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A NOTE ON EXTENDING LOCALLY FINITE COLLECTIONS

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Recently there has been a great deal of interest in extending refinements of locally finite and point finite collections on subsets of certain topological spaces. In particular the first named author showed that a subset S of a topological space X is P-embedded in X if and only if every locally finite cozero-set cover on S has a refinement that can be extended to a locally finite cozero-set cover of X. Since then many authors have studied similar types of embeddings (see [1], [2], [3], [4], [6], [8], [9], [10], [11], and [12]). Since the above characterization of P-embedding is equivalent to *extending* continuous pseudometrics from the subspace S up to the whole space X, it is natural to wonder when can a locally finite or a point finite open or cozero-set cover on S be *extended* to a locally finite or point-finite open or cozero-set cover on X. In general this a difficult requirement and not many results have been obtained along these lines. In [8, Theorem 8], Sennott showed the following:

THEOREM 1. *If X is a collectionwise normal perfectly normal space then every point-finite cozero-set cover of a closed subspace F of X can be extended to a point finite cozero-set cover of X that is locally finite on $X-F$.*

It is the object of this note to observe some additional results in this area. But first, we need the following observation.

THEOREM 2. *If X is a topological space then the following statements are equivalent:*

(1) *X is collectionwise normal.*

(2) *For every closed subset S of X every point-finite cozero-set cover of S has a refinement that can be extended to a locally finite cozero-set cover of X.*

Proof. From [9, Theorem 2.1 or 2, Theorem 14.5], it follows that (2) *implies* (1). The fact (1) *implies* (2) follows from [7, Theorem 2] which states that every point-finite open cover has a locally finite open refinement if the space is collectionwise normal.

Theorem 2 above should be compared to Theorem 1. In the first theorem the hypothesis of collectionwise normal and perfectly normal on X allow us to *extend* every point finite cozero-set cover on a closed subset S of X to a point finite cozero-set cover of X that is locally finite on $X-S$. We will now show that we can actually extend locally finite cozero-set covers of X provided the space X is normal expandable.

THEOREM 3. *If X is a normal expandable space then for every closed subset S of X every locally finite cozero-set cover of S can be extended to a locally finite cozero-set cover of X.*

Proof. Suppose that S is closed and that $\mathscr{G}=(G_\alpha)_{\alpha\in I}$ is a locally finite cozero-set cover of S. Since S is a closed subset of a normal space, for each $\alpha\in I$ there exists a cozero-set G_α^* of X such that $G_\alpha^* \cap S=G_\alpha$. Note that $\mathscr{G}$ is a locally finite family of X and hence $(cl_X G_\alpha)_{\alpha\in I}$ is locally finite in X. (Actually $cl_X G_\alpha=cl_S G_\alpha$.) Since X is expandable, there exists a locally finite family $\mathscr{H}=(H_\alpha)_{\alpha\in I}$ of open subsets of X such that $cl_X G_\alpha\subset H_\alpha$. Now X is normal, hence for each $\alpha\in I$ there exists a cozero-set W_α such that $cl_X G_\alpha\subset W_\alpha\subset H_\alpha$. For each $\alpha\in I$, let $U_\alpha=W_\alpha\cap G_\alpha^*$ and note that $\mathscr{U}=(U_\alpha)_{\alpha\in I}$ is a locally finite cozero-set family of X that extends $\mathscr{G}$. Now let $U=\bigcup_{\alpha\in I} U_\alpha$ and observe that U is an open set in X containing the closed set S hence, since X is normal, $X-U$ and S can be completely separated. Thus there exists a cozero-set A such that $X-U\subset A$ and $A\cap S=\varnothing$. Choose $\alpha_0\in I$ arbitrarily and let $A_{\alpha_0}=U_{\alpha_0}\cup A$ and set $A_\alpha=U_\alpha$ if $\alpha\neq\alpha_0$. Then $\mathscr{A}=(A_\alpha)_{\alpha\in I}$ is a locally finite cozero-set cover of X that extends $\mathscr{G}$.

Since a space is normal expandable if and only if it is strongly normal (countably paracompact and collectionwise normal; see [5]) we have the following:

COROLLARY. *If X is a strongly normal topological space then for every closed subset S of X, every locally finite cozero-set cover of S can be extended to a locally finite cozero-set cover of X.*

In [11], Smith and Krajewski defined a topological space X to be *almost expandable* if every locally finite collection of subsets of X is expandable to a point-finite open collection. Using the same techniques as in Theorem 3 and its corollary, one can prove the following.

THEOREM 4. *If X is a normal almost expandable topological space then for every closed subset S of X every locally finite cozero-set cover of S can be extended to a point-finite cozero-set cover of X.*

These results should be compared to the following known theorems concerning extensions of covers.

THEOREM 5 ([10, Theorem 2.10]). *If X is a topological space and if S is a subset of X then S is $P^{\aleph_0}$-embedded in X if and only if every countable cozero-set cover of S can be extended to a cozero-set cover of X.*

THEOREM 6 ([10, Theorem 2.11]). *A subspace S of a topological space X is T-embedded in X if and only if every finite cozero-set cover of S extends to a finite cozero-set cover of X.*

Finally we prove a theorem similar to Theorem 2.12 in [6]. In that result the hypothesis that X is expandable and each F_n is closed was needed. Here we can drop the hypothesis on X but need to add hypothesis on each F_n, namely:

THEOREM 7. *Suppose that X is a countable union of closed paracompact P-embedded subsets. Then X is paracompact.*

Proof. Suppose that $X=\bigcup_{n\in N} F_n$ and let $\mathscr{U}=(U_\alpha)_{\alpha\in I}$ be an open cover of X. For each $n\in\mathbf{N}$, let $\mathscr{U}_n=(U_\alpha \cap F_n)_{\alpha\in I}$. Then $\mathscr{U}_n$ is an open cover of the paracompact space F_n and hence has a locally finite cozero-set refinement $\mathscr{V}_n$. (A paracompact space is normal.) Since F_n is P-embedded in X there exists a locally finite cozero-set cover $\mathscr{W}_n=(W_\alpha^n)_{\alpha\in I}$ of X such that $\mathscr{W}_n \mid F_n$ refines $\mathscr{V}_n$. For each $\alpha\in I$, let $A_\alpha^n=W_\alpha^n \cap \mathscr{U}_\alpha$ and let $\mathscr{A}_n=(A_\alpha^n)_{\alpha\in I}$. Then $\mathscr{A}=\bigcup_{n\in N}\mathscr{A}_n$ is a σ-locally finite cozero-set cover that refines $\mathscr{U}$. Hence X is paracompact.

REFERENCES

1. R. A. Alo and H. L. Shapiro, *Countably paracompact, normal, and collectionwise normal spaces*, Indag. Math. **35** (1973), 347–351.
2. ——, *Normal Topological Spaces* (London, Cambridge University Press, 1974).
3. T. E. Gantner, *Extensions of uniformly continuous pseudometrics*, Trans. Amer. Math. Soc. **132** (1968), 147–157.
4. ——, *Extensions of uniformities*, Fund. Math. **66** (1970), 263–281.
5. M. Katetov, *Extension of locally finite covers*, Colloq. Math. **6** (1958), 145–151 (Russian).
6. L. L. Krajewski, *On expanding locally finite collections*, Canad. J. Math. **23** (1971), 58–68.
7. E. Michael, *Point-finite and locally finite covers*, Canad. J. Math. **7** (1955), 275–279.
8. L. I. Sennott, *Extending point-finite covers*, Proc. Third Prague Topological Symposium (1971), 393–397.
9. H. L. Shapiro, *Extensions of pseudometrics*, Canad. J. Math. **18** (1966), 981–998.
10. ——, *More on extending continuous pseudometrics*, Canad. J. Math., **22** (1970), 984–993.
11. J. C. Smith and L. L. Krajewski, *Expandability and collectionwise normality*, Trans. Amer. Math. Soc. **160** (1971), 437–51.
12. J. C. Smith and J. C. Nichols, *Embedding characterizations for expandable spaces*, Duke Math. J. **39** (1972), 489–496.