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Spectral Transformations of the Laurent Biorthogonal Polynomials, II. Pastro Polynomials

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Abstract. We continue to study the simplest closure conditions for chains of spectral transformations of the Laurent biorthogonal polynomials (LBP). It is shown that the 1-1-periodic q-closure condition leads to the LBP introduced by Pastro. We introduce classes of semi-classical and Laguerre-Hahn LBP associated to generic closure conditions of the chain of spectral transformations.

1 Introduction

In the previous paper [5] we started to study the closure conditions for spectral transformations of the Laurent biorthogonal polynomials (LBP) defined by the recurrence relation

$$P_{n+1}(z) + (d_n - z)P_n(z) = zb_nP_{n-1}(z), \quad n \geq 1 \tag{1.1}$$

with the initial conditions

$$P_0(z) = 1, \quad P_1(z) = z - d_0. \tag{1.2}$$

Let us recall some basic facts from the theory of spectral transformations of the LBP (for details see [5]).

It can be shown that there exists a linear Laurent functional $\mathcal{L}$ defined on all possible monomials z^n by the moments

$$c_0 = 1, \quad c_n = \mathcal{L}\{z^n\}, \quad \pm 1, \pm 2, \dots \tag{1.3}$$

(in general the moments c_n are complex numbers). This functional provides the orthogonality property

$$\mathcal{L}\{P_n(z)z^{-k}\} = h_n\delta_{kn}, \quad 0 \leq k \leq n, \tag{1.4}$$

where the normalization constants are

$$h_0 = 1, \quad h_n = \frac{b_1b_2\cdots b_n}{d_1d_2\cdots d_n}. \tag{1.5}$$

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The orthogonality property (1.4) can be rewritten as the biorthogonal relation [4], [3],

$$\mathcal{L}\{P_n(z)Q_m(1/z)\} = h_n\delta_{nm}, \tag{1.6}$$

where the polynomials $Q_n(z)$ are defined by the formula

$$Q_n(z) = \frac{z^n P_{n+1}(1/z) - z^{n-1}P_n(1/z)}{P_n(0)}. \tag{1.7}$$

Note that the polynomials $Q_n(z)$ are again LBP with moments $c_n^{\{Q\}} = c_{-n}$.

The transformation $\mathcal{Q}$ of the polynomials $P_n(z)$ into the polynomials $Q_n(z)$ is an involution, *i.e.*, $\mathcal{Q}^2 = I$, where I is the identity operator.

Another important involution is the transformation $\mathcal{T}$ of the polynomials $P_n(z)$ into the LBP $T_n(z)$ defined as

$$T_n(z) = \frac{z^n P_n(1/z)}{P_n(0)}. \tag{1.8}$$

The polynomials $T_n(z)$ are again the monic LBP with moments

$$c_n^{\{T\}} = \frac{c_{1-n}}{c_1}. \tag{1.9}$$

The j-associated polynomials $P_n^{(j)}(z)$ satisfy the recurrence relation

$$P_{n+1}^{(j)}(z) + d_{n+j}P_n^{(j)}(z) = z\big(P_n^{(j)}(z) + b_{n+j}P_{n-1}^{(j)}(z)\big) \tag{1.10}$$

with initial conditions $P_0^{(j)} = 1$, $P_1^{(j)}(z) = z - d_j$. Let us recall the scaling property of LBP [5].

Assume that the polynomials $P_n(z)$ are LBP with moments c_n and recurrence coefficents d_n, b_n. The polynomials $S\tilde{P}_n(z) = \kappa^{-n}P_n(\kappa z)$ are also LBP with moments $\tilde{c}_n = \kappa^{-n}c_n$ and recurrence parameters $\tilde{b}_n = b_n/\kappa$, $\tilde{d}_n = d_n/\kappa$.

We say that LBP are *regular* if $b_n\, d_n \neq 0$. For the regular LBP the following lemma holds (see, *e.g.* [6]):

Lemma 1 *Let $P_n(z)$ be a set of regular* LBP. *Assume that the identity*

$$S_1(z\,;n)P_n(z) + S_2(z\,;n)P_{n-1}(z) = 0, \quad n = 1, 2, \dots \tag{1.11}$$

takes place, where $S_{1,2}(z\,;\,n)$ are polynomials in z whose coefficients depend on n but whose degrees are fixed numbers (not depending on n). Then the polynomials $S_{1,2}$ vanish identically:

$$S_1(z\,;n) = S_2(z\,;n) \equiv 0. \tag{1.12}$$

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The main spectral transformations of the LBP are the Christoffel and Geronimus transformations.

By Christoffel transformation (CT) we mean the transformation

$$\tilde{P}_n(z) = \frac{P_{n+1}(z) - U_n P_n(z)}{z - \mu}, \tag{1.13}$$

where μ is an arbitrary parameter and

$$U_n = \frac{P_{n+1}(\mu)}{P_n(\mu)}. \tag{1.14}$$

It is easily verified that the polynomials $\tilde{P}_n(z)$ are again monic LBP having the moments

$$\tilde{c}_n = (c_1 - \mu)^{-1}(c_{n+1} - \mu c_n), \quad n = 0, \pm 1, \pm 2, \dots . \tag{1.15}$$

If the polynomials $P_n(z)$ satisfy the recurrence relation (1.1), the polynomials $\tilde{P}_n(z)$ satisfy the recurrence relation

$$\tilde{P}_{n+1}(z) + \tilde{d}_n \tilde{P}_n(z) = z\big(\tilde{P}_n(z) + \tilde{b}_n \tilde{P}_{n-1}(z)\big), \tag{1.16}$$

where

$$\tilde{b}_n = b_n \frac{b_{n+1} + U_n}{b_n + U_{n-1}}, \tag{1.17}$$

$$\tilde{d}_n = d_n \frac{d_{n+1} + U_{n+1}}{d_n + U_n}. \tag{1.18}$$

The reciprocal to the CT is the Geronimus transformation GT (for details see [5] and [7]).

In what follows we will denote by $\mathcal{C}(\mu)\{P_n(z)\}$ the effect of the Christoffel transformations of the LBP $P_n(z)$ (*i.e.*, (1.13)).

2 Closure Conditions for Chains of Spectral Transformations

From given polynomials $P_n(z)$, we can construct a chain of polynomial sets

$$P_n(z\,; \mu_1, \mu_2, \dots, \mu_N) = \mathcal{C}(\mu_N)\mathcal{C}(\mu_{N-1}) \cdots \mathcal{C}(\mu_1)\{P_n(z)\} \tag{2.1}$$

applying successively N CT at the points $\mu_1, \mu_2, \dots, \mu_N$. Choosing another set of points $\nu_1, \nu_2, \dots, \dots, \nu_M$ we can construct another chain of polynomials sets

$$P_n(z\,; \nu_1, \nu_2, \dots, \nu_M) = \mathcal{C}(\nu_M)\mathcal{C}(\nu_{M-1}) \cdots \mathcal{C}(\nu_1)\{P_n(z)\}. \tag{2.2}$$

By (generalized) q-closure condition we mean the following relation

$$P_n^{(j_1)}(z\,; \mu_1, \mu_2, \dots, \mu_N) = q^n P_n^{(j_2)}(z/q\,; \nu_1, \nu_2, \dots, \nu_M), \tag{2.3}$$

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where j_1, j_2 are arbitrary nonnegative integers and q is a fixed parameter. As usual, $P_n^{(j)}(z)$ denotes the j-associated polynomial defined by (1.10). The closure condition (2.3) is thus described by the 4 integers $(N, j_1 ; M, j_2)$.

We define the LBP $P_n(z)$ obtained as the solution of the closure condition (2.3) as *q-Laguerre-Hahn* LBP. If in addition, $j_1 = j_2 = 0$, we then obtain *q-semiclassical* LBP.

In terms of the recurrence coefficients b_n, d_n the closure condition (2.3) means

$$b^{(N)}_{n+j_1} = q b^{(M)}_{n+j_2}, \quad d^{(N)}_{n+j_1} = q d^{(M)}_{n+j_2}, \tag{2.4}$$

where by $b_n^{(N)}$ and $d_n^{(N)}$ we mean the coefficients obtained from b_n and d_n by the application of N CT at the points $\mu_1, \mu_2, \dots, \mu_N$. For the arbitrary scheme $(N, j_1 ; M, j_2)$, the relations (2.4) are very complicated non-linear difference equations. However for some special cases these equations can be resolved in terms of elementary functions.

3 The Simplest 1-1 Closure Condition and the Pastro LBP

In the previous paper [5] we considered the simplest 1-periodic closure condition $(1, 0 ; 0, 0)$ and showed that it leads to q-Appell LBP.

In this section we consider the closure condition $(1, 0 ; 1, 0)$ with two different prescribed points μ and ν. We show that the resulting polynomials contain two essential parameters and coincide with those introduced by Pastro [4].

In terms of the polynomials the closure condition means

$$P_n(z ; \mu) = q^n P_n(z/q ; \nu), \tag{3.1}$$

where

$$P_n(z ; \mu) = \frac{P_{n+1}(z) - U_n(\mu)P_n(z)}{z - \mu},$$

$$P_n(z ; \nu) = \frac{P_{n+1}(z) - U_n(\nu)P_n(z)}{z - \nu},$$

and $U_n(\mu) = P_{n+1}(\mu)/P_n(\mu)$, $U_n(\nu) = P_{n+1}(\nu)/P_n(\nu)$. In what follows we assume that $\mu\,\nu \neq 0$. Let $b_n^{(1)}$, $d_n^{(1)}$ be the recurrence coefficients corresponding to the polynomials $P_n(z ; \mu)$:

$$b_n^{(1)} = b_n \frac{b_{n+1} + U_n(\mu)}{b_n + U_{n-1}(\mu)}, \tag{3.2}$$

$$d_n^{(1)} = d_n \frac{d_{n+1} + U_{n+1}(\mu)}{d_n + U_n(\mu)}, \tag{3.3}$$

and $b_n^{(2)}$, $d_n^{(2)}$ be the recurrence coefficients corresponding to the polynomials $P_n(z ; \nu)$:

$$b_n^{(2)} = b_n \frac{b_{n+1} + U_n(\nu)}{b_n + U_{n-1}(\nu)}, \tag{3.4}$$

$$d_n^{(2)} = d_n \frac{d_{n+1} + U_{n+1}(\nu)}{d_n + U_n(\nu)}. \tag{3.5}$$

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We then have the following closure conditions for the recurrence coefficients

$$\frac{b_{n+1} + U_n(\mu)}{b_n + U_{n-1}(\mu)} = q\frac{b_{n+1} + U_n(\nu)}{b_n + U_{n-1}(\nu)}, \tag{3.6}$$

$$\frac{d_{n+1} + U_{n+1}(\mu)}{d_n + U_n(\mu)} = q\frac{d_{n+1} + U_{n+1}(\nu)}{d_n + U_n(\nu)}. \tag{3.7}$$

Moreover, from the recurrence relation (1.1), we have two additional relations for the coefficients U_n

$$U_n(\mu) + d_n = \mu\big(1 + b_n/U_{n-1}(\mu)\big), \tag{3.8}$$

$$U_n(\nu) + d_n = \nu\big(1 + b_n/U_{n-1}(\nu)\big), \quad n = 1, 2, 3, \dots \tag{3.9}$$

with initial conditions

$$U_0(\mu) = \mu - d_0, \quad U_0(\nu) = \nu - d_0. \tag{3.10}$$

From the conditions (3.6), (3.7), (3.8) and (3.9), we easily find that

$$U_n(\nu) = \beta U_n(\mu), \tag{3.11}$$

$$b_{n+1} + U_n(\mu) = \gamma_1 q^n\big(b_{n+1} + \beta U_n(\mu)\big), \tag{3.12}$$

$$d_n + U_n(\mu) = \gamma_2 q^n\big(d_n + \beta U_n(\mu)\big), \tag{3.13}$$

where $\beta, \gamma_1, \gamma_2$ are arbitrary constants. Assuming that $\beta \neq 1$, we arrive at the expression for the $U_n(\mu)$

$$U_n(\mu) = \frac{\mu\gamma_1}{q\gamma_2}\,\frac{1 - \gamma_2 q^n}{1 - \gamma_1 q^{n-1}}. \tag{3.14}$$

Taking into account the initial conditions (3.10), we get the restrictions for the parameters

$$\gamma_1 = q/\beta, \quad \gamma_2 = \mu/\nu \tag{3.15}$$

and finally the expressions for the recurrence coefficients

$$b_n = -\frac{\nu}{\beta}\,\frac{(1 - q^n)(1 - \mu q^{n-1}/\nu)}{(1 - q^n/\beta)(1 - q^{n-1}/\beta)}, \tag{3.16}$$

$$d_n = -\frac{\nu}{\beta}\,\frac{1 - \beta\mu q^n/\nu}{1 - q^n/\beta}. \tag{3.17}$$

In view of the scaling property of the LBP, we see that the common factor $-\frac{\nu}{\beta}$ is not an essential parameter (it can be removed by scaling transformation). Only two parameters: the ratio μ/ν and β are thus essential.

A simple comparision indicates that the recurrence coefficients (3.16) and (3.17) coincide with those of the polynomials introduced by Pastro [4]. These polynomials have the following explicit expression

$$P_n(z) = \left(\frac{\nu}{\beta}\right)^n \frac{(\beta\mu/\nu)_n}{(1/\beta)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, 1/\beta \\ \frac{\nu q^{1-n}}{\beta\mu} \end{matrix} ; q, \frac{zq}{\mu}\right), \tag{3.18}$$

where $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-shifted factorial and ${}_2\phi_1$ is the basic hypergeometric function [2].

In order to find explicit expression for the moments c_n, we use formula (1.15) of the transformation for the moments c_n under the CT. The closure condition (3.1) is thus equivalent to the condition

$$\frac{c_{n+1} - \nu c_n}{c_1 - \nu} = q^{-n}\frac{c_{n+1} - \mu c_n}{c_1 - \mu}, \quad n = 0, \pm 1, \pm 2, \dots . \tag{3.19}$$

Taking into account the fact that $c_0 = 1$, $c_1 = d_0$, we find from (3.19)that

$$c_n = \mu^n \frac{(\nu/(\beta\mu)\,;q)_n}{(1/\beta\,;q)_n}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{3.20}$$

where q-shifted factorials for negative n are defined [2] as

$$(a\,;q)_n = \frac{1}{(aq^{-n}\,;q)_n}.$$

We thus have the following:

Proposition 1 *The closure condition* (3.1) *characterizes the Pastro polynomials.*

An interesting property of the Pastro polynomials is their self-similarity property with respect to the involutions $\mathcal{Q}$ and $\mathcal{T}$. Indeed, consider what happens under the involution $\mathcal{Q}$. We have from (3.20) the expression for the new moments

$$c_n^{\{Q\}} = c_{-n} = \nu^{-n}\frac{(\beta q\,;q)_n}{(\beta\mu q/\nu\,;q)_n}. \tag{3.21}$$

Comparing (3.20) and (3.21), we see that the moments $c_n^{\{Q\}}$ are obtained from the moments c_n by the folowing substitution of the parameters: $\mu \to 1/\nu$, $\nu \to 1/\mu$, $\beta \to \nu/(\beta\mu q)$. Hence, the corresponding polynomials $Q_n(z)$ have the same structure, *i.e.*, they are again the Pastro polynomials. (This was noticed by Pastro himself [4]). It is shown similarly that the polynomials $T_n(z)$ defined by (1.8) are again Pastro polynomials with modified parameters.

Note that the Pastro polynomials are believed to be "the most general" LBP having an explicit expression in terms of basic hypergeometric functions (see also [1]).

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4 Another Example of 1-Closure

In this section we consider one more example of closure condition leading to coefficients b_n, d_n expressible in terms of elementary functions. Consider the scheme $(1, 0\,;\, 0, 1)$. In terms of the polynomials this means

$$C(\mu)\{P_n(z)\} = q^n P_n^{(1)}(z/q), \tag{4.1}$$

where $P_n^{(1)}(z)$ are the 1-associated LBP defined in (1.10).

In terms of the recurrence coefficients this condition reads

$$qb_{n+1} = b_n \frac{b_{n+1} + U_n}{b_n + U_{n-1}}, \tag{4.2}$$

$$qd_{n+1} = d_n \frac{d_{n+1} + U_{n+1}}{d_n + U_n}, \tag{4.3}$$

$$U_n + d_n = \mu(1 + b_n/U_{n-1}), \quad n = 1, 2, 3, \dots \tag{4.4}$$

with initial condition

$$U_0 + d_0 = \mu. \tag{4.5}$$

Omitting simple technical details, we merely give the generic solution of the system (4.2)–(4.4) which is:

$$d_n = -\frac{\mu\gamma_1}{q\gamma_2(1 + \gamma_1 q^{n-1})}, \tag{4.6}$$

$$b_{n+1} = -\frac{\mu\gamma_1(1 + \gamma_2 q^n)}{q\gamma_2(1 + \gamma_1 q^{n-1})(1 + \gamma_1 q^n)}, \quad n = 1, 2, 3, \dots \tag{4.7}$$

$$U_n = \frac{\mu\gamma_1}{q\gamma_2} \frac{b_{n+1}}{d_{n+1}}. \tag{4.8}$$

The initial condition (4.5) yields

$$d_0 = -\mu/\gamma_2, \quad b_1 = -\frac{\mu(\gamma_2 + 1)}{\gamma_2(\gamma_1 + 1)}. \tag{4.9}$$

Note that the corresponding polynomials depend in an essential way on 2 (arbitrary) parameters γ_1, γ_2 (the parameter μ can be reduced to $\mu = 1$ by a scaling transformation).

The polynomials $P_n(z)$ corresponding to these recurrence coefficients do not coincide with any known system of LBP. It would be interesting to find explicit expressions for these polynomials.

5 Difference Equation for q-Semiclassical LBP

In this section we return to the generic case $(N, 0\ ;\ M, 0)$ corresponding (in our terminology) to the q-semi-classical LBP.

Explicitly we have the condition

$$P_n(z\,;\mu_1,\mu_2,\dots,\mu_N) = q^n P_n(z/q\,;\nu_1,\nu_2,\dots,\nu_M). \tag{5.1}$$

In order to derive the difference equation for the q-semi-classical LBP we first need the following:

Lemma 2 *Let $P_n(z)$ be* LBP *satisfying the recurrence relation* (1.1). *The polynomial $P_{n+j}(z)$, $j = 1, 2, \dots$, can be expressed in terms of $P_n(z)$, $P_{n-1}(z)$ according to*

$$P_{n+j}(z) = A_j(z\,;n)P_n(z) + B_j(z\,;n)P_{n-1}(z), \tag{5.2}$$

where $A_j(z\,;n)$ and $B_j(z\,;n)$ are polynomials in z of degree j with coefficients depending on n.

The proof is almost obvious: for $j = 1$ we have from the recurrence relation $P_{n+1}(z) = (z - d_n)P_n(z) + zb_nP_{n-1}(z)$, *i.e.*, $A_1(z\,;n) = z - d_n$, $B_1(z\,;n) = zb_n$ are polynomials of the first degree. Then the Lemma is proven by induction in j.

As a consequence of this Lemma we have that

$$P_n(z\,;\mu_1,\mu_2,\dots,\mu_N) = \frac{A_N(z\,;n)P_n(z) + B_N(z\,;n)P_{n-1}(z)}{(z-\mu_1)(z-\mu_2)\cdots(z-\mu_N)}, \tag{5.3}$$

where $A_N(z\ ;\ n)$ and $B_N(z\ ;\ n)$ are some N-order polynomials in z. Analogous expressions can be written down for the polynomials $P_n(z\,;\nu_1,\nu_2,\dots,\nu_M)$.

Using these expressions we can extract from the condition (5.1) the following system

$$\begin{aligned} P_n(qz) &= X(z\,;n)P_n(z) + Y(z\,;n)P_{n-1}(z),\\ P_{n-1}(qz) &= V(z\,;n)P_n(z) + W(z\,;n)P_{n-1}(z), \end{aligned} \tag{5.4}$$

where $X(z,n)$, $Y(z,n)$, $V(z,n)$, $W(z,n)$ are rational functions in z of fixed degrees.

From (5.4), we then easily find the second-order difference equation

$$P_n(q^2z) = \Omega_1(z\,;n)P_n(zq) + \Omega_2(z\,;n)P_n(z), \tag{5.5}$$

where

$$\begin{aligned} \Omega_1(z\,;n) &= X(qz\,;n) + \frac{W(z\,;n)}{Y(z\,;n)},\\ \Omega_2(z\,;n) &= Y(qz\,;n)V(z\,;n) - \frac{W(z\,;n)X(z\,;n)}{Y(z\,;n)} \end{aligned} \tag{5.6}$$

are rational functions whose degrees do not depend on n.

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