 1.1.

THEOREM 1.2. *Let G be a unimodular amenable second countable locally compact group. Let $X \subset G$ be such that:*

(1) *$X^{-1}X$ is uniformly discrete, that is, $(X^{-1}X)^2 \cap V^{-1}V = \{e\}$ for some neighbourhood of the identity $V \subset G$;*

(2) *there is a proper G-invariant Borel probability measure ν on Ω_X.*

Then, for ν-almost every $Y \in \Omega_X$, there are an approximate lattice $\Lambda \subset YY^{-1}$ and a finite subset $F \subset G$ such that $Y \subset \Lambda F$. In addition, if μ_G denotes the Haar measure $\mathcal{P}_X^*\nu$ (§3.1), then F and Λ can be chosen so $|F| = O(\mu_G(V)^{-3})$, Λ is an $O(\mu_G(V)^{-12})$-approximate subgroup and Λ has internal dimension $O(\log \mu_G(V)^{-1})$.

Even in the case of an abelian ambient group, our results extend those of Lagarias [**Lag96**] as they apply more generally to subsets of finite co-volume rather than relatively dense subsets alone. Furthermore, examples of subsets satisfying the conditions of Theorem 1.2 but not those of Theorem 1.1 are plenty. A natural example of number-theoretic origin is the subset of $\mathbb{Z}^2$ made of pairs of co-prime integers—its co-volume is $6/\pi^2$ (note that, in this particular case, Theorem 1.2 can also be established by direct means).

We also take our investigation beyond the realm of amenable locally compact groups. We prove that a Lagarias-type theorem also holds in simple Lie groups, whose behaviour is the polar opposite of that of amenable groups.

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THEOREM 1.3. *Let k be a local field of characteristic* 0. *Let G be the group of k-points of a simple algebraic group* $\mathbb{G}$ *defined over k of k-rank at least* 2. *Let* $X \subset G$ *be such that:*
(1) XX^{-1} *and* $X^{-1}X$ *are uniformly discrete;*
(2) *there is a proper G-invariant Borel probability measure on* Ω_X.
Then, there is an approximate lattice $\Lambda \subset (X^{-1}X)^2$ *and a finite subset* $F \subset G$ *such that* $X \subset F\Lambda$.

An interesting example one may keep in mind is the case $G = \mathrm{PSL}_n(\mathbb{R})$, $n \geq 3$, as this case already contains all the difficulty of the proof. The proof of Theorem 1.3 shows that the subsets studied are extremely rigid and exhibit even more striking behaviour than in the amenable world. This reminisces the works of Margulis and Mozes [**MM98**], Mozes [**Moz97**], and Block and Weinberger [**BW92**] who take advantage of rigidity properties in symmetric spaces and non-amenable spaces to provide beautiful examples of aperiodic tilings. Theorem 1.3 also provides new information regarding the structure of approximate lattices in semi-simple groups and is an interesting counterpart to Hrushovski's [**Hru20**]. Compared with [**Hru20**], we are able to prove a structure theorem while assuming that a stronger 'finite co-volume' assumption holds *but*, as a trade-off, we do not assume that X is an approximate subgroup and assume only a weaker algebraic relation between XX^{-1}, $X^{-1}X$ and X.

1.1. *A dynamical reformulation and two questions.* The proof of Theorem 1.3 relies greatly on considering the restriction of the dynamics on the invariant hull to a naturally defined cross-section. Given a dynamical system (X, ν) equipped with a probability measure preserving ergodic action of a locally compact group G, we say that a Borel subset $B \subset X$ is a *cross-section* if for some neighbourhood of the identity $W \subset G$, the map $W \times B \to X$ defined by $(w, b) \mapsto wb$ is one-to-one and $\nu(GB) = 1$. For $x \in X$, we define the set of *hitting times* $T_h(x, B)$ as the subset $\{g \in G : gx \in B\}$. Since B is a cross-section, the subset $T_h(x, B)$ is uniformly discrete for every B.

It was shown by Björklund, Hartnick and Karasik in [**BHK25**] that the map

$$\begin{aligned} \phi : X &\longrightarrow \mathcal{C}(G) \\ x &\longmapsto T_h(x, B)^{-1} \end{aligned}$$

is a Borel G-equivariant map. There is moreover $X_0 \subset \mathcal{C}(G)$ such that ϕ takes values in Ω_{X_0} for ν-almost every $x \in X$. Set ν_0, the push-forward of ν via ϕ. Since cross-sections exist as soon as G acts freely, this produces an abundance of uniformly discrete subsets X_0 for which Ω_{X_0} admits a proper G-invariant Borel probability measure, see [**BHK25**] and references therein.

Using this language, Theorem 1.2 may be rephrased as follows.

THEOREM 1.4. *Let G be a unimodular amenable second countable locally compact group acting by probability measure preserving action on* (X, ν). *Suppose that* $B \subset G$ *is a cross-section of X and that the set of return times of B,*

$$\mathcal{R}(B) := \{g \in G : gB \cap B \neq \emptyset\} = \bigcup_{y \in B} T_{\mathrm{hit}}(y, B),$$

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is uniformly discrete. Then, there is an approximate lattice $\Lambda \subset G$ such that $T_h(x, B)$ is covered by finitely many translates of Λ for ν-almost every $x \in X$.

It would be interesting to be able to extend this theorem to the case of a simple Lie group acting on *X*. However, Theorem 1.3 does not allow us to do so because condition (1) requires both XX^{-1} and $X^{-1}X$ to be uniformly discrete. We therefore ask the following question.

Question 1.5. Let k be a local field of characteristic 0. Let G be the group of k-points of a simple algebraic group $\mathbb{G}$ defined over k of k-rank at least 2. Let X be such that:
(1) $X^{-1}X$ is uniformly discrete;
(2) there is $\mathcal{F}$ Borel of finite Haar measure such that $X\mathcal{F} = G$.
Is there an approximate lattice $\Lambda \subset \langle X \rangle$ and a finite subset $F \subset G$ such that $X \subset F\Lambda$?

It is not clear to us how one could approach such a question, and whether the tools from either [**Hru20**] or this paper (see also [**Mac23**]) can be applied.

Furthermore, as is apparent in the proof of Theorem 1.3, the techniques on which we rely are infinitary by nature. We are not able therefore to go from the local structure of Λ to its global structure, and we rely in particular on ideas related to Margulis' arithmeticity and superrigidity theorems. That is, why we are not able to exhibit the same quantitative conclusions in the simple case (Theorem 1.3), which provides a second difference with the case of an amenable group acting (Theorem 1.2). This prompts us to ask the following question.

Question 1.6. Let F be as in the conclusion of Theorem 1.3. Can $|F|$ be bounded above by a quantity depending on how uniformly discrete $X^{-1}X$ is and the co-volume of X (see Lemma 3.3 for a guess at what the co-volume might be)?

It is the guess of the author that answering Question 1.5 would eventually lead to an answer to Question 1.6.

1.2. *Proof strategy.* In §3, we exploit invariant measures on Ω_X to draw information on finite configurations present in *X*. Precisely, we explain the link between certain maps on the invariant hull—the so-called *periodization maps* inspired by the Siegel transform [**Sie45**] (and first used in our context in [**BH18**])—and quantities such as $|Y \cap K|$ for a given compact subset K and a generic $Y \in \Omega_X$. The main takeaway from this article is that periodization maps enable us—in many different ways—to relate ergodic-theoretic information with counting arguments.

In §4, we prove Theorems 1.1 and 1.2. The proof stems from an analogy with finite approximate subgroups. Konieczny was the first to show, in [**Kon23**], that this comparison can be used effectively when $G = \mathbb{R}^n$. There, he noticed that the subsets studied by Lagarias can be recovered as a union of subsets with small doubling in the sense of additive combinatorics simply by considering intersections with boxes of growing size. In general amenable groups, this method fails (cf. Remark 4.5). Indeed, in a general amenable group G, there is no sequence of K-approximate subgroups $(F_n)_{n\geq 0}$ (for some

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fixed K) whose union is G, see [Lin01, §5] for a case where an even stronger property fails, as well as the generalization of Gromov's polynomial growth theorem [Hru12, Theorem 7.1]. Nonetheless, if $(F_n)_{n\geq 0}$ denotes a Følner sequence, then the intersections $(X \cap F_n)_{n\geq 0}$ satisfy additive combinatorial conditions *as a family* as soon as they are sufficiently large—even though each individual set $X \cap F_n$ might be far from enjoying small doubling properties. We then draw estimates on the size of $X \cap F_n$ from applications of ergodic theorems to periodization maps—and, more precisely, to $Y \in \Omega_X \mapsto |Y \cap K|$ for K compact.

The proof strategy for Theorem 1.3 resembles the approach of Theorem 1.2. However, since the additive-combinatorial tools from [Tao08] mostly fail in simple Lie groups—due in part to the absence of Følner sets—we resort to using much more powerful ergodic-theoretic results pertaining to the rigidity of measure-preserving actions of simple Lie groups. The main step of our approach consists in extending Margulis' superrigidity theorem to certain large subsets of X. It builds upon previous results of the author established in his proof of a Meyer-type theorem in higher-rank semi-simple groups [Mac23] along with a crucial theory of transverse subsets in invariant hulls studied by Björklund, Hartnick and Karasik in [BHK25]. Nevertheless, amidst the most technical parts of this argument, we make crucial use of simple and beautiful ideas from combinatorics such as covering lemmas and Massicot–Wagner-type arguments (see the proof of Theorem 6.1). We introduce the building blocks of that approach in §5 and, then, prove Theorem 1.3 in §6. Compared with the approach in [Mac23, §5], we make sure to solely use counting arguments rather than stronger structural results regarding infinite approximate subgroups.

2. *Background material and general properties*

2.1. *Notation.* Given two subsets X, Y of a group G, we will denote $XY := \{xy \in G : x \in X, y \in Y\}$, $X^{-1} := \{x^{-1} \in G : x \in X\}$ and $X^n := \{x_1 \cdots x_n \in G : x_1, \ldots, x_n \in X\}$. Write $\langle X \rangle$, the group generated by X.

If G is a locally compact group, μ_G will denote a Haar measure. Let X be a compact space equipped with a continuous action of some locally compact group G. Given two finite Borel measures μ and ν on G and X, respectively, we define the convolution $\mu * \nu$ as

$$\mu * \nu(\phi) := \int_{G\times X} \phi(gx)\, d\mu(g)\, d\nu(x)$$

for every $\phi \in C_c^0(X)$.

2.2. *Bi-Følner sequences.* Take G a locally compact group and μ_G a Haar measure. A Følner sequence of a second countable locally compact group G is an increasing sequence $(F_n)_{n\geq 0}$ of compact subsets such that for any other compact subset K and any $\epsilon > 0$, there is $n_0 \geq 0$ with

$$\frac{\mu_G(F_n \Delta (kF_n))}{\mu_G(F_n)} \leq \epsilon$$

for every $n \geq n_0$ and $k \in K$.

It is well known that a locally compact group admits a Følner sequence if and only if G is amenable (see [Gre69] for a general introduction to amenable groups). In what follows, we will use the fact that one may in fact find Følner sequences satisfying stronger assumptions in unimodular amenable second countable locally compact groups. Indeed, Ornstein and Weiss proved the following proposition.

PROPOSITION 2.1. [OW87] *Let G be a unimodular amenable second countable locally compact group. There is a sequence of symmetric compact subsets $(F_n)_{n\geq 0}$ such that for any $K \subset G$ compact, we have*

$$\lim \frac{\mu_G((KF_nK)\Delta F_n)}{\mu_G(F_n)} = 0.$$

We call such a sequence a bi-Følner sequence.

2.3. *Elementary tools from additive combinatorics.* The combinatorial tools we will use in what follows are mostly elementary. A key role will be played by covering lemmas.

LEMMA 2.2. (Ruzsa's covering lemma, [Ruz92]) *Let X, Y be two subsets of a group G. Suppose that $F \subset Y$ is maximal such that the subsets fX for $f \in F$ are pairwise disjoint. Then,*

$$Y \subset FXX^{-1}.$$

In particular, if there is a bound $C > 0$ on the size of subsets F' of Y such that the subsets fX for $f \in F'$ are pairwise disjoint, then

$$Y \subset F''XX^{-1}$$

for some subset F'' of size at most C.

We will also use a straightforward generalization of this lemma.

LEMMA 2.3. (Ruzsa's covering lemma for families of sets) *Let $(X_i)_{i\in I}$ be a family of subsets of G and let Y be a subset of a group G. Suppose that there is a bound $C > 0$ on the size of subsets F' of Y such that for every $i \in I$, the subsets fX_i for $f \in F'$ are pairwise disjoint. Then,*

$$Y \subset F''\bigg(\bigcup_{i\in I} X_i X_i^{-1}\bigg)$$

for some subset F'' of size at most C.

Proof. Let F'' be a subset of maximal cardinality satisfying the assumption. Then, by maximality, for every $y \in Y$, there are $i \in I$ and $f \in F''$ such that $fX_i \cap yX_i \neq \emptyset$, that is, $y \in fX_iX_i^{-1}$. Since F'' has size at most C, this concludes the proof. □

2.4. *Internal dimension of a model set.* Suppose that Λ is commensurable with a model set associated to a cut-and-project scheme (G, H, Γ) (see [BH18, §2.3]). By the Gleason–Yamabe theorem [Yam53], there is an open subgroup $U \subset H$ and a compact normal subgroup $K \subset U$ such that U/K is a connected Lie group without non-trivial compact normal subgroup. The subgroup thus obtained is essentially unique, that is, it

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depends on the commensurability class of Λ only (see [Mac25, Proposition 3.6(2)]). Equivalently, it is the connected Lie group H' of minimal dimension such that there exists a cut-and-project scheme (G, H', Γ') whose model sets are commensurable with Λ. We thus define the *internal dimension* of Λ as the dimension of U/K. A recent result of An, Jing, Tran and Zhang relying on the non-abelian Brunn–Minkowski inequality [JTZ21] implies the following theorem.

THEOREM 2.4. [AJTZ21, Theorem 1] *If Λ is a K-approximate subgroup and commensurable to a model set, then Λ has internal dimension at most $O(\log_2^2 K)$.*

Building upon that result as well as a structure theorem for amenable approximate subgroups from [Mac25], we proved furthermore the following proposition.

PROPOSITION 2.5. *If Λ is an approximate lattice and a K-approximate subgroup of some second countable locally compact amenable group, then Λ has internal dimension at most $O(\log(K))$.*

Proof. Let R be a connected solvable group of dimension d. For any compact subset $A \subset R$, we have $\mu(A^2) \geq 2^{d/2}\mu(A)$, see [AJTZ21, §3] and references within. Hence, if A is a K-approximate subgroup, then $d = O(\log K)$. By [Mac25, Theorem 1.6], the internal space of minimal dimension is a connected solvable Lie group. The result follows now from standard results about approximate subgroups [Mac25, §2.2]. □

3. *Counting points using periodization maps and consequences*

3.1. *The invariant hull.* Let G be a second countable locally compact group. The Chabauty space of G is the set $\mathcal{C}(G)$ of all closed subsets of G—possibly empty—equipped with the topology generated by the open subsets

$$U^V := \{X \in \mathcal{C}(G) : V \cap X \neq \emptyset\}$$

and

$$U_K := \{X \in \mathcal{C}(G) : K \cap X = \emptyset\}$$

for every V open and K compact. The Chabauty space is a compact metrizable space whose Borel structure is generated by either one of the two families $(U^V)_{V \text{ open}}$ or $(U_K)_{K \text{ compact}}$ (e.g. [BH18] and references therein). Furthermore, the natural action $(g, X) \mapsto gX$ of G on $\mathcal{C}(G)$ is continuous.

Given a uniformly discrete subset $X \subset G$, we define the *invariant hull* by

$$\Omega_X := \overline{G \cdot X}.$$

The invariant hull is a compact metrizable G-space that encodes many properties of Ω_X. For instance, X is relatively dense if and only if $\emptyset \notin \Omega_X$ (see [BH18, Proposition 4.4]). We will be particularly interested in the situation where there is a G-invariant Borel probability measure ν on Ω_X such that $\nu(\{\emptyset\}) = 0$. We will call such a measure *proper*.

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In what follows, we will use the so-called *periodization maps* to draw quantitative information about X from ergodic theorems on Ω_X. Precisely, the periodization map $\mathcal{P}_X$ sends a continuous function with compact support $\phi \in C_c^0(G)$ to the continuous map

$$\begin{aligned}\mathcal{P}_X\phi : \Omega_X &\longrightarrow \mathbb{R}\\ Y &\longmapsto \sum_{y\in Y}\phi(y)\end{aligned}$$

with support contained in $\Omega_X \setminus \{\emptyset\}$.

The map $\mathcal{P}_X$ is G-equivariant and allows to pull-back measures. We mention the following crucial observation: if ν is a proper G-invariant Borel probability measure on Ω_X, then $\mathcal{P}_X^*\nu$ is a non-trivial Haar measure on G [**BH18**, Corollary 5.7].

Finally, the definition of $\mathcal{P}_X$ extends, in a straightforward manner, to positive functions ϕ on G with $\mathcal{P}_X\phi$ possibly taking infinite values. Note that, when $\phi = \mathbf{1}_V$ for some open relatively compact subset V and X is uniformly discrete, $\mathcal{P}_X\phi$ takes finite values, is upper-semi-continuous and equal to $Y \mapsto |Y \cap V|$.

3.2. *Counting points with the periodization map.* Our next result is a handy observation that will allow us to translate cardinality estimates into convolution estimates involving quantities reminiscent of the periodization maps.

LEMMA 3.1. *Let X be a countable subset of a unimodular locally compact group. Let $A, B \subset G$ be two Borel subsets. Then,*

$$\int_B \sum_{x\in X} \mathbf{1}_{Ax}(g)\, d\mu_G(g) \leq \mu_G(A)|X \cap A^{-1}B|, \tag{3.1}$$

$$\mu_G(A)|X \cap B| \leq \int_{AB} \sum_{x\in X} \mathbf{1}_{Ax}(g)\, d\mu_G(g). \tag{3.2}$$

Proof. To prove (3.1), notice that if $Ax \cap B \neq \emptyset$, then $x \in A^{-1}B$. So,

$$\begin{aligned}\int_B \sum_{x\in X} \mathbf{1}_{Ax}(g)\, d\mu_G(g) &= \sum_{x\in X}\int_B \mathbf{1}_{Ax}(g)\, d\mu_G(g)\\ &= \sum_{x\in X\cap A^{-1}B}\int_B \mathbf{1}_{Ax}(g)\, d\mu_G(g)\\ &\leq \mu_G(A)|X\cap A^{-1}B|.\end{aligned}$$

Let us now prove (3.2). If $x \in B$, then $Ax \in AB$. Therefore,

$$\begin{aligned}\mu_G(A)|X \cap B| &\leq \int_{AB} \sum_{x\in X\cap B} \mathbf{1}_{Ax}(g)\, d\mu_G(g)\\ &\leq \int_{AB} \sum_{x\in X} \mathbf{1}_{Ax}(g)\, d\mu_G(g).\end{aligned}$$ □

3.3. *Invariant hull and commensurability.* In [**Mac23**, §2.2.2], we proved a commensurability criterion for uniformly discrete subsets of finite co-volume.

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LEMMA 3.2. *Let X, Y be two uniformly discrete subsets of a locally compact group G. Suppose that there is a Borel probability measure ν on Ω_X such that:*

(a) $\mathcal{P}_X^*\nu \geq \mu_G$ *for a Haar measure* μ_G *on* G*;*

(b) $V \subset G$ *is an open subset such that there is* $C > 0$ *with* $|XY \cap gV| \leq C$ *for every* $g \geq 0$.

Then, Y *is covered by* $\mu_G(V)^{-1}C$ *right-translates of* $X^{-1}X$.

Although our hypotheses are slightly weaker, the proof of Lemma 3.2 is identical to the proof of [**Mac23**, Proposition 4]. This is in fact also a special case of a more general result at the heart of the proof of superrigidity and property (T) for $\star$-approximate lattices that we will invoke once more later on, see §5.

3.4. *Commensurability and finite co-volume.* We will prove a technical result about commensurable subsets and finite co-volume. This will turn out to be useful in the proof of Theorem 1.2.

LEMMA 3.3. *Let $X \subset G$ be uniformly discrete of a locally compact group and suppose that Ω_X admits a proper Borel probability measure ν such that $\mathcal{P}_X^*\nu \geq \mu_G$, where μ_G denotes some Haar measure. Let Y be uniformly discrete and let F be a finite subset such that $X \subset FY$. Then, there is a Borel subset $\mathcal{F}$ of finite Haar measure such that $Y^{-1}Y\mathcal{F} = G$ and the multiplication map $Y \times \mathcal{F} \to G$ is one-to-one.*

Proof. By adapting an argument from [**Hru20**, Lemma A.3], there is $\mathcal{F}$ Borel such that $Y^{-1}Y\mathcal{F} = G$ and the multiplication map $Y \times \mathcal{F} \to G$ is one-to-one. Indeed, let V be a compact neighbourhood of the identity such that $VV^{-1} \cap Y^{-1}Y = \{e\}$ and let $(g_n)_{n\geq 0}$ be a sequence of elements of G such that $G = \bigcup_{n\geq 0} Vg_n$. Define inductively $B_n := Vg_n \setminus \bigcup_{m<n} Y^{-1}YB_m$ and $B := \bigcup_{n\geq 0} B_n$. Then, $BB^{-1} \cap Y^{-1}Y = \{e\}$ and $Y^{-1}YB = G$. We can in fact choose such an $\mathcal{F}$ with null boundary. Therefore, the multiplication map $X \times \mathcal{F} \to G$ is $|F|$-to-one. So for every $g \in G$, $|gX \cap V^{-1}| \leq |F|$, where V denotes the interior of $\mathcal{F}$. We obtain $|Y \cap V^{-1}| \leq |F|$ for every $Y \in \Omega_X$ since V^{-1} is open. Hence,

$$|F| \geq \int_{\Omega_X} |Y \cap V^{-1}|\, d\nu(Y) = \mathcal{P}_X^*\nu(V^{-1}).$$

However, $\mathcal{P}_X^*\nu$ dominates a Haar measure. Thus, V has finite Haar measure, and so has $\mathcal{F}$. □

4. *Lagarias-type result in amenable groups*

4.1. *Building measures on the invariant hull.* Our first result already shows that counting points is related to invariant measures on the hull.

PROPOSITION 4.1. *Let X be a uniformly discrete subset of a locally compact second countable group G. Suppose that there is a Følner sequence $(F_n)_{n\geq 0}$ such that*

$$\liminf \frac{|X^{-1} \cap F_n|}{\mu_G(F_n)} = \alpha > 0.$$

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Then, there is a proper G-invariant Borel probability measure μ on Ω_X such that $\mathcal{P}_X^*\mu \geq \alpha\mu_G$.

Proof. Take V a compact neighbourhood of the identity such that $X^{-1}X \cap V^{-1}V = \{e\}$. Then, we can reformulate (3.2) of Lemma 3.1 as follows:

$$\mu_G(V)|X^{-1} \cap F_n| \leq \int_{VF_n} \mathcal{P}_X \mathbf{1}_V(gX)\, d\mu_G(g). \tag{4.1}$$

Consider $\phi \in C_c^0(G)$ with non-negative values and $\phi \geq \mathbf{1}_V$. Let μ_n denote the measure with density $\mu_G(VF_n)^{-1}\mathbf{1}_{VF_n}$ against the Haar measure and write

$$\liminf \frac{|X \cap F_n|}{\mu_G(F_n)} = \alpha.$$

Since F_n is a Følner sequence, $\mu_G(VF_n) \sim \mu_G(F_n)$ and $(VF_n)_{\geq 0}$ is a Følner sequence. Then, (4.1) implies that

$$\begin{aligned}\liminf \int_{\Omega_X} \mathcal{P}_X\phi(Y)\, d(\mu_n * \delta_X)(Y) &\geq \liminf \int_{VF_n} \frac{\mathcal{P}_X\mathbf{1}_V(g)}{\mu_G(VF_n)}\, d\mu_G(g)\\ &\geq \alpha\mu_G(V) > 0,\end{aligned}$$

where δ_X denotes the Dirac mass at X. Therefore, any weak-* limit ν of $\mu_n * \delta_X$ satisfies

$$\int_{\Omega_X} \mathcal{P}_X\phi(Y)\, d\nu(Y) \geq \alpha\mu_G(V). \tag{4.2}$$

Since $\mathcal{P}_X\phi(Y)$ has compact support contained in $\Omega_X \setminus \{\emptyset\}$, ν is not supported on $\{\emptyset\}$. However, ν is a G-invariant Borel probability measure since $(VF_n)_{n\geq 0}$ is a Følner sequence. Note moreover that (4.2) is valid for any such ϕ, so

$$\mathcal{P}_X^*\nu(V) = \int_{\Omega_X} \mathcal{P}_X\mathbf{1}_V(Y)\, d\nu(Y) \geq \alpha\mu_G(V).$$

Thus, a proper G-invariant ergodic Borel probability measure ν' such that $\nu'(U^V) \geq \alpha\mu_G(V)$ must appear in the ergodic decomposition of ν. The result now follows from the fact that $\mathcal{P}_X^*\nu'$ is a Haar measure. □

4.2. *The small doubling criterion.* Our proof will rely on the following criterion.

PROPOSITION 4.2. *Let X be a subset of a unimodular amenable second countable locally compact group G. If:*

(1) *there is a neighbourhood of the identity $W \subset G$ with $(X^{-1}X)^2 \cap W^{-1}W = \{e\}$;*

(2) *we have*

$$\limsup \frac{|X \cap F_n|}{\mu_G(F_n)} = \alpha > 0$$

for some bi-Følner sequence F_n;

then there is a subset $S \subset XX^{-1}$ such that S^n is uniformly discrete for every $n \geq 0$ and there is a finite subset F with $X \subset SF$. Moreover, S^2 is an $O(\alpha^{-4}\mu_G(W)^{-4})$-approximate subgroup and F has size at most $O(\alpha^{-1}\mu_G(W)^{-1})$.

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Our strategy will be to rephrase conditions (1) and (2) in additive-combinatorial terms. However, in contrast to the case treated by Konieczny in [Kon23], we will not be able to write X as an inductive limit of finite approximate subgroups. Rather, we will show that the subsets $(X \cap F_n)_{n\geq 0}$ satisfy a small doubling condition *as a family*. We will build upon an idea of Tao [Tao08] used in his proof of a non-commutative generalization of Plünnecke's lemma.

LEMMA 4.3. *Let $A_0, \ldots, A_{k+1}$ be finite subsets of a group G. Suppose that there is $K \geq 0$ such that for every $l \in \{0, \ldots, k\}$, we have*

$$|A_l^{-1}A_{l+1}| \leq K|A_{l+1}|.$$

For every $l \in \{1, \ldots, k\}$, set

$$S_l := \{g \in G : |A_l \cap gA_l| \geq (4K)^{-1}|A_l|\}.$$

Then,

$$|A_0^{-1}S_1 \cdots S_k A_{k+1}| \leq 4^k K^{2k+1}|A_{k+1}|.$$

Proof. Let $m : A_0^{-1}A_1 \times \cdots \times A_k^{-1}A_{k+1} \to G$ denote the multiplication map. Take an element $a \in A_0^{-1}S_1 \cdots S_k A_{k+1}$ and choose $\alpha_0 \in A_0, \alpha_{k+1} \in A_{k+1}, s_1 \in S_1, \ldots, s_k \in S_k$ such that $a = \alpha_0^{-1}s_1 \cdots s_k\alpha_{k+1}$. Now, for every $l \in \{1, \ldots, k\}$, take $\alpha_l \in A_l \cap s_lA_l$. We have $\alpha_l^{-1}s_l\alpha_{l+1} \in A_l^{-1}A_{l+1}$. Moreover,

$$a = m(\alpha_0^{-1}s_1\alpha_1, \ldots, \alpha_k^{-1}s_k\alpha_{k+1}).$$

So a belongs to the range of the map m and $m^{-1}(a)$ has size at least $|A_1 \cap s_1A_1| \cdots |A_k \cap s_kA_k|$. Therefore,

$$|m^{-1}(a)| \geq (4K)^{-k}|A_1| \cdots |A_k|.$$

However,

$$|A_0^{-1}A_1 \times \cdots \times A_k^{-1}A_{k+1}| \leq K^{k+1}|A_1| \cdots |A_{k+1}|.$$

So, there are at most $4^kK^{2k+1}|A_{k+1}|$ such elements a. □

Proof of Proposition 4.2. We will rephrase condition (2) as follows. Choose $\alpha > 0$ such that $\limsup |X \cap F_n|/\mu_G(F_n) > \alpha$. Upon considering a subsequence of $(F_n)_{n\geq 0}$, we may assume that

$$|X \cap F_nW| > \alpha\mu_G(F_nW)$$

for every $n \geq 0$.

We will be able to rephrase condition (1) in a similar fashion. Fix $\beta_0 > 1$. First of all, upon considering a further subsequence, we may assume that the sequence $(F_n)_{n\geq 0}$ satisfies the Shulman assumption [Shu88]. In other words,

$$\text{for every } m > n \geq 0, \quad \mu_G(F_n^{-1}F_m) \leq \beta_0\mu_G(F_m).$$

Note that $\beta_0 > 0$ may be chosen as close to 1 as one wishes.

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Let W be as in condition (1). Since $(F_n)_{n\geq 0}$ is bi-Følner, we may furthermore assume that

$$\text{for every } m > n \geq 0, \quad \mu_G(W^{-1}F_n^{-1}F_mW^2) \leq \beta_0\mu_G(F_m)$$

for the same β_0 as above.

We have

$$\begin{aligned}\mu_G(W)|X^{-1}X \cap W^{-1}F_n^{-1}F_mW| &= \mu_G((X^{-1}X \cap W^{-1}F_n^{-1}F_mW)W)\\ &\leq \mu_G(W^{-1}F_n^{-1}F_mW^2)\\ &\leq \beta_0\mu_G(F_m).\end{aligned}$$

Define now the subsets $\tilde{F}_n := F_nW$ and $A_n := X \cap \tilde{F}_n$ for every $n \geq 0$. The above discussion implies

$$\text{for every } n \geq 0, \qquad |A_n| \geq \alpha\mu_G(F_n), \tag{4.3}$$

$$\text{for every } m > n \geq 0, \quad |A_n^{-1}A_m| \leq \beta\mu_G(F_m), \tag{4.4}$$

where $\beta = \beta_0/\mu_G(W)$.

Choose a non-principal ultrafilter U on $\mathbb{N}$. Let $P_W(G)$ be the set of subsets Y of G such that the multiplication map $Y \times W \to G$ is one-to-one. For $Y \in P_W(G)$, define

$$M(Y) = \lim_{n\to U} \frac{\mu_G(YW \cap \tilde{F}_nW)}{\mu_G(\tilde{F}_n)}.$$

Since $(F_n)_{n\geq 0}$ is Følner, it easily seen that M is a left-invariant finitely additive measure defined on $P_W(G)$. According to (4.3), we have $M(X) \geq \alpha\mu_G(W)$. By (4.4), we have $M(X^{-1}X) \leq \beta\mu_G(W)$. Define

$$S := \left\{g \in G : M(gX \cap X) \geq \frac{\alpha}{2\beta}M(X)\right\}.$$

CLAIM 4.4. (Massicot and Wagner, [**MW15**, Proof of Theorem 12]) *There is a finite subset F of size $O(\beta\alpha^{-1})$ such that $X^{-1} \subset FS$.*

Proof. Let $m \geq 0$ and suppose that there is no subset F of size at most m such that $X^{-1} \subset FS$. Therefore, we can choose by induction $f_1, \ldots, f_m \in X^{-1}$ such that for every $1 \leq i \leq m$, we have

$$f_i \notin \bigcup_{j=1}^{i-1} f_jS.$$

Equivalently, for every $1 \leq i < j \leq m$,

$$M(f_iX \cap f_jX) < \frac{\alpha}{2\beta}M(X).$$

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Hence,

$$\begin{aligned} M(X^{-1}X) &\geq M\bigg(\bigcup_{i=1}^{m} f_i X\bigg) \\ &\geq \sum_{i=1}^{m} M(f_i X) - \sum_{1\leq i<j\leq m} M(f_i X \cap f_j X) \\ &> mM(X) - \frac{m(m-1)}{2}\frac{\alpha}{2\beta}M(X) \\ &= m\bigg(1 - \frac{(m-1)\alpha}{4\beta}\bigg)M(X). \end{aligned}$$

This yields

$$\beta > \alpha m\bigg(1 - \frac{(m-1)\alpha}{4\beta}\bigg).$$

Additionally, this inequality fails for $m = \lfloor \alpha/2\beta \rfloor$. So, for $m = \lfloor \alpha/2\beta \rfloor$, $X^{-1} \subset FS$ for some $F \subset X^{-1}$ of cardinality at most m. □

We will now show that S^n is uniformly discrete for every $n \geq 0$. To do so, we will prove that $S^n \cap W$ is finite for every $n \geq 0$. Notice first that $S \subset \bigcup_{n\geq 0} S_n$, where

$$S_n := \bigg\{ g \in G : |A_n \cap gA_n| \geq \frac{\alpha}{4\beta}|A_n| \bigg\}$$

for every $n \geq 0$. More precisely, each element of S is contained in infinitely many S_n terms. Indeed, for every $Y \in P_W(G)$, we have

$$|Y \cap \tilde{F}_n| \leq \mu_G(YW \cap F_n W^2).$$

Additionally,

$$\begin{aligned} \mu_G(YW \cap F_n W^2) &\leq |Y \cap \tilde{F}_n| + \mu_G(YW \cap (F_n W^2 \setminus F_n)) \\ &= |Y \cap \tilde{F}_n| + o(\mu_G(F_n)). \end{aligned}$$

However, if $g \in S$, for some $A \in U$ and all $n \in A$, we have

$$\mu_G((X \cap gX)W \cap \tilde{F}_n W) \geq \frac{\alpha}{3\beta}\mu_G(XW \cap \tilde{F}_n W),$$

which yields

$$|X \cap gX \cap \tilde{F}_n| + o(\mu_G(F_n)) \geq \frac{\alpha}{3\beta}|X \cap \tilde{F}_n|.$$

Since $|X \cap \tilde{F}_n| \geq \alpha\mu_G(\tilde{F}_n)$, if n is chosen sufficiently large with respect to U, then

$$|X \cap gX \cap \tilde{F}_n| \geq \frac{\alpha}{4\beta}|X \cap \tilde{F}_n|.$$

So our claim is proved up to considering a subsequence.

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Choose any finite subset $F' \subset S^n \cap W$ such that the subsets fX are pairwise disjoint when f runs through F'. Then, there are $k_n > \cdots > k_0 \geq 0$ such that $F' \subset S_{k_1} \cdots S_{k_n}$. Choose an integer k_{n+1} such that $k_n < k_{n+1}$. We thus have by Lemma 4.3 (note that we had to specify k_0 to invoke Lemma 4.3, although it is not needed from now on),

$$|F'||A_{k_{n+1}}| = |F'A_{k_{n+1}}| \leq |S_{k_1} \cdots S_{k_n} A_{k_{n+1}}| \leq 4^n \frac{\beta^{2n+1}}{\alpha^{2n+1}} |A_{k_{n+1}}|.$$

Therefore, $|F'| \leq 4^n(\beta^{2n+1}/\alpha^{2n+1})$. So take F' as above of maximal size. By Ruzsa's covering lemma, we have $S^n \cap W \subset F'XX^{-1}$. Since $F' \subset W$, we finally have

$$S^n \cap W \subset F'(XX^{-1} \cap W^2) = F'.$$

Let us finally prove that S^2 is an approximate lattice. Notice first that

$$\begin{aligned}\limsup \frac{|S \cap F_n F|}{\mu_G(F_n)} &\geq \limsup \frac{|SF^{-1} \cap F_n|}{|F|\mu_G(F_n)} \\ &\geq \limsup \frac{|X \cap F_n|}{|F|\mu_G(F_n)} \\ &\geq \limsup \frac{|X \cap F_n|}{|F|\mu_G(F_n)} > 0.\end{aligned}$$

Since $(F_n)_{n\geq 0}$ is bi-Følner, $(F_nF)_{n\geq 0}$ is a Følner sequence and $\mu_G(F_nF) \sim \mu_G(F_n)$. Hence, according to Proposition 4.1, there is a proper G-invariant Borel probability measure ν on Ω_S. By Lemma 3.2, that S^5 is uniformly discrete implies that S^2 is an approximate subgroup and, hence, an approximate lattice. The bounds in the statement of Proposition 4.2 are a consequence of the bounds on the size of F' and those found in Lemma 3.2. □

Remark 4.5. In Konieczny's [**Kon23**], a more direct approach when $X \subset \mathbb{R}^n$ relies on the fact that the intersections $X \cap [-R; R]^n$ for $R \to \infty$ are l-approximate subgroups for some fixed $l \geq 0$ independent of R. Unfortunately, this fact is specific to nilpotent ambient groups G, as can be derived from the polynomial growth theorem for approximate subgroups (see [**CHT24**, Appendix]).

4.3. *Double counting and ergodic theorems.*

PROPOSITION 4.6. *Let X be a subset of a unimodular amenable locally compact second countable group G. Suppose that XX^{-1} is uniformly discrete and that there is $\mathcal{F}$ of finite Haar measure such that $X\mathcal{F} = G$. Then, for some bi-Følner sequence $(F_n)_{n\geq 0}$,*

$$\liminf \frac{|X \cap F_n|}{\mu_G(F_n)} \geq \frac{1}{\mu_G(\mathcal{F})}.$$

Proposition 4.6 will be an easy consequence of a Fubini-type argument that we write down below.

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LEMMA 4.7. *Let X be a subset such that $\mathcal{F}X = G$ for $\mathcal{F}$ of finite co-volume and $X^{-1}X \cap V^{-1}V = \{e\}$ for a neighbourhood V of the identity. Choose $\epsilon > 0$. Then, there is $K \subset \mathcal{F}$ compact such that for every $B \subset G$ Borel with $\mu_G(B) < \infty$, we have*

$$|kX \cap BV| \geq \frac{(1-\epsilon)\mu_G(B)}{\mu_G(K)}$$

for some $k \in K$.

Proof. Since μ_G is bi-invariant, for every $h \in G$,

$$\begin{aligned}\int_{\mathcal{F}} \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(g) &= \sum_{x \in X} \int_{\mathcal{F}} \mathbf{1}_V(hgx)\, d\mu_G(g) \\ &= \sum_{x \in X} \int_{\mathcal{F}x} \mathbf{1}_V(hg)\, d\mu_G(g) \\ &\geq \int_{\mathcal{F}X} \mathbf{1}_V(hg)\, d\mu_G(g) = \mu_G(V).\end{aligned}$$

If now $K \subset \mathcal{F}$, we have

$$\int_K \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(g) \geq \mu_G(V) - \int_{\mathcal{F}\setminus K} \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(g).$$

However,

$$\sum_{x \in X} \mathbf{1}_V(hgx) = |X \cap g^{-1}h^{-1}V|.$$

Additionally, if $\mathbf{1}_V(hgx) > 0$, then $x \in g^{-1}h^{-1}V$. Hence,

$$\sum_{x \in X} \mathbf{1}_V(hgx) \leq |X^{-1}X \cap V^{-1}V| = 1.$$

So,

$$\int_K \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(g) \geq \mu_G(V) - \mu_G(\mathcal{F} \setminus K).$$

Consider K compact such that $\mu_G(\mathcal{F} \setminus K) \leq \epsilon\mu_G(V)$. Then,

$$\int_{B^{-1}} \int_K \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(g)\, d\mu_G(h) \geq (1-\epsilon)\mu_G(B)\mu_G(V).$$

However, Lemma 3.1 implies

$$\int_{B^{-1}} \sum_{x \in X} \mathbf{1}_V(hgx)\, d\mu_G(h) \leq |gX \cap BV|\mu_G(V).$$

Therefore,

$$\int_K \mu_G(V)|gX \cap BV|\, d\mu_G(g) \geq (1-\epsilon)\mu_G(B)\mu_G(V).$$

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Now, if $\sup_{g\in K} |gX \cap BV| = +\infty$, then the result is certainly satisfied. Otherwise, since $g \mapsto |gX \cap BV|$ takes integer values, we have

$$\sup_{g\in K} |gX \cap BV| = \max_{g\in K} |gX \cap BV|.$$

Then,

$$\Big(\sup_{g\in K} |gX \cap BV|\Big)\mu_G(K) \geq \int_K |gX \cap BV| \, d\mu_G(g) \geq (1-\epsilon)\mu_G(B),$$

which concludes the proof. □

Proof of Proposition 4.6. Let $(K_n)_{n\geq 0}$ be an exhaustion of G by compact subsets and let V be a compact neighbourhood of the identity such that $XX^{-1} \cap V^{-1}V = \{e\}$. Fix $n \geq 0$. The set X^{-1} satisfies the hypotheses of Lemma 4.7. Let K_n' be given by Lemma 4.7 applied to X^{-1} and $\epsilon = 1/n$.

Since G is amenable, there is a symmetric compact subset F_n such that

$$\mu_G(K_n K_n' F_n V K_n) \leq \Big(1 + \frac{1}{n}\Big)\mu_G(F_n).$$

Define now $F_n' := K_n'^{-1} F_n V$ for every $n \geq 0$. Then, $(F_n')_{\geq 0}$ is a bi-Følner sequence, so is $(F_n'^{-1})_{n\geq 0}$, and

$$|X \cap F_n'^{-1}| = |X^{-1} \cap F_n'| \geq \Big(1 - \frac{1}{n}\Big)\frac{\mu_G(F_n)}{\mu_G(\mathcal{F})} \geq \frac{(1-1/n)}{1+1/n}\frac{\mu_G(F_n'^{-1})}{\mu_G(\mathcal{F})}. \qquad \square$$

Proof of Theorem 1.1. By Proposition 4.6 for some bi-Følner sequence $(F_n)_{n\geq 0}$,

$$\liminf \frac{|X \cap F_n|}{\mu_G(F_n)} \geq \frac{1}{\mu_G(\mathcal{F})}.$$

So, Theorem 1.1 is a consequence of Proposition 4.2. □

As another consequence of Proposition 4.6, the invariant hull Ω_X admits a proper G-invariant Borel probability measure (see Proposition 4.1). Conversely, when we merely have information on the hull rather than on a 'fundamental domain' $\mathcal{F}$, we can invoke mean ergodic theorems to conclude the following proposition.

PROPOSITION 4.8. *Let X be a uniformly discrete subset of G such that Ω_X admits a proper ergodic G-invariant Borel probability measure ν. Let V be a neighbourhood of the identity such that $X^{-1}X \cap V^{-1}V = \{e\}$. Let μ_G be the Haar measure given by $\mathcal{P}_X^*\nu$. Then, there is a bi-Følner sequence $(F_n)_{n\geq 0}$ such that*

$$\liminf \frac{|Y \cap VF_n|}{\mu_G(VF_n)} \geq 1$$

for ν-almost every $Y \in \Omega_X$.

Proof. Let $(F_n)_{n\geq 0}$ be a Følner sequence and let μ_n be the probability measure with density $\mu_G(F_n)^{-1}\mathbf{1}_{F_n}$ against the Haar measure. By the mean ergodic theorem, we know that $\phi_n : Y \to \int_G |gY \cap V| \, d\mu_n(g)$ converges in the L^2-norm to its average

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$\int_{\Omega_X} \mathcal{P}_X \mathbf{1}_V(Y)\, d\nu(Y)$. However, $\mathcal{P}_X \mathbf{1}_V = \mathbf{1}_{U^V}$. So ϕ_n converges in the L^2-norm to $\nu(U^V)$. Upon considering a subsequence, we may assume that the convergence is ν-almost everywhere. Therefore, for ν-almost all Y, we have $\int_G |gY \cap V|\, d\mu_n(g) \to \nu(U^V)$. According to Lemma 3.1, we find

$$\liminf \frac{|Y \cap VF_n|}{\mu_G(VF_n)} \geq \frac{\nu(U^V)}{\mu_G(V)} = 1. \qquad \square$$

It is interesting to compare the two formulae obtained in Proposition 4.8 and Lemma 4.7. Indeed, we see that the Haar measure $\mu_G = \mathcal{P}_X^* \nu$ seems to correspond with a Haar measure normalized such that X has co-volume 1 *in both meanings of co-volume*.

Proof of Theorem 1.2. According to Proposition 4.6, there is a bi-Følner sequence $(F_n)_{n\geq 0}$ such that

$$\liminf \frac{|Y \cap VF_n|}{\mu_G(VF_n)} \geq 1$$

for ν-almost every $Y \in \Omega_X$, where $\mu_G := \mathcal{P}_X^* \nu$. Fix one such $Y_0 \in \Omega_X$. By Proposition 4.2 applied to Y_0, there is an approximate subgroup S contained in $Y_0 Y_0^{-1}$ and a finite subset F such that $Y_0 \subset SF$. So, S is an approximate lattice according to Lemma 3.3. $\square$

5. *The invariant hull, equivariant families and cross-sections*

In this section, we establish additional results regarding the invariant hull, which we use in the proof of Theorem 1.3. In the following, fix G a second countable locally compact group and $X \subset G$ such that Ω_X admits a proper ergodic G-invariant Borel probability measure. Fix moreover one such G-invariant measure ν. Additionally, assume that both $X^{-1}X$ and XX^{-1} are uniformly discrete. We start with a general result concerning equivariant families of discrete subsets.

5.1. *Equivariant families of discrete subsets.*

LEMMA 5.1. *Let G, X and Ω_X be as in the first paragraph of this section. Let S be a Borel G-space with a probability measure-preserving action of a unimodular locally compact second countable group G. Let $\Phi : S \to \Omega_X$ be a Borel G-equivariant map taking non-empty values for almost every s. Let $Y \subset G$ be such that XY^{-1} is uniformly discrete. Then, for every $Y' \in \Omega_Y$, Y' is covered by finitely many left-translates of $\bigcup_{s\in S} \Phi(s)^{-1}\Phi(s)$. If, moreover, the action is ergodic, then for almost every $s \in S$, Y' is covered by finitely many left-translates of $\Phi(s)^{-1}\Phi(s)$.*

The proof of Lemma 5.1 is a first instance of the overall strategy. Below, we will show how using periodization maps will allow us to reformulate the problem into a simple counting question, which we will then solve using elementary covering arguments.

The proof method will essentially follow the method found in [**Mac23**]. There, we proved a similar result valid more generally for maps that are not G-equivariant, but satisfy a functional identity.

Proof. Let ν_0 denote a G-invariant probability measure on S. Take $F \subset (Y')^{-1}$ finite such that for ν_0-almost every $s \in S$, the subsets $\{\Phi(s)f : f \in F\}$ are pairwise disjoint. Let $V \subset G$ be a neighbourhood of the identity such that $YX^{-1}XY^{-1} \cap V^{-1}V = \{e\}$. Therefore, for every $g \in G$ and $X' \in \Omega_X$, we have $|X'Y^{-1} \cap gV| \leq 1$ for every $X' \in \Omega_X$. Since F is finite and $G \cdot Y$ is dense in Ω_Y, there is $h \in G$ such that $F \subset Y^{-1}h$. Now,

$$\begin{aligned}\int_S \sum_{x \in \Phi(s)F} \mathbf{1}_{Vh}(x)\, d\nu_0(s) &= \int_S \sum_{x \in \Phi(s)Fh^{-1}} \mathbf{1}_V(x)\, d\nu_0(s)\\ &\leq \int_S \sum_{x \in \Phi(s)Y^{-1}} \mathbf{1}_V(x)\, d\nu_0(s)\\ &\leq 1.\end{aligned}$$

However,

$$\begin{aligned}\int_S \sum_{x \in \Phi(s)F} \mathbf{1}_{Vh}(x)\, d\nu_0(s) &= \sum_{f \in F} \int_S \sum_{x \in \Phi(s)f} \mathbf{1}_{Vh}(x)\, d\nu_0(s)\\ &= \sum_{f \in F} \int_S \sum_{x \in \Phi(s)} \mathbf{1}_{Vhf^{-1}}(x)\, d\nu_0(s)\\ &= \sum_{f \in F} \mathcal{P}_X^* \Phi_* \nu_0(Vhf^{-1}) = |F| \mathcal{P}_X^* \Phi_* \nu_0(V),\end{aligned}$$

where the interversion in the first line was because of the disjointness property of F and we use in the last line that $\mathcal{P}_X^* \Phi_* \nu_0$ is a Haar measure of a unimodular locally compact group. Thus, $|F| \leq \mathcal{P}_X^* \Phi_* \nu_0(V)^{-1}$. Hence,

$$(Y')^{-1} \subset \bigcup_{s \in S} \Phi^{-1}(s)\Phi(s)F'$$

for some $F' \subset (Y')^{-1}$ of size at most $\mathcal{P}_X^* \Phi_* \nu_0(V)^{-1}$ by the covering lemma (Lemma 2.3).

Suppose now that ν_0 is ergodic. For every $g \in G$, the set

$$S_g := \{s \in S : g \in \Phi^{-1}(s)\Phi(s)\}$$

is G-invariant and Borel. Indeed, since $\Phi(s)^{-1}\Phi(s)$ is a uniformly discrete subset, we have $g \in \Phi^{-1}(s)\Phi(s)$ if and only if

$$\Phi(s) \in \bigcap_{n \geq 0} \bigcup_{d \in D} U^{dV_n} \cap U^{dV_n g},$$

where $(V_n)_{n \geq 0}$ is a neighbourhood basis at e and D is a dense subset of G. Because ν is ergodic, we thus have that S_g is either null or co-null. Now, $\bigcup_{s \in S} \Phi^{-1}(s)\Phi(s)$ is contained in $X^{-1}X$, which is countable. Define

$$\tilde{S} = \bigcap_{g : \nu(S_g) = 1} S_g \setminus \bigg(\bigcup_{g : \nu(S_g) = 0} S_g \bigg).$$

Then, for every $s \in \tilde{S}$, we have $\bigcup_{s' \in \tilde{S}} \Phi^{-1}(s')\Phi(s') = \Phi^{-1}(s)\Phi(s)$. Hence,

$$Y' \subset \bigcup_{s' \in \tilde{S}} \Phi^{-1}(s')\Phi(s')F' = \Phi^{-1}(s)\Phi(s)F'$$

for any $s \in \tilde{S}$. □

5.2. *Canonical transverse and hitting times.* The proof of Theorem 1.3 will make central use of the following notion studied by Björklund, Hartnick and Karasik in [**BHK25**].

Definition 5.2. Let G, X, Ω_X and ν be as in the first paragraph of this section. The *canonical transverse* is defined as the subset

$$\mathcal{T}_e := \{Y \in \Omega_X : e \in Y\}.$$

A Borel subset $B \subset \mathcal{T}_e$ is said to be *null* if $\nu(GB) = 0$.

Let us give another characterization of $\mathcal{T}_e$. If $Y \in \mathcal{T}_e$, then there is a sequence $(g_n)_{n \geq 0}$ of elements of G such that $g_n X \to Y$. However, $e \in Y$, so we can find a sequence $x_n \in X$ such that $g_n x_n \to e$. In turn, this means that $x_n^{-1} X \to Y$. Hence,

$$\mathcal{T}_e = \overline{\{x^{-1}X : x \in X\}}.$$

In general, $\mathcal{T}_e$ is a compact subset of Ω_X and satisfies:

(1) $G\mathcal{T}_e = \Omega_X$;

(2) $W \times \mathcal{T}_e \to \Omega_X$ is one-to-one if $W^{-1}W \cap Y_1 Y_2^{-1} = e$ for every $Y_1, Y_2 \in \mathcal{T}_e$.

However, under the assumptions of Theorem 1.3, there is a neighbourhood W of the identity such that $W^{-1}W \cap x_1^{-1}XX^{-1}x_2 = \{e\}$ for every $x_1, x_2 \in X$. So condition (2) holds for this choice of W. Therefore, for any Borel subset $B \subset \mathcal{T}_e$, GB is a Borel subset of Ω_X [**Zim84**, Corollary A.6]. In other words, $\mathcal{T}_e$ is a cross-section.

Remark 5.3. Rather than simply defining null subsets of the canonical transverse, the notion of transverse measure that formalizes the idea of restricting a measure on Ω_X to a measure on $\mathcal{T}_e$ is studied in [**BHK25**]. In fact, there is on $\mathcal{T}_e$ a finite Borel measure η such that the restriction of ν to $U^W = W\mathcal{T}_e$ is equal to the product measure $(\mu_G)_{|W} \otimes \eta$ (see e.g. [**BHK25**, §4]). In particular, if $B \subset \mathcal{T}_e$ is measurable and $g \in G$ satisfies $gB \subset \mathcal{T}_e$, then $\eta(gB) = \eta(B)$. Moreover, a measurable subset $B \subset \mathcal{T}_e$ is null in the sense of Definition 5.2 if and only if $\eta(B) = 0$.

We will only use a simple consequence of the existence of a transverse measure.

LEMMA 5.4. *Let G, X, Ω_X and ν be as in the first paragraph of this section. Let $\phi : \mathcal{T}_e \to H$ be a measurable map taking values in a second countable topological space H. Then, there is $h \in H$ such that for any neighbourhood $V \subset H$ of h, the subset $\phi^{-1}(V)$ is non-null.*

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Lemma 5.4 essentially reduces to saying that the support of the push-forward of the transverse measure is not empty. However, this is guaranteed by the second countability of H.

Definition 5.5. Let G, X, Ω_X and ν be as in the first paragraph of this section. For $Y \in \Omega_X$ and $B \subset \mathcal{T}_e$, the set of *hitting times* is the subset of G defined as follows:

$$T_h(Y, B) := \{g \in G : gY \in B\}.$$

We list now a number of properties satisfied by T_h.

LEMMA 5.6. *Let G, X, Ω_X and ν be as in the first paragraph of this section. With X, $\mathcal{T}_e$ as above, take $Y \in \Omega_X$ and $B \subset \mathcal{T}_e$. Then:*

(1) *for every $g \in G$,*

$$T_h(gY, B) = T_h(Y, B)g^{-1};$$

(2) *if $g \in T_h(Y, B)T_h(Y, B)^{-1}$, then there is $Y' \in B$ such that $gY' \in B$, that is,*

$$g \in \mathcal{R}(B);$$

(3) *we have the inclusions,*

$$T_h(Y, B)T_h(Y, B)^{-1} \subset \bigcup_{Z \in B} Z^{-1} \subset X^{-1}X;$$

(4) *and, for $W \subset G$ open,*

$$\{Y \in \Omega_X : T(Y, B) \cap W \neq \emptyset\} = W^{-1}B.$$

Proof. We have the equivalences,

$$g_0 \in T_h(gY, B) \Leftrightarrow g_0gY \in B \Leftrightarrow g_0g \in T_h(Y, B).$$

So, $T_h(gY, B) = T_h(Y, B)g^{-1}$. This proves part (1). Take $g_1, g_2 \in T_h(Y, B)$. Set $Y_1 := g_1Y$ and $Y_2 := g_2Y$. Then, $Y_1, Y_2 \in B$ and $g_1g_2^{-1}Y_2 = Y_1$. So taking $Y' = Y_2$ works and part (2) is proved. Furthermore, since $e \in Y_1$, there is $y \in Y_2$ such that $g_1g_2^{-1}y = e$, that is, $g_1g_2^{-1} \in Y_2^{-1}$. However, $Y_2 \in B$ so the first inclusion of part (3) is proved. If, now, we take any $Z \in B$, then $Z \in \mathcal{T}_e$. In other words, $e \in Z$. By [**BH18**], we have $Z \subset X^{-1}X$. This proves part (3). Finally, remark that

$$\{Y \in \Omega_X : T(Y, B) \cap W \neq \emptyset\} = \{Y \in \Omega_X : WY \cap B \neq \emptyset\} = W^{-1}B.$$

And part (4) is proved. □

We are now in a position to apply Lemma 5.1 to obtain information on sets of hitting times.

LEMMA 5.7. *Let G, X, Ω_X and ν be as in the first paragraph of this section. Take $B \subset \mathcal{T}_e$ that is not a null subset. Suppose that $Z_0 \subset G$ is any subset such that XZ_0^{-1} is uniformly discrete. Then, for any $Z \in \Omega_{Z_0}$ with $e \in Z$, there is $F \subset G$ finite such that*

$$Z \subset FT_h(Y, B)T_h(Y, B)^{-1}$$

for almost every $Y \in \Omega_X$.

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Proof. By Lemma 5.1, it suffices to show that

$$\begin{aligned} \Omega_X &\longrightarrow \mathcal{C}(G) \\ Y &\longmapsto T_h(Y, B)^{-1} \end{aligned}$$

is a well-defined G-equivariant Borel map, takes values in $\Omega_{X'}$ for X' such that $X'Z_0$ is uniformly discrete and takes non-empty values almost always. We will see this as a consequence of Lemma 5.6. Part (1) of Lemma 5.6 implies G-equivariance and part (2) implies measurability. Since ν is ergodic, there is Y' such that for almost every $Y \in \Omega_X$, $T_h(Y, B)^{-1} \in \Omega_{X'}$ with $X' = T_h(Y', B)$. In particular, $X'^{-1}X' \subset X^{-1}X$ so $X'Z_0^{-1}$ is uniformly discrete. It remains to prove that $T_h(Y, B)$ takes non-empty values for ν-almost every Y. However, $T_h(Y, B)$ is non-empty as soon as $Y \in G \cdot B$. Since B cannot be null, $G \cdot B$ is a G-invariant Borel subset of Ω_X of full measure and, hence, $\nu(G \cdot B) = 1$ as ν is ergodic. □

5.3. *Cocycles on the invariant hull.* The kind of cocycles we are going to work with were first considered in [**BH20**] to study certain Kazhdan-type and Haager-up-type properties of uniform model sets. In [**Mac23**], we considered related cocycles on the extended invariant hull to show that superrigidity theorems hold for ⋆-approximate lattices. These cocycles are built from Borel sections of the invariant hull.

Definition 5.8. Let G, X, Ω_X and ν be as in the first paragraph of this section. A map $s : \Omega_X \to G$ is a *Borel section* if it is Borel, defined ν-almost everywhere and for ν-almost every $Y \in \Omega_X$, we have $s(Y) \in Y$. Given a Borel section s, one can build a cocycle defined for every $g \in G$ and ν-almost every $Y \in \Omega_X$ by

$$\alpha_s(g, Y) := s(gY)^{-1} g s(Y).$$

As it is stated here, our definition implies that if ν is not proper, then no Borel section exists. We refer to [**BH20**, **Mac23**] for this and more concerning these specific cocycles on invariant hulls.

5.4. *Restricting maps to the canonical transverse.* When X is a subgroup and, thus, $\Omega_X \setminus \{\emptyset\}$ is a transitive space, Zimmer uses a result of Mackey to show that cocycle superrigidity implies Margulis' superrigidity theorem, see [**Mar91**, **Zim84**]. Mackey's result asserts that a cocycle on a transitive space can be completely understood simply by looking at its restriction at one point. Precisely, given $Y \in \Omega_X \setminus \{\emptyset\}$, the map that sends a cocycle α to $\alpha_{|\{x\}\times \mathrm{Stab}(x)}$ provides a bijection between cohomology classes of cocycles $\Omega_X \setminus \{\emptyset\} \times G \to H$ and group homomorphisms $\mathrm{Stab}(x) \to H$ [**Zim84**, 4.2.15, 4.2.16 and 5.2.5].

When X is not assumed to be a subgroup, $\Omega_X \setminus \{\emptyset\}$ is not a transitive space any more. However, as observed by Björklund, Hartnick and Karasik (private communication), we can partially generalize Mackey's result by considering the restriction of cocycles of certain types to the canonical transverse.

LEMMA 5.9. (Björklund, Hartnick and Karasik) *Let G, X, Ω_X and ν be as in the first paragraph of this section. Let $s : \Omega_X \to G$ denote a Borel section. Let $\tau : \langle X \rangle \to \mathbb{H}(k)$ be a group homomorphism, where k is a local field and $\mathbb{H}$ is a simple algebraic group defined over k. Suppose that the cocycle $\tau \circ \alpha_s$ satisfies the conclusions of Zimmer's cocycle superrigidity theorem. Namely:*

(i) *either $\tau \circ \alpha_s$ is cohomologous with a cocycle taking values in a compact subgroup of $\mathbb{H}(k)$;*

(ii) *or there are a continuous group homomorphism $\pi : G \to \mathbb{H}(k)$ and a measurable map $\phi : \Omega_X \to \mathbb{H}(k)$ such that for every $g \in G$ and for ν-almost every $Y \in \Omega_X$,*

$$\tau \circ \alpha_s(g, Y) = \phi(gY)^{-1}\pi(g)\phi(Y).$$

Then, there are a G-invariant Borel subset Ω of full measure and a measurable map $\psi : \mathcal{T}_e \cap \Omega \to H$ such that:

(1) *in case (i), $\psi(gY)\tau(g)\psi(Y)^{-1}$ takes values in a compact subgroup of $\mathbb{H}(k)$ when Y ranges through $\Omega \cap \mathcal{T}_e$ and g ranges through $T_h(Y, \Omega \cap \mathcal{T}_e)$;*

(2) *in case (ii), there is a continuous group homomorphism $\pi : G \to \mathbb{H}(k)$ such that for every $Y \in \Omega \cap \mathcal{T}_e$ and $g \in T_h(Y, \Omega \cap \mathcal{T}_e)$,*

$$\tau(g) = \psi(gY)^{-1}\pi(g)\psi(Y).$$

Proof. We explain the proof under assumption (ii), the proof of part (1) will then follow with minor changes. By a Fubini argument, for ν-almost every $Y \in \Omega_X$ and almost every $g \in G$, we have

$$\tau \circ \alpha_s(g, Y) = \phi(gY)^{-1}\pi(g)\phi(Y). \tag{5.1}$$

According to [**Zim84**, Lemma B.8], we can therefore find a G-invariant Borel subset $\Omega \subset \Omega_X$ of measure 1 and a Borel map $s' : \Omega \to G$ such that for every $Y \in \Omega$ and almost every $g \in G$, (5.1) holds at $(g, s'(Y)Y)$—in other words,

$$\tau \circ \alpha_s(g, s'(Y)Y) = \phi(gs'(Y)Y)^{-1}\pi(g)\phi(s'(Y)Y). \tag{5.2}$$

Note moreover that $s'(Y) = e$ for almost every $e \in \Omega$. By a further Fubini argument, the G-invariant subset $\{Y \in \Omega : s'(gY) = e$ for almost every $g \in G\}$ is Borel and co-null in Ω. So, we may assume that it is equal to Ω. For every $Y \in \Omega$, set $s''(Y) = s'(Y)^{-1}s(s'(Y)Y)$. Since s is a Borel section, we get

$$s''(Y) = s'(Y)^{-1}s(s'(Y)Y) \in s'(Y)^{-1}s'(Y)Y = Y.$$

So s'' is a Borel section as well. Fix $Y \in \Omega$. For almost every $g \in G$, we have $s'(gY) = e$. Hence, (5.2) holds at $(s'(gY)gs'(Y)^{-1}, s'(Y)Y)$ for almost every $g \in G$. So, for every $Y \in \Omega$ and almost every $g \in G$, (5.2) becomes

$$\tau \circ \alpha_{s''}(g, Y) = \tau \circ \alpha_s(s'(gY)gs'(Y)^{-1}, Y) = \psi(gY)^{-1}\pi(g)\psi(Y), \tag{5.3}$$

where $\psi(Y) := \pi(s'(Y))^{-1}\phi(s'(Y)Y)$. Therefore, for every $Y \in \Omega$ and $h \in G$, we have

$$\begin{aligned}\pi(h)\psi(Y) &= \pi(g)^{-1}\psi(ghY)\tau \circ \alpha_{s''}(gh, Y) && \text{for almost every } g \in G\\ &= \pi(g)^{-1}\psi(ghY)\tau \circ \alpha_{s''}(g, hY)\tau \circ \alpha_{s''}(h, Y) && \text{for almost every } g \in G\\ &= \psi(hY)\tau \circ \alpha_{s''}(h, Y),\end{aligned}$$

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where we have used (5.3) in the first line, the cocycle identity to deduce the second line and (5.3) again to get to the last line. Notice now that for every $Y \in \Omega$, we have $s(s'(Y)Y)^{-1}s'(Y)s(Y) \in Y^{-1}Y \subset X^{-1}X$. So, we can unfold the definitions of s'' and $\alpha_{s''}$ to find that, for every $g \in G$ and every $Y \in \Omega$, we have

$$\tau \circ \alpha_s(g, Y) = \chi(gY)^{-1}\pi(g)\chi(Y),$$

where

$$\chi(Y) := \pi(s'(Y))^{-1}\phi(Y)\tau(s(s'(Y)Y)^{-1}s'(Y)s(Y)).$$

Finally, since for $Y \in \Omega \cap \mathcal{T}_e$ and $g \in T_h(Y, \Omega \cap \mathcal{T}_e)$ we have $s(Y) = s(gY) = e$, we get Lemma 5.9. □

Remark 5.10. One might note that the Borel section chosen at the beginning of Lemma 5.9 does not appear in the latter part of the statement. This highlights the easily established fact that if s' is some other Borel section, then $\tau \circ \alpha_s$ and $\tau \circ \alpha_{s'}$ are cohomologous. In addition, the map ϕ realizing the cohomology relation takes values in $\tau(X^{-1}X)$.

We will follow a similar strategy to prove a related result about restrictions of cocycles acting on Banach spaces. We will then use this in our considerations on property (T).

LEMMA 5.11. *Let G, X, Ω_X and ν be as in the first paragraph of this section. Suppose that $\langle X\rangle$ acts by isometries on a Banach space $(B, \ ||\cdot||)$. Let $\phi : \Omega_X \to B$ be a measurable map such that for every $g \in G$ and ν-almost every $Y \in \Omega_X$, we have*

$$\alpha_s(g, Y)\phi(Y) = \phi(gY).$$

Then, there are a Borel G-invariant subset $\Omega \subset \Omega_X$ of full measure and $\psi : \Omega \cap \mathcal{T}_e \to B$ a measurable map such that for every $Y \in \Omega \cap \mathcal{T}_e$ and every $g \in T_h(Y, \Omega \cap \mathcal{T}_e)$, we have

$$g \cdot \psi(Y) = \psi(gY).$$

Moreover, if ϕ takes values in the unit ball of B, then so does ψ.

Proof. Let μ_G denote the Haar measure on G. By a Fubini-type argument, there is $\Omega_0 \subset \Omega_X$ of full measure such that for every $Y \in \Omega_0$ and almost every $g \in G$, we have $\alpha_s(g, Y)\phi(Y) = \phi(gY)$. By [**Zim84**, Lemma B.8], we can find $\Omega \subset \Omega_X G$-invariant and of measure 1 and a Borel map $s' : \Omega \to G$ such that for every $Y \in \Omega$ and almost every $g \in G$, we have

$$\alpha_s(g, s'(Y)Y)\phi(s'(Y)Y) = \phi(gs'(Y)Y).$$

Set $s''(Y) := s'(Y)^{-1}s(s'(Y)Y)$ for every $Y \in \Omega$. Then, s'' is a Borel section of Ω (Definition 5.8) that satisfies $s''(Y) = e$ for every $Y \in \Omega \cap \mathcal{T}_e$. Furthermore, one can ensure—proceeding as in the proof of Lemma 5.9—that for every $Y \in \Omega$ and almost every $g \in G$, we have

$$\alpha_{s''}(g, Y)\phi(s'(Y)Y) = \phi(s'(gY)gY).$$

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Let $\psi : \Omega \to B$ be defined by $\psi(Y) = \phi(s'(Y)Y)$. For every $Y \in \Omega$, we have

$$\alpha_{s''}(g, Y)^{-1}\psi(gY) = \psi(Y)$$

for almost every $g \in G$. Moreover, we have for every $Y \in \Omega$ and $h \in G$,

$$\begin{aligned}\psi(hY) &= \alpha_{s''}(g, hY)^{-1}\psi(ghY) && \text{for almost every } g \in G\\ &= \alpha_{s''}(h, Y)\alpha_{s''}(gh, Y)^{-1}\psi(ghY) && \text{for almost every } g \in G\\ &= \alpha_{s''}(h, Y)\psi(Y),\end{aligned}$$

where we went from the first line to the second using the cocycle identity. We conclude as in the proof of Lemma 5.9. □

6. *Lagarias-type result in simple Lie groups*

We will now prove Theorem 1.3. In the following, we will consider a subset X of the group of points G of an absolutely simple algebraic group defined over a local field k of characteristic 0. Assume also that Ω_X admits a proper ergodic G-invariant Borel probability measure, say ν. We will assume moreover that both $X^{-1}X$ and XX^{-1} are uniformly discrete. We start with a general result concerning equivariant families of discrete subsets.

6.1. *Superrigidity.*

We will now prove the superrigidity theorem needed in the last part of the proof.

THEOREM 6.1. *Let $X \subset G$ be such that $X^{-1}X$, XX^{-1} are uniformly discrete and Ω_X admits a proper G-invariant Borel probability measure ν. Then, there is $S \subset X^{-1}X$ such that for every $\tau : \langle X\rangle \to \mathbb{H}(k)$ group homomorphism with k a local field and $\mathbb{H}$ a simple algebraic group defined over k such that $\dim_k(\mathbb{H}) \leq \dim(G)$, we have:*

(1) *either $\tau(X)$ is relatively compact;*
(2) *or there exists $\pi : G \to \mathbb{H}(k)$ a continuous group homomorphism such that $X \subset FS$ for some finite set F and $\tau_{|S} = \pi_{|S}$.*

Proof. Fix a Borel section $s : \Omega_X \to G$. Since $\dim_k(\mathbb{H}) \leq \dim(G)$ and G is a property (T) simple algebraic group, we can apply [**Mac23**, Proposition 14]. So, $\tau \circ \alpha_s$ satisfies the conclusions of Zimmer's cocycle superrigidity. Therefore, we may apply Lemma 5.9.

If we have a measurable map $\psi : \Omega \cap \mathcal{T}_e \to \mathbb{H}(k)$ and a compact subgroup $K \subset \mathbb{H}(k)$ such that $\psi(gY)\tau(g)\psi(Y)^{-1} \subset K$ for every $Y \in \Omega \cap \mathcal{T}_e$ and $g \in T_h(Y, \Omega \cap \mathcal{T}_e)$ (part (1) of Lemma 5.9), we will prove that condition (1) of Theorem 6.1 holds. There is $B \subset \mathcal{T}_e$ non-null such that $K' := \overline{\psi(B)}$ is compact (Lemma 5.4). So, if we have $g \in G$, $Z \in B$ and $gZ \in B$, then $\tau(g) \in K'^{-1}KK'$. Therefore, for every $Y \in \Omega_X$, we have

$$\tau(T(Y, B)T(Y, B)^{-1}) \subset K'^{-1}KK'.$$

However, X is covered by finitely many translates of $\bigcup_Y T(Y, B)T(Y, B)^{-1}$ (Lemma 5.7), so $\tau(X)$ is relatively compact.

Suppose that we are given a measurable map $\psi : \Omega \cap \mathcal{T}_e \to \mathbb{H}(k)$ and a continuous group homomorphism $\pi : G \to \mathbb{H}(k)$ such that for every $Y \in \Omega \cap \mathcal{T}_e$ and

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$g \in T_h(Y, \Omega \cap \mathcal{T}_e)$, we have $\tau(g) = \psi(gY)^{-1}\pi(g)\psi(Y)$ (corresponding to part (2) of Lemma 5.9). We will prove that condition (2) of Theorem 6.1 must hold. We will follow the strategy of [**Mac23**]. By Lemma 5.4, there is $h \in \mathbb{H}(k)$ such that for every compact neighbourhood V of h, there is $B_V \subset \mathcal{T}_e$ non-null such that $\psi(B_V) \subset V$. Upon modifying ψ, we may assume without loss of generality that $h = e$. Now, for any $Y_0 \in \mathcal{T}_e$ and any $Y \in \Omega_X$, we have that YY_0^{-1} is uniformly discrete. Therefore, for any V neighbourhood of e, we know by Lemma 5.7 that there is a finite set $F = F(Y_0, V)$ such that

$$Y_0 \subset FT_h(Y, B_V)T_h(Y, B_V)^{-1}.$$

Take $y \in Y_0$, then there is $f \in F$ such that $f^{-1}y \in T_h(Y, B_V)T_h(Y, B_V)^{-1}$. By Lemma 5.6, there is $Y_1 \in B_V$ such that $f^{-1}yY_1 \in B_V$ as well. In particular, this yields

$$\tau(y) = \tau(f)\tau(f^{-1}y) = \tau(f)\psi(f^{-1}yY_1)\pi(f^{-1}y)\psi(Y_1) \in \tau(F)V\pi(F)^{-1}\pi(y)V.$$

Write $K_V = \tau(F)V\pi(F)^{-1}$, then we have the equality

$$\tau(y) \in K_V\pi(y)V$$

for every $y \in Y_0$.

We have shown in [**Mac23**, §3.4] that if $\gamma \in G$ satisfies that the projection of $\pi(Y_0 \cap Y_0\gamma^{-1})$ to $K \setminus H$ has dense closure in the (right-)visual boundary $\partial(K \setminus H)$—where K denotes a maximal compact subgroup—then $\tau(\gamma) = \pi(\gamma)$. Since π is surjective according to our assumption on dimensions, it in fact suffices to show: (*) the projection of $Y_0 \cap Y_0\gamma^{-1}$ to $K \setminus G$ has dense closure in the (right-)visual boundary $\partial(K \setminus G)$—where K denotes a maximal compact subgroup of G this time. We will show that this condition is fulfilled for many $\gamma \in G$ by invoking pointwise ergodic theorems.

Note first that the map

$$\begin{aligned} I_\gamma : \Omega_X &\longrightarrow \Omega_X^{\text{ext}} \\ Y &\longmapsto Y \cap Y\gamma^{-1} \end{aligned}$$

is a well-defined G-equivariant map. Let us show that it is Borel when $\gamma \in X$. As a consequence of the uniform discreteness of XX^{-1}, there is a neighbourhood of the identity $W \subset G$ such that for every $Y \in \Omega_X$ and every $g \in G$, we have $|YX^{-1} \cap gW| \leq 1$. So,

$$\begin{aligned} |Y \cap gW| + |Y\gamma^{-1} \cap gW| - |Y \cap Y\gamma^{-1} \cap gW| &= |Y \cup Y\gamma^{-1} \cap gW| \\ &\leq |YX^{-1} \cap gW| \\ &\leq 1. \end{aligned}$$

Therefore, $Y \cap Y\gamma^{-1} \cap gW$ is non-empty if and only if both $Y \cap gW$ and $Y\gamma^{-1} \cap gW$ are non-empty. In other words,

$$\{Y \in \Omega_X : Y \cap Y\gamma^{-1} \cap gW \neq \emptyset\} = U^{gW} \cap U^{gW\gamma}.$$

So, I_γ is Borel (see e.g. [**Mac23**, §2.1.2]). As a consequence, the push-forward ν_γ of ν is a G-invariant ergodic measure. It is either proper or concentrated on $\{\emptyset\}$.

Fix $D \subset G$ countable such that the sets of $\xi \in \partial(K \setminus G)$ such that $x_0 \cdot d^n \to \xi$ is dense. Here, x_0 denotes the class of e in $(K \setminus G)$. Take $\gamma \in G$ and suppose that ν_γ is

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proper. Then, the semi-group generated by d^{-1} acts ergodically on (Ω_X, ν) according to the Howe–Moore property (e.g. [**Zim84**, 2.2.20]. So, for every $W \subset G$ open and relatively compact, and almost every $Y \in \Omega_X$, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |d^{-k}(Y \cap Y\gamma^{-1}) \cap W| = \nu(U^W).$$

In particular, there is a sequence of elements (y_n) of $Y \cap Y\gamma^{-1}$ such that $d(y_n, e) \to \infty$ as n goes to ∞ and $y_n \in d^{i_n} W$ for some $i_n \geq 0$. By considering the above for W running over a countable basis of neighbourhoods of the identity, we find that γ satisfies condition (*) for almost every $Y \in \Omega_X$.

Let us now show that there are many $\gamma \in G$ such that ν_γ is proper. Define

$$S := \{x \in X^{-1}X : \nu_x \text{ is proper}\}$$

and fix W a relatively compact open neighbourhood of the identity. If X^{-1} is not covered by finitely many right-translates of S, then we can find a sequence $(x_n)_{n\geq 0}$ of elements of X^{-1} such that for any two $i \neq j$, $\nu_{x_i x_j^{-1}}$ is concentrated on $\{\emptyset\}$. In particular,

$$\int_{\Omega_X} |Yx_j \cap Yx_i \cap W| \, d\nu(Y) = \int_{\Omega_X} |Y \cap Yx_i x_j^{-1} \cap Wx_j^{-1}| \, d\nu(Y) = 0$$

for every $i, j \geq 0$. However, for every $N \geq 0$ and $Y \in \Omega_X$, we have

$$|YX^{-1} \cap W| \geq \left| \bigcup_{i=0}^{N} Yx_i \cap W \right| \geq \sum_{i=0}^{N} |Yx_i \cap W| - \sum_{0\leq i<j\leq N} |Yx_j \cap Yx_i \cap W|. \tag{6.1}$$

Write μ_G the pull-back of ν via the periodization map. We know that μ_G is a Haar measure. In addition, since XX^{-1} is uniformly discrete, $|YX^{-1} \cap W| \leq C$ for some constant C. Integrating (6.1) against ν, we find

$$C \geq \sum_{i=0}^{N} \mu_G(Wx_i^{-1}) = (N+1)\mu_G(W)$$

for every $N \geq 0$. This is a contradiction. So, there is $F \subset X^{-1}$ finite such that $X^{-1} \subset SF$. Since $\tau_{|S} = \pi_{|S}$, we conclude that S is the subset for which we are looking. □

6.2. *Finite generation and property (T).* The goal of this section is to prove the following key result.

PROPOSITION 6.2. *With X as in Theorem 1.3, the subgroup $\langle X \rangle$ is finitely generated.*

To prove Proposition 6.2, we follow the same strategy as in [**Mac23**] with the additional input of Lemma 5.11 to help us restrict the cocycle we consider to the canonical transverse.

PROPOSITION 6.3. *Let $\langle X \rangle$ act on a separable Hilbert space $\mathcal{H}$ with almost invariant vectors. Then, there is $\xi \in \mathcal{H}$ a unit vector such that $X \cdot \xi$ is relatively compact in the norm topology.*

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Proof. Following Zimmer's approach [**Zim84**] (see also [**Mac23**]), we can find a non-trivial $\phi : \Omega_X \to G$ such that for every $g \in G$ and ν-almost every $Y \in \Omega_X$, we have

$$\pi(\alpha_s(g, Y))(\phi(Y)) = \phi(gY).$$

In particular, $||\phi(Y)|| = ||\phi(gY)||$. So $||\phi(Y)||$ is constant for almost every Y by ergodicity of ν. We may therefore assume that ϕ takes values in the set of unit vectors of $\mathcal{H}$. By Lemma 5.11, there are thus $\Omega \subset \Omega_X$ that is G-invariant and of full measure, and $\psi : \Omega \cap \mathcal{T}_e \to \mathcal{H}$ measurable such that

$$\pi(g) \cdot \psi(Y) = \psi(gY)$$

for every $Y \in \Omega \cap \mathcal{T}_e$ and $g \in T_h(Y, \Omega \cap \mathcal{T}_e)$.

Take $\xi \in \mathcal{H}$ as provided by Lemma 5.4. Then, for every $\epsilon > 0$, the subset

$$\mathcal{T}_{e,\epsilon} := \psi^{-1}(B(\xi, \epsilon))$$

is non-null. According to Lemma 5.7,

$$X \subset FT_h(Y, \mathcal{T}_{e,\epsilon})T_h(Y, \mathcal{T}_{e,\epsilon})^{-1}$$

for some finite subset F and some $Y \in \Omega$. For every $g \in T_h(Y, \mathcal{T}_{e,\epsilon})T_h(Y, \mathcal{T}_{e,\epsilon})^{-1}$, we can find $Y' \in \mathcal{T}_{e,\epsilon}$ such that $gY' \in \mathcal{T}_{e,\epsilon}$. So,

$$\psi(gY') = \pi(g)(\psi(Y')).$$

Since $\psi(Y'), \psi(gY') \in B(\xi, \epsilon)$, we have $\pi(g)(\xi) \in B(\xi, 2\epsilon)$. To sum up, we have $\pi(X)(\xi) \subset \pi(F)(B(\xi, 2\epsilon))$. Since there is such an F for every ϵ and $\mathcal{H}$ is complete, $\pi(X)(\xi)$ is relatively compact. □

Proof of Proposition 6.2. The proof of [**Mac23**, Proposition 13] holds almost verbatim. We reproduce it here for the sake of completeness. Let $\mathcal{H}$ denote $\bigoplus_{\Delta \subset \langle X \rangle \text{ f.g.}} L^2(\langle X \rangle / \Delta)$. The sum of quasi-regular representations $(\pi, \mathcal{H})$ almost has invariant vectors (e.g. [**BdlHV08**]). So, we can find $\phi \in \mathcal{H}$ such that $X \cdot \phi$ has compact closure in the norm topology. Upon projecting ϕ to a well-chosen factor of $\mathcal{H}$, we may assume that $\phi \in L^2(\langle X \rangle / \Delta)$ for some finitely generated subgroup Δ of $\langle X \rangle$. Now, define

$$X(\delta, \phi) := \{x \in X^{-1}X : ||x \cdot \phi - \phi|| \leq \delta\}.$$

Then, $(X(\delta, \phi))_{\delta > 0}$ is a family of subsets such that X is covered by finitely many translates of $X(\delta, \phi)$ and $X(\delta, \phi)$ is contained in $X^{-1}X$ (see [**Mac23**]). Now, let $p : \langle X \rangle \to \langle X \rangle / \Delta$ denote the natural projection. Take $\gamma \in \langle X \rangle$ such that $\phi(p(\gamma)) = \alpha > 0$. So, for every $x \in X(\alpha/2, \phi)$, we have

$$|\phi(p(x^{-1}\gamma)) - \phi(p(\gamma))| \leq ||\pi(x)(\phi) - \phi|| < \alpha/2,$$

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meaning $x^{-1}\gamma \in \phi^{-1}([\alpha/2; +\infty))$. Since $\phi^{-1}([\alpha/2; +\infty))$ is finite, we can find a finite set F of representatives of $\phi^{-1}([\alpha/2; +\infty))$ in $\langle X\rangle$. Then, $x^{-1}\gamma\Delta \cap F\Delta \neq \emptyset$ and $X(\alpha/2, \phi)$ is contained in $F\Delta\gamma^{-1}$. However, there is a finite subset $F' \subset \langle\Lambda\rangle$ such that $X \subset F'X(\alpha/2, \phi) \subset F'F\Delta\gamma^{-1}$. Since

$$X \subset F' \cup F \cup \Delta \cup \{\gamma^{-1}\} \subset \langle X\rangle,$$

$F' \cup F \cup \Delta \cup \{\gamma^{-1}\}$ generates $\langle X\rangle$. However, Δ is finitely generated. So, $\langle X\rangle$ is finitely generated. □

6.3. *Proof of Theorem 1.3.*

Proof. Suppose as we may that G is a closed subgroup of $\mathrm{GL}_n(k)$ for some n. According to Proposition 6.2, the subgroup $\langle X\rangle$ is finitely generated. So the field K generated by entries of elements of $\langle X\rangle$ is finitely generated. Let $S \subset XX^{-1}$ be the subset provided by Theorem 6.1. We will show that S^m is uniformly discrete for every $m \geq 0$. Let W be a compact neighbourhood of the identity and take an integer $m \geq 0$. If $S^{2m} \cap W$ is infinite, then the set E of entries of elements in $S^{2m} \cap W$ is infinite as well. By a result of Breuillard and Gelander [**BG03**], there is a local field l and a field embedding $\sigma : K \to l$ such that $\sigma(E)$ is unbounded. Denote by σ_0 the natural group homomorphism $\mathrm{GL}_m(K) \to \mathrm{GL}_m(l)$ given by applying σ entry-wise. Then, $\sigma_0(S^{2m} \cap W)$ is unbounded. As

$$\sigma_0(S^{2m} \cap W) \subset \sigma_0((XX^{-1})^{2m}),$$

the subset $\sigma_0(X)$ must be unbounded. Moreover, the Zariski-closure of $\sigma_0(\langle X\rangle)$ is semi-simple and has dimension at most $\dim G$. By Theorem 6.1, there is a continuous group homomorphism $\pi : G \to \mathrm{GL}_m(l)$ such that $(\sigma_0)_{|\langle S\rangle} = \pi_{|\langle S\rangle}$. However,

$$\sigma_0(S^{2m} \cap W) = \pi(S^{2m} \cap W) \subset \pi(W).$$

So, $\sigma_0(S^{2m} \cap W)$ is bounded. This is a contradiction.

Therefore, $S^{2m} \cap W$ is finite for every $m \geq 0$. So, S^m is uniformly discrete for every $m \geq 0$. However, there is F finite such that $X \subset FS$. By Lemma 3.3, there is thus a Borel subset $\mathcal{F}$ of finite Haar measure such that $S^2\mathcal{F} = G$. Since S^5 is uniformly discrete, we find that S^2 is an approximate subgroup as a consequence of Lemma 3.2. Hence, S^2 is an approximate lattice. □

Remark 6.4. This last part of the argument is the only one that is not quantitative. We do not know if this argument can be made quantitative as we heavily rely on the superrigidity theorem which is, by essence, global.

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REFERENCES

[AJTZ21] J. An, Y. Jing, C.-M. Tran and R. Zhang. On the small measure expansion phenomenon in connected noncompact nonabelian groups. *Preprint*, 2021, arXiv:2111.05236.

[BdlHV08] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T) (New Mathematical Monographs, 11)*. Cambridge University Press, Cambridge, 2008.

[BG03] E. Breuillard and T. Gelander. On dense free subgroups of Lie groups. *J. Algebra* **261**(2) (2003), 448–467.

[BG08] J. Bourgain and A. Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)* **167**(2) (2008), 625–642.

[BG13] M. Baake and U. Grimm. *Aperiodic Order: A Mathematical Invitation. Volume 1 (Encyclopedia of Mathematics and its Applications, 149).* Cambridge University Press, Cambridge, 2013; with a foreword by Roger Penrose.

[BGT12] E. Breuillard, B. Green and T. Tao. The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 115–221.

[BH18] M. Björklund and T. Hartnick. Approximate lattices. *Duke Math. J.* **167**(15) (2018), 2903–2964.

[BH20] M. Björklund and T. Hartnick. Analytic properties of approximate lattices. *Ann. Inst. Fourier (Grenoble)* **70**(5) (2020), 1903–1950.

[BHK25] M. Björklund, T. Hartnick and Y. Karasik. Intersection spaces and multiple transverse recurrence. *J. Anal. Math.* **156** (2025), 97–150.

[BW92] J. Block and S. Weinberger. Aperiodic tilings, positive scalar curvature, and amenability of spaces. *J. Amer. Math. Soc.* **5**(4) (1992), 907–918.

[CHT24] M. Cordes, T. Hartnick and V. Tonić. *Foundations of Geometric Approximate Group Theory.* (*Memoirs of the American Mathematical Society*) American Mathematical Society, Providence, RI, 2024.

[dB81a] N. G. de Bruijn. Algebraic theory of Penrose's non-periodic tilings of the plane. II. *Nederl. Akad. Wetensch. Indag. Math.* **43**(1) 1981, 39–52, 53–66.

[dB81b] N. G. de Bruijn. Algebraic theory of Penrose's non-periodic tilings of the plane. *Kon. Nederl. Akad. Wetensch. Proc. Ser. A* **43**(84) (1981), 1–7.

[ER22] U. Enstad and S. Raum. A dynamical approach to non-uniform density theorems for coherent systems. *Preprint*, 2022, arXiv:2207.05125.

[EV25] U. Enstad and J. Velthoven. Coherent systems over approximate lattices in amenable groups. *Ann. Inst. Fourier (Grenoble)* **75**(6) (2025), 2293–2315.

[Gre69] F. P. Greenleaf. *Invariant Means on Topological Groups and Their Applications (Van Nostrand Mathematical Studies, 16)*. Van Nostrand Reinhold Co., New York–Toronto–London, 1969.

[Hel08] H. A. Helfgott. Growth and generation in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. *Ann. of Math. (2)* **167**(2) (2008), 601–623.

[HK22] N. Heuer and D. Kielak. Quasi-BNS invariants. *Preprint*, 2022, arXiv:2210.10607.

[Hru12] E. Hrushovski. Stable group theory and approximate subgroups. *J. Amer. Math. Soc.* **25**(1) (2012), 189–243.

[Hru20] E. Hrushovski. Beyond the lascar group. *Preprint*, 2022, arXiv:2011.12009.

[HW22] T. Hartnick and S. Witzel. Higher finiteness properties of arithmetic approximate lattices: the rank theorem for number fields. *Preprint*, 2022, arXiv:2204.01496.

[JTZ21] Y. Jing, C.-M. Tran and R. Zhang. A nonabelian Brunn–Minkowski inequality. *Preprint*, 2021, arXiv:2101.07782.

[Kon23] J. Konieczny. Characterisation of Meyer sets via the Freiman–Ruzsa theorem. *J. Number Theory* **253** (2023), 278–294.

[Lag96] J. C. Lagarias. Meyer's concept of quasicrystal and quasiregular sets. *Comm. Math. Phys.* **179**(2) (1996), 365–376.

[Lin01] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.* **146**(2) (2001), 259–295.

[LO15] N. Lev and A. Olevskii. Quasicrystals and Poisson's summation formula. *Invent. Math.* **200**(2) (2015), 585–606.

[Mac20] S. Machado. Approximate lattices and Meyer sets in nilpotent Lie groups. *Discrete Anal.* **1** (2020), 18 pp.

[Mac22] S. Machado. Discrete approximate subgroups of Lie groups. *PhD Thesis*, University of Cambridge, 2022.

[Mac23] S. Machado. Approximate lattices in higher-rank semi-simple groups. *Geom. Funct. Anal.* **33**(4) (2023), 1101–1140.

[Mac25] S. Machado. Closed approximate subgroups: compactness, amenability and approximate lattices. *Forum Math. Sigma* **13** (2025), Paper no. e5.

https://doi.org/10.1017/etds.2025.10253 Published online by Cambridge University Press

[Mar91] G. A. Margulis. *Discrete Subgroups of Semisimple Lie Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 17)*. Springer-Verlag, Berlin, 1991.

[Mey72] Y. Meyer. *Algebraic Numbers and Harmonic Analysis. (North-Holland Mathematical Library, 2)* North-Holland Publishing Company, Amsterdam, 1972.

[MM98] G. A. Margulis and S. Mozes. Aperiodic tilings of the hyperbolic plane by convex polygons. *Israel J. Math.* **107** (1998), 319–325.

[Moo97] R. V. Moody. Meyer sets and their duals. *NATO ASI Ser. C Math. Phys. Sci.-Adv. Study Inst.* **489** (1997), 403–442.

[Moz97] S. Mozes. Aperiodic tilings. *Invent. Math.* **128**(3) (1997), 603–611.

[MW15] J.-C. Massicot and F. O. Wagner. Approximate subgroups. *J. Éc. polytech. Math.* **2** (2015), 55–64.

[OW87] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Anal. Math.* **48** (1987), 1–141.

[Pet12] G. Petridis. New proofs of Plünnecke-type estimates for product sets in groups. *Combinatorica* **32**(6) (2012), 721–733.

[PRS22] F. Pogorzelski, C. Richard and N. Strungaru. Leptin densities in amenable groups. *J. Fourier Anal. Appl.* **28**(6) (2022), 85.

[Ruz92] I. Z. Ruzsa. Arithmetical progressions and the number of sums. *Period. Math. Hungar.* **25**(1) (1992), 105–111.

[Ruz99] I. Z. Ruzsa. An analog of Freiman's theorem in groups. *Astérisque* **258** (1999), 323–326; Structure theory of set addition.

[Shu88] A. Shulman. Maximal ergodic theorems on groups. *Dep. Lit. NIINTI* **2184** (1988).

[Sie45] C. L. Siegel. A mean value theorem in geometry of numbers. *Ann. of Math. (2)* **46**(2) (1945), 340–347.

[Tao08] T. Tao. Product set estimates for non-commutative groups. *Combinatorica* **28**(5) (2008), 547–594.

[Yam53] H. Yamabe. A generalization of a theorem of Gleason. *Ann. of Math. (2)* **58**(2) (1953), 351–365.

[Zim84] R. J. Zimmer. *Ergodic Theory and Semisimple Groups (Monographs in Mathematics, 81)* Birkhauser, Boston, MA, 1984.