ON DUALITY IN COMPLEX LINEAR PROGRAMMING

Dedicated to the memory of Hanna Neumann

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Introduction

In [3], Levinson proved a duality theorem for linear programming in complex space. Ben-Israel [1] generalized this result to polyhedral convex cones in complex space. In this paper, we give a simple proof of Ben-Israel's result based directly on the duality theorem for linear programming in real space. The explicit relations shown between complex and real linear programs should be useful in actually computing a solution for the complex case. We also give a simple proof of Farkas' theorem, generalized to polyhedral cones in complex space ([1], Theorem 3.5); the proof depends only on the classical form of Farkas' theorem for real space.

Notation and preliminary results

Denote by R^n (resp. C^n) n-dimensional real (resp. complex) space; denote by $R^{m \times n}$ (resp. $C^{m \times n}$) the vector space of all $m \times n$ real (resp. complex) matrices; denote by $R^n_+ = \{x \in R^n : x_i \geqq 0, 1 \leqq i \leqq n\}$ the *non-negative orthant* of R^n; and for $x, y \in R^n$, $x \geqq y$ denotes $x - y \in R^n_+$. If A is a matrix, then A^T, $\bar{A}$, A^H denote its *transpose, complex conjugate, conjugate transpose.*

In this paper, a *cone* in R^n means a closed polyhedral convex cone (in terminology of [1]), defined here as a finite intersection of closed half-spaces in R^n, each half-space containing 0 in its boundary. Thus S is a cone in R^n iff there is an integer r and $K \in R^{r \times n}$ such that

(1) $$S = \{x \in R^n : Kx \geqq 0\}.$$

(Since trivially $S + S \subset S$ and $\alpha S \subset S$ for $\alpha \in R_+$, S is a convex cone by usual definition.)

The dual cone S^* is defined as

(2) $$S^* = \{y \in R^n : x \in S \Rightarrow y^T x \geqq 0\}.$$

Therefore, if S is defined by the matrix K,

$$S^* = \{y \in R^n : Kx \geqq 0 \Rightarrow y^T x \geqq 0\}.$$

Since, by Farkas' theorem [2]

$$[Kx \geqq 0 \Rightarrow y^T x \geqq 0] \Leftrightarrow [\exists z \geqq 0 : y = K^T z], \tag{3}$$

$$S^* = \{y \in R^n : \exists z \geqq 0 : y = K^T z\} \tag{4}$$

Since $v \in (S^*)^* \Leftrightarrow [y \in S^* \Rightarrow v^T y \geqq 0]$

$$\Leftrightarrow [z \geqq 0 \Rightarrow v^T K^T z \geqq 0] \qquad \text{by (4)}$$

$$\Leftrightarrow Kv \geqq 0,$$

$$(S^*)^* = S. \tag{5}$$

Each vector $z \in C^n$ may be written $z = x + iy$ when $x, y \in R^n$; this defines a natural map ρ of C^n onto $R^n \times R^n = R^{2n}$. Define $S \subset C^n$ to be a cone iff ρS is a cone in R^{2n}. (This is not the definition in [1], but is equivalent to it, and its use simplifies the proofs). Thus, by (1),

$$x + iy \in S \Leftrightarrow [K_1 K_2] \begin{pmatrix} x \\ y \end{pmatrix} \geqq 0 \tag{6}$$

where K_1 and K_2 are real matrices of appropriate dimensions.
Setting

$$\bar{z} = x - iy \text{ and } K = K_1 + iK_2,$$

$$\begin{aligned} z \in S &\Leftrightarrow K_1(z + \bar{z}) - iK_2(z - \bar{z}) \geqq 0 \\ &\Leftrightarrow \bar{K}z + K\bar{z} \geqq 0 \\ &\Leftrightarrow \operatorname{Re}(\bar{K}z) \geqq 0. \end{aligned} \tag{7}$$

Define S^* as the dual cone of S iff $\rho(S^*) = (\rho S)^*$. Then

$$\begin{aligned} u + iv \in S^* &\Leftrightarrow \exists w \geqq 0 : \begin{pmatrix} K_1^T \\ K_2^T \end{pmatrix} w = \begin{pmatrix} u \\ v \end{pmatrix} \\ &\Leftrightarrow \exists w \geqq 0 : u + iv = K^T w. \end{aligned} \tag{8}$$

Note that w is a *real* non-negative vector.

One version of the duality theorem for real linear programming states that the dual of (P) is (D), where

(P): Minimize $c^T x$ subject to $Hx - b \geqq 0$

(D): Maximize $b^T y$ subject to $H^T y = c$ and $y \geqq 0$.

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Complex duality

We first extend the duality theorem for linear programming to (closed convex polyhedral) cones in real space. Let $S \subset R^n$ and $T \subset R^m$ be cones, defined, using (1), by matrices M and K; let $C \in R^n$, $b \in R^m$, $A \in R^{m \times n}$. Consider the following two problems:

(P1): Minimize $c^T x$ subject to $Ax - b \in T$ and $x \in S$.

(D1): Maximize $b^T y$ subject to $-A^T y + c \in S^*$ and $y \in T^*$.

THEOREM 1. *If either (P1) or (D1) has an optimal solution, then both have optimal solutions, and Minimum (P1) = Maximum (D1).*

PROOF. Using (1), (P1) is equivalent to

(P1′): Minimize $c^T x$ subject to $K(Ax-b) \geqq 0$ and $Mx \geqq 0$. Substituting

$$\begin{pmatrix} A \\ M \end{pmatrix} \text{ for } H, \begin{pmatrix} b \\ 0 \end{pmatrix} \text{ for } b, \text{ and } \begin{pmatrix} w \\ v \end{pmatrix} \text{ for } y, \text{ in (P)},$$

the duality theorem of linear programming shows that the dual of (P1′) is

(D1′): Maximize $b^T K^T w$ subject to $A^T K^T w + M^T v = c$, $w \geqq 0$, $v \geqq 0$.

Since from (4), $y = K^T w \in T^*$ and $z = M^T v \in S^*$, (D1′) is the same as (D1).

Consider the following two problems in complex space, where $S \subset C^n$ and $T \subset C^m$ are cones defined by matrices M and K, $c \in C^n$, $b \in C^m$, $A \in C^{m \times n}$.

(P2): Minimize Re $c^H z$ subject to $Az - b \in T$ and $z \in S$

(D2): Maximize Re $b^H w$ subject to $- A^H w + c \in S^*$ and $w \in T^*$.

THEOREM 2. *If either (P2) or (D2) has an optimal solution, then both have optimal solutions, and Minimum (P2) = Maximum (D2).*

PROOF. (P2) can be written as a problem in real space as follows; denoting real and imaginary parts by the suffixes r and i:

(P2′): Minimize $c_r^T z_r + c_i^T z_i$

$$\text{Subject to } \begin{pmatrix} A_r z_r - A_i z_i - b_r \\ A_i z_r + A_r z_i - b_i \end{pmatrix} \in \rho T$$

$$\begin{pmatrix} z_r \\ z_i \end{pmatrix} \in \rho S.$$

By Theorem 1, its dual is

(P2′): Maximize $b_r^T w_1 + b_i^T w_2$.

$$\text{Subject to } \left(- \begin{pmatrix} A_r^T & A_i^T \\ -A_i^T & A_r^T \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} c_r \\ c_i \end{pmatrix} \right) \in (\rho S)^*$$

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$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in (\rho T)^*;$$

where w_1 and w_2 denote the two real vectors involved.

Setting $w = w_1 + iw_2$, i.e., $w_r = w_1$ and $w_i = w_2$, and noting that $(\rho S)^* = \rho(S^*)$, it is seen that (D2′) is identical with (D2).

Generalized Farkas Theorem

THEOREM 3. *Let* $S \subset R^n$ *be a (closed polyhedral convex) cone,* $b \in R^n$, $A \in R^{m \times n}$. *Then*

$$[Ax \in S \Rightarrow b^T x \geqq 0] \Leftrightarrow [\exists u \in S^*: A^T u = b].$$

PROOF. Define S, using (1), by a matrix K. Then

$$\begin{aligned} [Ax \in S \Rightarrow b^T x \geqq 0] &\Leftrightarrow [KAx \geqq 0 \Rightarrow b^T x \geqq 0] \\ &\Leftrightarrow [\exists z \geqq 0: b = (KA)^T z] \qquad \text{by (3)} \\ &\Leftrightarrow [\exists z \geqq 0: b = A^T(K^T z)] \\ &\Leftrightarrow [\exists u = K^T z \in S^*: A^T u = b] \quad \text{by (4).} \end{aligned}$$

THEOREM 4. *Let* $S \subset C^n$ *be a (closed polyhedral convex) cone,* $b \in C^n$, $A \in C^{m \times n}$. *Then*

$$[Az \in S \Rightarrow \operatorname{Re} b^H z \geqq 0] \Leftrightarrow [\exists w \in S^*: A^H w = b].$$

PROOF. $$Az \in S \Rightarrow \operatorname{Re} b^H z \geqq 0$$

iff $$\begin{pmatrix} A_r & -A_i \\ A_i & A_r \end{pmatrix} \begin{pmatrix} z_r \\ z_i \end{pmatrix} \in \rho S \Rightarrow \begin{pmatrix} b_r \\ b_i \end{pmatrix}^T \begin{pmatrix} z_r \\ z_i \end{pmatrix} \geqq 0$$

iff $$\exists \begin{pmatrix} u \\ v \end{pmatrix} \in (\rho S)^* = \rho(S^*): \begin{pmatrix} A_r^T & A_i^T \\ -A_i^T & A_r^T \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_r \\ b_i \end{pmatrix}$$

by Theorem 3,

iff $$\exists w = u + iv \in S^*: A^H w = b.$$

References

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