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COMPLETELY BOUNDED BANACH–MAZUR DISTANCE†

by CHUN ZHANG

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Analogous to the Banach–Mazur distance between Banach spaces, we study the completely bounded Banach–Mazur distance between operator spaces.

$$d_{cb}(X, Y) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb}, \quad T : X \mapsto Y \text{ is any linear isomorphism}\}.$$

In many cases of Banach spaces and Hilbert spaces we show that the infimum is attained when T is the identity map, and X, Y have the same base space. This provides a machinery to compute and estimate $d_{cb}(X, Y)$. Later, using symmetric norming functions we construct counterexamples to show that distinct infinite dimensional homogeneous operator spaces may have finite cb-distance, and that two homogeneous Hilbertian operator spaces may not coincide even if they coincide over all 2-dimensional subspaces.

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1. Introduction

In the study of the local theory of Branch spaces, people introduced various ways to measure the difference of structures between Banach spaces and their subspaces. Examples are the bounded approximation constant and the projection constant in **[18]**. Another Banach space constant which plays a considerably important role is the Banach–Mazur distance which is defined as follows.

Definition 1.1. Let X, Y be Banach spaces, then the *Banach–Mazur distance* between them is defined to be

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\| : \quad X \mapsto Y \text{ is a linear isomorphism}\}$$

This distance makes some impact on operator space theory too. Paulsen's paper **[13]** contains an application of this distance to computations in operator spaces. There is an analogy of this distance in operator space theory. The completely bounded norm of a linear map between operator spaces $T : X \mapsto Y$ is defined to be

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$$\|T\|_{cb} = \sup\{\|T_n\| : \quad n = 1, 2, 3, \ldots\},$$

where $T_n : M_n(X) \mapsto M_n(Y)$ is defined by $T_n((x_{ij})) = (T(x_{ij}))$. The following defines the *completely bounded Banach–Mazur distance* which is briefly called *cb-distance* in the sequel.

Definition 1.2. For any operator spaces E, F, we define

$$d_{cb}(E, F) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb}: \quad T : E \mapsto F \text{ is any linear isomorphism}\}. \tag{1}$$

In **[15]**, Pisier gave a general estimate that for any n dimensional operator spaces $E, F, d_{cb}(E, F) \leq n$. This n is the best constant that can possibly be obtained, since we shall see an example where the cb-distance between two n dimensional operator spaces is exactly equal to n. Nevertheless, it is still tempting to compute the cb-distance more precisely for some particular classes of operator spaces. In Section 2, we show that for any two homogeneous operator spaces with the same underlying Banach spaces, the cb-distance of equation (1) is attained by the identity map when the two cb-norms are comparable. In Section 3 we study the case of homogeneous Hilbertian operator spaces. We are able to show that for any two such spaces the distance is attained by the identity operator. The theorems are applied to the computations of the cb-distance for several examples. Section 4 is a study of homogeneous Hilbertian operator spaces that are induced from symmetric norming functions. We define a distance $d(\Phi_1, \Phi_2)$ between symmetric norming functions, then using the results of Section 3 we prove that the cb-distance is equivalent to $d(\Phi_1, \Phi_2)$. This theorem provides a machinery for us to construct examples that clarify some puzzles about homogeneous operator spaces.

It's simple but worth remarking that for any operator spaces X, Y, Z,

$$d_{cb}(X, Z) \leq d_{cb}(X, Y)d_{cb}(Y, Z).$$

This immediately follows from the inequality for cb-norms

$$\|AB\|_{cb} \leq \|A\|_{cb}\|B\|_{cb}.$$

In the category of all operator spaces, there is a special type of them called homogeneous operator spaces which was introduced by Pisier **[14]**. In this paper we study the cb-distance between homogeneous operator spaces. The notation i_X is used very often to denote the identity map on X.

Definition 1.3. An operator space X is said to be *homogeneous* if every $T : X \mapsto X$ satisfies $\|T\|_{cb} = \|T\|$.

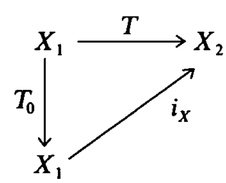
2. The case of Banach spaces

Proposition 2.1. *Let X be a Banach space which is assigned two homogeneous operator space structures, $X_1 = (X, \{\| \ \|_n\}_{n=1}^{\infty})$, $X_2 = (X, \{\| \ \|'_n\}_{n=1}^{\infty})$. Suppose that*

$$\|(x_{ij})\|'_n \le \|(x_{ij})\|_n \quad \textit{for all } (x_{ij}) \in M_n(X), \quad n = 1, 2, 3 \ldots$$

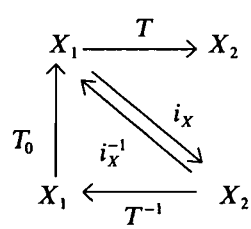
Then, $d_{cb}(X_1, X_2) = \|i_X\|_{cb} \|i_X^{-1}\|_{cb}$, where $i_X : X_1 \mapsto X_2$ is the identity operator on X.

Proof. Let us first notice a fact that for any invertible linear map T in $B(X_1, X_2)$, where $B(X_1, X_2)$ is the set of all bounded linear maps from X_1 into X_2, then T is completely bounded and $\|T\|_{cb} = \|T\|$. To see this, consider the diagram



where T_0 is the same map as T but regarded as a map from X_1 into itself. Whence we have $\|T\|_{cb} = \|i_X \circ T_0\|_{cb} \le \|i_X\|_{cb} \|T_0\|_{cb} = \|T_0\| = \|T\|$, noticing that i_X is completely contractive.

Now look at



where T_0 bears the same sense as in the last diagram. Then, $\|i_X\|_{cb} \|i_X^{-1}\|_{cb} \le \|i_X^{-1}\|_{cb} = \|T_0 \circ T^{-1}\| \le \|T_0\|_{cb} \|T^{-1}\|_{cb} = \|T\|_{cb} \|T^{-1}\|_{cb}$. □

Paulsen [13] introduced a numercial invariant $\alpha(X)$ for any Banach space X which measures the difference between MIN(X) and MAX(X). The reader will also find the definitions of spaces $MIN(X)$ and $MAX(X)$ in [13].

$$\alpha(X) = \sup\{\|(x_{ij})\|_{max} : \|(x_{ij})\|_{min} \le 1\}.$$

We immediately see from the definition that

$$\alpha(X) = \|i_X\|_{cb},$$

where $i_X : MIN(X) \mapsto MAX(X)$.

Corollary 2.2. *$d_{cb}(MIN(X), MAX(X)) = \alpha(X)$ for any finite dimensional normed space X.*

Proof. The operator spaces $MIN(X), MAX(X)$ are homogeneous since every map $T : E \mapsto MIN(X)$ must satisfy $\|T\|_{cb} = \|T\|$, and every map *from* a $MAX(X)$ also satisfies the same equation. Since $MIN(X)$ and $MAX(X)$ are homogeneous and comparable,

$$d_{cb}(MIN(X), MAX(X)) = \|i_X\|_{cb} \|i_X^{-1}\|_{cb} = \|i_X\|_{cb} = \alpha(X). \qquad \square$$

Proposition 2.1 enables us to calculate the cb-distance when two operator space norms are comparable. We don't know whether or not the conclusion of this proposition is true for any two homogeneous operator spaces norms on X. But if X is a Hilbert space, we are able to show that it is true. This is done in the next section.

3. The case of Hilbert spaces

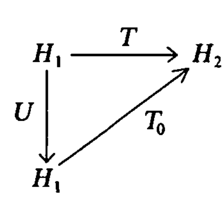
In this section, we do the computations of cb-distances between Hilbertian operator spaces (which means the underlying spaces are Hilbert spaces). Without loss of generality, we always suppose the two operator spaces, whose cb-distance we are computing, are on the same Hilbert space. The following theorem is both a tool of computation and a preparation for our main theorem.

Theorem 3.1. *Let H be an n dimensional Hilbert space, H_1, H_2 be homogeneous operator spaces which have H as the underlying space. Then*

$$d_{cb}(H_1, H_2) = \|i_H\|_{cb} \|i_H^{-1}\|_{cb},$$

where i_H is the identity operator on H but considered from H_1 to H_2.

Proof. Observe that if we decompose any fixed invertible operator $T : H_1 \mapsto H_2$ as $T = T_0 U$



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where U is a unitary, then $\|T\|_{cb} = \|T_0\|_{cb}$. Indeed, $\|T\| = \|T_0 U\|_{cb} \le \|T_0\|_{cb}\|U\|_{cb} = \|T_0\|_{cb}$, and conversely $\|T_0\|_{cb} = \|TU^{-1}\| \le \|T\|_{cb}$. Similarly, $\|VT_{cb} = \|T\|_{cb}$ when V is a unitary.

Now let $T = U|T|$ be a polar decomposition. Since $|T|$ is positive, it is diagonalizable, say, $|T| = V^* DV$, where V is a unitary and $D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\}$. Then $T = UV^*DV$, and $\|T\|_{cb} = \|D\|_{cb}$. Let S be the permutation of the canonical basis of H such that $Se_1 = e_2, Se_2 = e_3, \ldots, Se_{n-1} = e_n, Se_n = e_1$, then by permuting D repeatedly, we get

$$\left(\frac{1}{n}\sum_{i=o}^{n-1} S^i\right)D = \frac{d_1 + d_2 + \cdots + d_n}{n} i_H,$$

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S^i\right)D^{-1} = \frac{d_1^{-1} + d_2^{-1} + \cdots + d_n^{-1}}{n} i_H^{-1}.$$

By the geometric-arithmetic mean inequality

$$\frac{(d_1 + \cdots + d_n)(d_1^{-1} + \cdots + d_n^{-1})}{n^2} = \frac{\sum_{i,j=1}^n d_i d_j^{-1}}{n^2} \ge \sqrt[n^2]{\prod_{i,j=1}^n d_i d_j^{-1}} = 1.$$

Finally,

$$\begin{aligned}\|i_H\|_{cb}\|i_H^{-1}\|_{cb} &\le \frac{(d_1 + \cdots + d_n)(d_1^{-1} + \cdots + d_n^{-1})}{n^2}\|i_H\|_{cb}\|i_H^{-1}\|_{cb} \\ &= \left\|\left(\frac{1}{n}\sum_{i=0}^{n-1} S^i\right)D\right\|_{cb}\left\|\left(\frac{1}{n}\sum_{i=0}^{n-1} S^i\right)D^{-1}\right\|_{cb} \\ &\le \left\|\frac{1}{n}\sum_{i=0}^{n-1} S^i\right\|_{cb}\|D\|_{cb}\left\|\frac{1}{n}\sum_{i=0}^{n-1} S^i\right\|_{cb}\|D^{-1}\|_{cb} \\ &\le \|D\|_{cb}\|D^{-1}\|_{cb}.\end{aligned}$$

□

It is known (see [6, 7]) that for any $T \in \mathcal{B}(H)$, if $\|T\|_{HS} < \infty$ ($\|\ \|_{HS}$ denotes the Hilbert Schmidt norm), then T is completely bounded when it is viewed as an element in any of the sets $CB(H_{row}, H_{col})$, $CB(H_{col}, H_{row})$, (see [6]), $CB(MIN(H), H_{row})$, $CB(H_{row}, MAX(H))$, (see [9]), where H_{row} and H_{col} are the row and column Hilbert spaces, respectively. They are denoted by R_n and C_n respectively when $dim(H) = n$. Furthermore, $\|T\|_{cb} = \|T\|_{HS}$ in all of these cases. In particular, $\|i_H\|_{cb} = \sqrt{n}$ if H is n dimensional.

Corollary 3.2.

$$d_{cb}(R_n, C_n) = n.$$

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Proof. Since R_n, C_n are homogeneous (see [6]), $d_{cb}(R_n, C_n) = \| i_H \|_{cb} \| i_H^{-1} \|_{cb} = \| i_H \|_{HS} \| i_H^{-1} \|_{HS} = \sqrt{n}\sqrt{n} = n$.

This result was also obtained independently by Mathes in [10]. The result of this corollary gives an example that the constant n in Pisier's theorem is attainable. Other similar computations will yield $d_{cb}(MIN(H), R_n) = d_{cb}(MIN(H), C_n) = \sqrt{n}$.

A more general class of homogeneous Hilbertian operator spaces can be obtained by the idea of complex interpolation of Banach spaces. The reader is referred to [1, 14] for the definitions and basic properties of interpolations of Banach spaces and operator spaces respectively. For a compatible pair of operator spaces E_0 and E_1, following Pisier [14] we use $(E_0, E_1)_\theta$ as the notation for the interpolated spaces between E_0 and E_1.

It is known that if E_0 and E_1 are homogeneous with the same underlying Banach space E, then $(E_0, E_1)_\theta$ is homogeneous for all θ. Indeed, let $T : (E_0, E_1) \mapsto (E_0, E_1)$ be any completely bounded map. Let $\| T_\theta \|_{cb}$ denote the cb-norm of T when it is viewed as $T : (E_0, E_1)_\theta \mapsto (E_0, E_1)_\theta$. Then we have

$$\| T_\theta \|_{cb} \leq (\| T_0 \|_{cb})^\theta (\| T_1 \|_{cb})^{1-\theta} = \| T \|^\theta \| T \|^{1-\theta} = \| T \|.$$

Where T_0, T_1 are both T regarded as $T_0 : E_0 \mapsto E_0$, $T_1 : E_1 \mapsto E_1$.

Notation. Let $T : X \mapsto Y$ be a completely bounded map, in the following discussion, we use $\| T \|_{cb(X,Y)}$ for its cb-norm when it is necessary to make it clear where the cb-norm is taken.

Proposition 3.3. *Let H_n be an n dimensional Hilbert space. For every $\theta \in [0, 1]$,*
(i) $d_{cb}(R_n, (R_n, C_n)_\theta) = d_{cb}(C_n (R_n, C_n)_{1-\theta}) = n^\theta$.
(ii) $d_{cb}(MIN(H_n), (MIN(H_n), MAX(H_n))_\theta) = \alpha(H_n)^\theta$;
$d_{cb}(MAX(H_n), (MIN(H_n), MAX(H_n))_{1-\theta}) = \alpha(H_n)^\theta$.

Proof.
$$\begin{aligned} d_{cb}(R_n, (R_n, C_n)_\theta) &= \| i_{H_n} \|_{cb} \| i_{H_n}^{-1} \|_{cb} \\ &\leq \| i_{H_n} \|^{1-\theta}_{cb(R_n,R_n)} \| i_{H_n} \|^\theta_{cb(R_n,C_n)} \| i_{H_n}^{-1} \|^{1-\theta}_{cb(C_n,C_n)} \| i_{H_n}^{-1} \|^\theta_{cb(C_n,R_n)} \\ &= 1 \cdot n^{\frac{\theta}{2}} \cdot 1 \cdot n^{\frac{\theta}{2}} = n^\theta. \end{aligned}$$

Similarly,

$$d_{cb}((R_n, C_n)_\theta, C_n) \leq n^{1-\theta}.$$

Thus $n = d_{cb}(R_n, C_n) \leq d_{cb}(R_n (R_n, C_n)_\theta) d_{cb}((R_n, C_n)_\theta, C_n) \leq n^\theta n^{1-\theta} = n$. Thus we conclude that $d_{cb}(R_n (R_n, C_n)_\theta) = n^\theta$. By replacing R_n by $MIN(H_n)$, and C_n by $MAX(H_n)$ in the proof of (i) we get a proof of (ii). □

Proposition 3.4. *For all $\theta \in [0, 1]$,*
$d_{cb}(C_n, (MIN(H_n), MAX(H_n))_\theta) = d_{cb}(R_n, (MIN(H_n), MAX(H_n))_\theta) = \sqrt{n}$.

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Proof.

$$d_{cb}(R_n(MIN(H_n), MAX(H_n))_\theta) = \|i_{H_n}\|_{cb}\|i_{H_n}^{-1}\|_{cb}$$
$$\leq \|i_{H_n}\|^{1-\theta}_{cb(R_n, MIN(H_n))}\|i_{H_n}\|^{\theta}_{cb(R_n MAX(H_n))}\|i_{H_n}^{-1}\|^{1-\theta}_{cb(MIN(H_n), R_n)}\|i_{H_n}^{-1}\|^{\theta}_{cb(MAX(H_n), R_n)}$$
$$= 1 \cdot n^{\frac{\theta}{2}} \cdot n^{\frac{1-\theta}{2}} \cdot 1 = \sqrt{n}.$$

Similarly, $d_{cb}(C_n, (MIN(H_n), MAX(H_n))_\theta) \leq \sqrt{n}$.

If $d_{cb}(R_n, (MIN(H_n), MAX(H_n))_\theta) < \sqrt{n}$, then

$$n = d_{cb}(R_n, C_n) \leq d_{cb}(R_n(MIN(H_n), MAX(H_n)_\theta))d_{cb}((MIN(H_n), MAX(H_n)_\theta, C_n))$$
$$< \sqrt{n}\sqrt{n} = n,$$

a contradiction which implies that $d_{cb}(R_n, (MIN(H_n), MAX(H_n))_\theta) = \sqrt{n}$.

In exactly the same way we can prove the other equality. □

We turn our attention now to the cb-distance between infinite dimensional Hilbertian operator spaces. In the previous two propositions, if the dimension n is replaced by ∞, then all the cb-distances turn to ∞. This gives rise to a natural question whether or not the cb-distance between any distinct infinite dimensional homogeneous Hilbertian operator spaces is infinity. We will give a counterexample in the next section.

If H is an infinite dimensional homogeneous Hilbertian operator space, H_1 is a subspace of the same dimension, then H, H_1 are completely isometric. Therefore if H and K are homogeneous with the same underlying space, and if i is the identity operator $i : H \mapsto K$, then $\|i|_{H_1}\|_{cb} = \|i\|_{cb}$.

Lemma 3.5. *Let H, K be homogeneous operator spaces with the same underlying Hilbert space, $K_1 \subset K$ be a subspace of the same dimension as K. Let P be the orthogonal projection of H onto K_1, then $\|i\|_{cb(H,K)} = \|P\|_{cb(H,K)}$.*

Proof. Obviously $\|P\|_{cb} \leq \|i\|_{cb}$. Now let $H_1 = i^{-1}(K_1)$, then

$$\|i\|_{cb} = \|i|_{H_1}\|_{cb} = \|P|_{H_1}\|_{cb} \leq \|P\|_{cb}.$$

Thus $\|P\|_{cb} = \|i\|_{cb}$. □

Theorem 3.6. *Let H_1, H_2 be any homogeneous operator spaces with the same underlying infinite dimensional Hilbert space H, then*

$$d_{cb}(H_1, H_2) = \|i_H\|_{cb}\|i_H^{-1}\|_{cb}.$$

Proof. For any invertible operator $T \in \mathcal{B}(H)$, there is a polar decomposition $T = U \cdot |T|$, where $U : H_2 \mapsto H_2$ is a unitary. Automatically $\| \, |T| \, \|_{cb} = \| T \|_{cb}$. So we reduce to the case when T is positive.

Let $\Gamma : C^*(T) \mapsto C(\sigma(T))$ be the Gelfand transform, where $C^*(T)$ denotes the C*-algebra generated by T, $\sigma(T)$ is the spectrum of T.

For any fixed $\epsilon > 0$, let $\sigma(T) = \bigcup_{i=1}^n A_i$ be a partition of $\sigma(T)$, such that the diameter of each A_i is smaller than ϵ. Letting χ_{A_i} be the characteristic function of A_i we define

$$\phi(z) = \sum_{i=1}^{n} z_i \chi_{A_i}$$

where z_i is a point in A_i, and

$$S = \Gamma^{-1}(\phi),$$

then $\| T - S \| = \| z - \phi \| \le \epsilon$. Notice that S is a diagonalizable operator with eigenvalues $z_1, z_2, \dots, z_n$. If K_i is the corresponding eigenspace of $z_i, i = 1, 2, \dots, n$, then $S = \sum_{i=1}^n z_i P_{K_i}$ and $S^{-1} = \sum_{i=1}^n z_i^{-1} \tilde{P}_{K_i}$. Here $\tilde{P}_{K_i}$ is the same operator as P_{K_1}, but the former is considered to be $H_2 \mapsto H_1$. By the continuity of inversion we can suppose $\| T^{-1} - S^{-1} \| \le \epsilon$. Among $K_1, K_2, \dots K_n$ there is at least one K_i such that $dim(K_i) = dim(H)$, say K_i, then

$$\| S \|_{cb} \| S^{-1} \|_{cb} \ge \| z_i P_{K_i} \|_{cb} \| z_i^{-1} \tilde{P}_{K_i} \|_{cb} = \| P_{K_i} \|_{cb} \| \tilde{P}_{K_i} \|_{cb} = \| i_H \|_{cb} \| i_H^{-1} \|_{cb}.$$

therefore

$$\| T \|_{cb} \| T^{-1} \|_{cb} \ge (\| S \|_{cb} - \epsilon)(\| S^{-1} \|_{cb} - \epsilon) \ge (1 - \epsilon)^2 \| i_H \|_{cb} \| i_H^{-1} \|_{cb}. \qquad \square$$

Corollary 3.7. *Two homogeneous Hilbertian operator spaces H_1, H_2 are completely isometric if and only if $d_{cb}(H_1, H_2) = 1$.* $\square$

4. Homogeneous operator spaces induced by symmetric norming functions

In this section we take a look at the homogeneous Hilbertian operator spaces induced from symmetric norming functions. These operator spaces were first introduced by B. Mathes and V. Paulsen (see **[11]**). We use the idea to construct the counterexamples referred to in the last section. We also prove a theorem which estimates the cb-distance between any two of these spaces.

Apart from the usual operator norm, there are various other norms that can be defined on $\mathcal{B}(H)$. We are interested in a particular class of these called symmetric norming functions. We restrict these functions to be defined on the finite rank operators only.

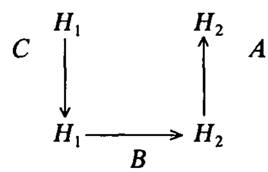
Definition 4.1. A norm Φ on $\mathcal{F}(H)$ is called a *symmetric norming function* if

$$\Phi(ABC) \le \|A\| \cdot \Phi(B) \cdot \|C\| \quad \text{for all } A, C \in \mathcal{B}(H), B \in \mathcal{F}(H),$$

where $\| \ \|$ is the usual operator norm, and $\mathcal{F}(H)$ is the set of all finite rank operators on H.

Note. First, the definition immediately implies that $\Phi(UTV) = \Phi(T)$ when U, V are unitaries, and vice versa. i.e. if $\Phi(UTV) = \Phi(T)$ for all unitaries U, V and finite rank operator T, then Φ is a symmetric norming function. (See [7, Theorem 3.1 & Corollary 3.1] for a justification). Secondly, the natural domain of Φ, $\{T \in \mathcal{B}(H) : \Phi(T) < \infty\}$ is an ideal of $\mathcal{B}(H)$.

If H_1, H_2 are homogeneous Hilbertian operator spaces with the same underlying space H, then the cb-norm with respect to $CB(H_1, H_2)$ is a symmetric norming function on $\mathcal{B}(H)$. Indeed, for any $A, C \in \mathcal{B}(H)$, $B \in CB(H_1, H_2)$,



$\|ABC\|_{cb} \le \|A\|_{cb}\|B\|_{cb}\|C\|_{cb} = \|A\|\,\|T\|_{cb}\|C\|.$

If H is an l^2 space, T is a compact operator on H then a suitable choice of unitaries U, V can result in UTV being a diagonal operator or even a positive diagonal operator with the diagonal elements in non-increasing order. And notice that if $T \in CB(H_1, H_2)$, $UTV = diag\{\alpha_1, \alpha_2, \ldots\}$ in non-increasing order, then $\|T\|_{cb} = \|diag\{\alpha_1, \alpha_2, \ldots\}\|_{cb}$. So the symmetric norming function $\| \ \|_{cb}$ is completely determined by a "unitarily invariant" norm on an infinite sequence space.

Conversely, from any symmetric norming function Φ on $\mathcal{B}(H)$, we can induce a homogeneous Hilbertian operator space norm on H in the following way (see [**11**]), for any $(x_{ij}) \in M_n(H)$

$$(\|(x_{ij})\|_{\Phi} = \sup\{\|(Tx_{ij})\|_{max} : \Phi(T) \le 1\}.$$

We denote H with this matrix norm structure $H_{\Phi,max}$. It is known that $H_{\Phi,max}$ is homogeneous (see [**11**]).

By the above remark the cb-norm in the sense of $CB(H_{\Phi,max}, H_{max})$ is a symmetric norming function on H, we denote it by Φ_{cb}, i.e.

$$\Phi_{cb}(T) = \|T\|_{cb} \quad \text{for } T \in CB(H_{\Phi,max}, H_{max}).$$

Theorem 3.2 of [**11**] states that there is a constant $C > 0$ such that

$$C\Phi(T) \leq \Phi_{cb}(T) \leq \Phi(T) \quad \text{for all finite rank } T.$$

It is natural to study the behaviour of $d_{cb}(H_{\Phi_1,max}, H_{\Phi_2,max})$. As one may expect it is related to the distance between Φ_1 and Φ_2.

Definition 4.2. Let Φ_1, Φ_2 be symmetric norming functions, we define

$$d(\Phi_1, \Phi_2) = \sup_{T \in \mathcal{F}(H)} \frac{\Phi_1(T)}{\Phi_2(T)} \cdot \sup_{T \in \mathcal{F}(H)} \frac{\Phi_2(T)}{\Phi_1(T)}.$$

Theorem 4.3. *Let Φ_1, Φ_2 be symmetric norming functions, then there is a constant $C > 0$ such that*

$$Cd(\Phi_1, \Phi_2) \leq d_{cb}(H_{\Phi_1,max}, H_{\Phi_2,max}) \leq d(\Phi_1, \Phi_2).$$

Proof. As we know, $d_{cb}(H_{\Phi_1,max}, H_{\Phi_2,max}) = \|i_H\|_{cb(\Phi_1,\Phi_2)} \|i_H^{-1}\|_{cb(\Phi_2,\Phi_1)}$. Suppose

$$\sup_{T \in \mathcal{F}(H)} \frac{\Phi_1(T)}{\Phi_2(T)} = k,$$

then $\{T \in \mathcal{F}(H) : \Phi_2(T) \leq 1\} \subseteq \{T \in \mathcal{F}(H) : \Phi_1(T) \leq k\}$. Thus

$$\begin{aligned} \sup\{\|(Tx_{ij})\|_{max} : \quad \Phi_2(T) \leq 1\} &\leq \sup\{\|(Tx_{ij})\|_{max} : \quad \Phi_1(T) \leq k\} \\ &= k \sup\{\|(Tx_{ij})\|_{max} : \quad \Phi_1(T) \leq 1\}. \end{aligned}$$

So $\|(x_{ij})\|_{\Phi_2,max} \leq k \|(x_{ij})\|_{\Phi_1,max}$ for any $(x_{ij}) \in M_n(H)$. This means

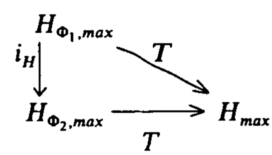
$$\|i_H\|_{cb(\Phi_1,\Phi_2)} \leq k = \sup_{T \in \mathcal{F}(H)} \frac{\Phi_1(T)}{\Phi_2(T)}.$$

Similarly,

$$\|i_H^{-1}\|_{cb(\Phi_2,\Phi_1)} \leq \sup_{T \in \mathcal{F}(H)} \frac{\Phi_2(T)}{\Phi_1(T)}.$$

Thus $d_{cb}(H_{\Phi_1,max}, H_{\Phi_2,max}) \leq d_{cb}(\Phi_1, \Phi_2)$.

For the other inequality, take any $T \in \mathcal{F}(H)$ and look at the diagram



Then

$$\|T\|_{cb(H_{\Phi_1,max},H_{max})} \leq \|i_H\|_{cb(\Phi_1,\Phi_2)} \cdot \|T\|_{cb(H_{\Phi_2,max},H_{max})},$$

i.e.

$$(\Phi_1)_{cb}(T) \leq \|i_H\|_{cb(\Phi_1,\Phi_2)} \cdot (\Phi_2)_{cb}(T),$$

or

$$\|i_H\|_{cb(\Phi_1,\Phi_2)} \geq \frac{(\Phi_1)_{cb}(T)}{(\Phi_2)_{cb}(T)} \geq \frac{C_1\Phi_1(T)}{\Phi_2(T)}$$

for all $T \in \mathcal{F}(H)$ and for some $C_1 > 0$. Similarly

$$\|i_H^{-1}\|_{cb(\Phi_2,\Phi_1)} \geq \frac{C_2\Phi_2(T)}{\Phi_1(T)}$$

for all $T \in \mathcal{F}(H)$ and for some $C_2 > 0$. Let $C = C_1C_2$, then

$$\|i_H\|_{cb(\Phi_1,\Phi_2)} \|i_H^{-1}\|_{cb(\Phi_2,\Phi_1)} \geq C \sup \frac{\Phi_1(T)}{\Phi_2(T)} \cdot \sup \frac{\Phi_2(T)}{\Phi_1(T)},$$

where the sup's are over all $T \in \mathcal{F}(H)$. Consequently we have

$$Cd(\Phi_1, \Phi_2) \leq d_{cb}(H_{\Phi_1,max}, H_{\Phi_2,max}). \qquad \square$$

This theorem generalizes [**11**, Theorem 3.2], and it has the following applications in constructing examples as well.

Example. This example gives infinite dimensional homogeneous Hilbertian operator spaces whose cb-distances to each other are finite. For every $r \in (0, 1)$, we define a symmetric norming function via, for any $(\lambda_1, \lambda_2, \lambda_3, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$, let

$$\phi_r(\lambda_1, \lambda_2, \lambda_3, \ldots) = \sup_n \left\{ \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{1 + r + \cdots + r^{n-1}} \right\}.$$

Then all ϕ_r's are equivalent to the l^1 norm, since

$$\sum_{i=1}^{\infty} \lambda_i \geq \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{1 + r + \cdots r^{n-1}} \geq \frac{\sum_{i=1}^{n} \lambda_i}{\frac{1}{1-r}} = (1 - r) \sum_{i=1}^{n} \lambda_i.$$

Hence

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$$(1-r)\|T\|_1 \le \phi_r(T) \le \|T\|_1, \quad \text{for all } T.$$

Now if $r_1 > r_2$, then

$$d_{cb}(H_{\phi_{r_1},max}, H_{\phi_{r_2},max}) = \|i_H\|_{cb}.$$

By Theorem 4.3 we see that

$$\|i_H\|_{cb} \le \frac{1}{1-r} < +\infty.$$

On the other hand, $\{H_{\phi_r,max},\ r \in (0,1)\}$ is not a singleton, i.e. $H_{\phi_r,max}$ are not all identical. Otherwise,

$$H_{\phi_1,max} = \bigvee_{r\in(0,1)} H_{\phi_r,max} = H_{\phi_r,max}.$$

where the symbol $\bigvee$ means to sup the norms of all $H_{\phi_r,max}$. We have reached a contradiction since $\phi_1(\lambda_1, \lambda_2, \ldots) = \lambda_1$, i.e. ϕ_1 is the usual operator norm on H. But it is easily seen that $H_{\phi_1,max} = MAX(H)$. Indeed, for every $(x_{ij}) \in M_n(H)$

$$\begin{aligned}\|(x_{ij})\|_{\phi_1,max} &= \sup\{\|(Tx_{ij})\|_{max}: \quad \|T\| \le 1\}\\ &\le \sup\{\|(Tx_{ij})\|_{max}: \quad T \text{ is a unitary}\}\\ &= \|(x_{ij})\|_{max}.\end{aligned}$$

□

For a Banach space X, it is well known that if all its two dimensional subspaces are isometric to a Hilbert space, then X is a Hilbert space. Here is a similar (but not very much the same) question in the case of operator space: if H_1, H_2 are homogeneous on Hilbert space H, and if the identity map $i_H : H_1 \mapsto H_2$ is a complete isometry when restricted to each two dimensional subspace, are H_1 and H_2 identical as operator spaces? The next example answers this question.

Proposition 4.4. *Let Φ_1, Φ_2 be symmetric norming functions, and fix n. If $\Phi_1(T) = \Phi_2(T)$ for all operators T with rank $\le n$. Then two induced operator space structures $H_{\Phi_1,max}$, $H_{\Phi_2,max}$ agree on all n dimensional subspaces.*

Proof. Suppose K is any n dimensional subspace of H, $(x_{ij}) \in M_n(K)$, then

$$\begin{aligned}\|(x_{ij})\|_{\Phi_1,max} &= \sup\{\|(Tx_{ij})\|_{max}: \quad \Phi_1(T)\le 1\}\\ &= \sup\{\|(Tx_{ij})\|_{max}: \quad \Phi_1(T)\le 1, \quad rank(T)\le n\}\\ &= \sup\{\|(Tx_{ij})\|_{max}: \quad \Phi_2(T)\le 1, \quad rank(T)\le n\}\\ &= \sup\{\|(Tx_{ij})\|_{max}: \quad \Phi_2(T)\le 1\}\\ &= \|(x_{ij})\|_{\Phi_2,max}.\end{aligned}$$

Note that the second equality is true because K is n dimensional, and we can consider the decomposition $H = K \oplus K^{\perp}$. □

Example. Let Φ_1, Φ_2 be

$$\Phi_1(\lambda_1, \lambda_2, \ldots) = \lambda_1 + \lambda_2$$

$$\Phi_2(\lambda_1, \lambda_2, \ldots) = \sum_{i=1}^{\infty} \lambda_i$$

for all $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then clearly $\Phi_1(T) = \Phi_2(T)$ if T is rank 2. Thus by the previous lemma, $H_{\Phi_1,max}$, $H_{\Phi_2,max}$ are identical on all two dimensional subspaces. But Φ_1, Φ_2 are not equivalent norms, hence by Theorem 4.3, $H_{\Phi_1,max}$, $H_{\Phi_2,max}$ are not even completely bounded equivalent. □

This example is easily generalized from dimension 2 to dimension n.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF HOUSTON
HOUSTON, TX 77204
U.S.A.