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AN ALGORITHM FOR THE PERMANENT OF CIRCULANT MATRICES

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1. Introduction. The *permanent* of an $n \times n$ matrix $A = (a_{ij})$ is the matrix function

$$\text{per } A = \sum_{\pi \in S_n} a_{1\pi(1)} \cdots a_{n\pi(n)} \tag{1}$$

where the summation is over all permutations in the symmetric group, S_n. An $n \times n$ matrix A is a *circulant* if there are scalars $a_1, \ldots, a_n$ such that

$$A = \sum_{i=1}^{n} a_i P^{i-1} \tag{2}$$

where P is the $n \times n$ permutation matrix corresponding to the cycle $(12 \cdots n)$ in S_n. In general the computation of the permanent function is quite difficult chiefly because it is not invariant under addition of a multiple of one row to another. Using the principle of "inclusion and exclusion", Ryser [6, p. 27] gave an expansion for the permanent. Also the Laplace expansion is available for the permanent [2, p. 20]. Neither of these methods are particularly efficient. In [4] Minc considered the permanents of matrices with entires either 0 or 1. Minc also studied tridiagonal circulants in [5]. Metropolis, Stein, and Stein [3] have given recurrence relations for evaluating the permanents of circulant matrices (2) where the first k scalars are 1 and the remaining ones are 0. Permanents of circulant matrices were also studied by Tinsley [7].

2. The algorithm. If we consider the scalars as indeterminates over an underlying field every term of the permanent (1) of a circulant matrix (2) is a monomial in the scalars $a_1, \ldots, a_n$. Our algorithm deletes appropriate monomials from the set of all n^n such monomials until only those appearing in the permanent remain. This is easily programmed because the monomials need only be considered one at a time and may be indexed by the n^n n-tuples chosen from $1, \ldots, n$ and ordered lexicographically. It is convenient to state the algorithm in terms of these indices.

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Algorithm. If $I=(i_1,\ldots,i_n)$ is an n-tuple with entries chosen from $1,\ldots,n$ then discard I if

(i) $\sum_{j=1}^{n} i_j \neq 0 \pmod{n}$,

or if

(ii) $i_{j+k} \equiv i_j - k \pmod{n}$ for any k and $j=1,\ldots,n-1$.

Condition (ii) excludes the occurrence of terms in the permanent of (2) with the following pattern

$$\cdots a_i \underbrace{*\cdots*}_{k-2\text{ entries}} a_{i+k+1} \cdots \tag{3}$$

where a_{i+n} is considered to be a_i if necessary. For example, if $n=4$ condition (ii) of the algorithm discards a monomial whenever one of the following patterns occurs:

$$\cdots 14 \ldots, \ldots 21 \ldots, \ldots 32 \ldots, \ldots 43 \cdots$$
$$\cdot\, 1{*}3 \,\cdot, \cdot\, 2{*}4 \,\cdot, \cdot\, 3{*}1 \,\cdot, \cdot\, 4{*}2 \,\cdot$$
$$1{*}{*}2,\ 2{*}{*}3,\ 3{*}{*}4,\ 4{*}{*}1.$$

Condition (i) leaves the following 4-tuples:

1111	1214	1313	$\underline{1412}$	$\underline{2114}$	$\underline{2213}$	2312	2411
1124	1223	$\underline{1322}$	$\underline{1421}$	$\underline{2123}$	2222	$\underline{2321}$	2424
$\underline{1133}$	$\underline{1232}$	$\underline{1331}$	$\underline{1434}$	$\underline{2132}$	2231	2334	$\underline{2433}$
$\underline{1142}$	1241	1344	$\underline{1443}$	$\underline{2141}$	$\underline{2244}$	$\underline{2343}$	$\underline{2442}$

$\underline{3113}$	$\underline{3212}$	$\underline{3311}$	$\underline{3414}$	4112	$\underline{4211}$	$\underline{4314}$	4413
3122	$\underline{3221}$	$\underline{3324}$	3423	$\underline{4121}$	$\underline{4224}$	$\underline{4323}$	$\underline{4422}$
3131	$\underline{3234}$	3333	$\underline{3432}$	4134	4233	$\underline{4332}$	$\underline{4431}$
$\underline{3144}$	$\underline{3243}$	3342	3441	$\underline{4143}$	4242	$\underline{4341}$	4444

Condition (ii) eliminates all of the above 4-tuples which are underlined.

Hence, if $n=4$ the permanent of (2) will be

$$\sum_{i=1}^{4} a_i^4 + 2a_1^2a_3^2 + 2a_2^2a_4^2 + 4\sum_{i=1}^{4} a_i^2 a_{i+1} a_{i+3}.$$

Let R_n denote the set of n-tuples left by the algorithm. We remark that the n-tuples in R_n need not be formally distinct; e.g., 1313 and 3131 are both in R_4. The number of formally distinct diagonal products in the permanent of an arbitrary circulant has been determined by Brualdi and Newman [1].

3. Proofs

THEOREM. *Let A be a circulant matrix* (2) *with scalars $a_1, \ldots, a_n$. Then*

$$\text{per } A = \sum a_{i_1} \cdots a_{i_n}$$

where the summation is over all $(i_1, \ldots, i_n) \in R_n$.

Proof. We are concerned with determining conditions for which $a_{i_1} \cdots a_{i_n}$ is a term of the permanent of the $n \times n$ matrix (2). Thus, a_{i_k} always denotes an element of the kth row of (2). The ith column of (2) is

$$\begin{bmatrix} a_i \\ a_{i-1} \\ \cdot \\ \cdot \\ \cdot \\ a_{i-n+1} \end{bmatrix}$$

where subscripts are taken modulo n. If the Laplace expansion along the first row is used to find per A the entry a_{i-k+1} cannot be chosen from row k to appear in any monomial beginning with a_i. In any monomial of the permanent the pattern (3) cannot appear since we may expand along any row.

Therefore any $(i_1, \ldots, i_n)$ in R_n satisfies

$$i_{j+k} \neq i_j - k \quad \text{for} \quad k = 1, \ldots, n-1.$$

Again, subscripts are taken modulo n when necessary.

Write $i_{j+k} = i_j - k + x_{jk} \pmod n$ where $x_{jk} \neq 0$, $1 \leq x_{jk} \leq n-1$, and $k \neq 0$. We would like to show that $s \neq t$ implies $x_{js} \neq x_{jt}$.

Suppose $x_{js} = x_{jt}$. Then

$$x_{js} = i_{j+s} - i_j + s = i_{j+t} - i_j + t = x_{jt}.$$

Hence

$$i_{j+s} = i_{j+t} - (s-t),$$

but unless $s = t$

$$i_{j+s} = i_{j+t+(s-t)} \neq i_{j+t} - (s-t).$$

So assuming $x_{js} = x_{jt}$ leads to a contradiction. Hence the contrapositive is true and $s \neq t$ implies $x_{js} \neq x_{jt}$.

Step (i) is included in the algorithm because it is easy to implement. In fact, (ii) implies (i) as we now show:

$$\begin{aligned} \sum_{k=0}^{n-1} i_{j+k} = i_j + \sum_{k=1}^{n-1} i_{j+k} &= \left(i_j + \sum_{k=1}^{n-1} (i_j - k + x_{jk}) \right) \pmod n \\ &= \left(n i_j - \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} x_{jk} \right) \pmod n \\ &= (n i_j - \tfrac{1}{2} n(n-1) + \tfrac{1}{2} n(n-1)) \pmod n \\ &\equiv 0 \pmod n. \end{aligned}$$

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We have shown why the n-tuples mentioned in (i) and (ii) must be discarded. It remains to show that no more should be excluded. Condition (ii) says there are n choices for a_{i_1}, $n-1$ choices for a_{i_2} and in general $n-k+1$ choices for a_{i_k}. That is, condition (ii) does not eliminate exactly $n!$ terms. But there are $n!$ terms in the permanent so precisely the right number of monomials has been excluded.

4. Numerical results. Dr. Joan Cooper wrote a Fortran programme for our algorithm which was implemented on an ICL 1904A at the University of Newcastle, N.S.W., Australia. The following various 7×7 circulants were computed using 2.54 seconds of core time.

First row of circulant matrix A							per A	row sum of $A=r$	per (A/r)
3	1	1	0	1	0	0	4416	6	0.0157750
1	1	1	0	0	0	0	31	3	0.0141747
1	1	0	0	0	0	0	2	2	0.0156250
1	1	1	1	1	1	1	5040	7	0.0061199
0	1	1	0	1	0	0	24	3	0.0109739
1	1	1	0	1	0	0	144	4	0.0087891
1	1	−1	0	0	0	0	1	1	1.0

We believe the algorithm is not shown to best advantage as most of the elapsed time is due to reading the 7-tuples of the example from disc.

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