Canad. J. Math. Vol. **59** (6), 2007 pp. 1207–1222

H^p-Maximal Regularity and Operator Valued Multipliers on Hardy Spaces

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Abstract. We consider maximal regularity in the H^p sense for the Cauchy problem $u'(t) + Au(t) = f(t)$ $(t \in \mathbb{R})$, where A is a closed operator on a Banach space X and f is an X-valued function defined on $\mathbb{R}$. We prove that if X is an AUMD Banach space, then A satisfies H^p-maximal regularity if and only if A is Rademacher sectorial of type $< \frac{\pi}{2}$. Moreover we find an operator A with H^p-maximal regularity that does not have the classical L^p-maximal regularity. We prove a related Mikhlin type theorem for operator valued Fourier multipliers on Hardy spaces $H^p(\mathbb{R}; X)$, in the case when X is an AUMD Banach space.

1 Introduction and Background

Let X be a complex Banach space. Let $-A$ be the infinitesimal generator of a bounded analytic semigroup on X. We consider the following Cauchy problem on $\mathbb{R}$:

$$u'(t) + A(u(t)) = f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where f is an X-valued function defined on $\mathbb{R}$.

Let $D(A)$ denote the domain of A and let $1 < p < \infty$. We let A_l be the tensor extension of A on $L^p(\mathbb{R}; X)$ given by $(A_l u)(t) = A(u(t))$, with domain $D(A_l) = L^p(\mathbb{R}; D(A))$. Then we let B_l be the derivation operator on $L^p(\mathbb{R}; X)$ given by $B_l u = u'$, with domain the Sobolev space $D(B_l) = W^{1,p}(\mathbb{R}; X)$. We say that A has L^p-maximal regularity if there exists a constant $C > 0$ such that

$$\|A_l u\|_p \leq C \|A_l u + B_l u\|_p, \quad u \in D(A_l) \cap D(B_l).$$

Here $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R}; X)$. It is well known that this property does not depend on $1 < p < \infty$. Moreover if A is invertible, this is equivalent to the property that for any $f \in L^p(\mathbb{R}; X)$, there is a unique $u \in D(A_l) \cap D(B_l)$ verifying (1.1). Thus L^p-maximal regularity means that (1.1) can be solved in $L^p(\mathbb{R}; X)$. We refer the reader to [2,11,17,18,21,25] for recent results and developments on abstract L^p-maximal regularity and related topics. See also the excellent survey [19] and the references therein.

The starting point of this work is the paper [25] by Lutz Weis giving a characterization of L^p-maximal regularity in terms of the Rademacher boundedness of the

Received by the editors February 27, 2005; revised April 14, 2006.
The first author was supported by the NSF of China, Specialized Research Fund for the Doctoral Program of Higher Education and Tsinghua Basic Research Foundation (Grant No. jCpy2005056)
AMS subject classification: 42B30, 47D06.
Keywords: L^p-maximal regularity, H^p-maximal regularity, Rademacher boundedness.

https://doi.org/10.4153/CJM-2007-051-5 Published online by Cambridge University Press

resolvent of A, in the case when X is a UMD Banach space. Here is a brief presentation.

Let $(\varepsilon_k)_{k\geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. That is to say, the ε_k are $\{-1, 1\}$-valued, pairwise independent, random variables such that $\mathbb{P}\{\varepsilon_k = 1\} = \mathbb{P}\{\varepsilon_k = -1\} = \frac{1}{2}$ for any $k \geq 1$. We let Rad be the linear span of the signs ε_k. Then Rad $\otimes X$ is the space of all finite sums $\sum_{k\geq 1} \varepsilon_k x_k$, with $x_k \in X$. For any $1 \leq p < \infty$, we let $\mathrm{Rad}_p(X)$ be the closed subspace of $L^p(\Omega; X)$ spanned by Rad $\otimes X$, that we equip with the induced norm. We recall that for any $1 \leq p, q < \infty$, the two norms $\|\cdot\|_{\mathrm{Rad}_p(X)}$ and $\|\cdot\|_{\mathrm{Rad}_q(X)}$ are equivalent on Rad $\otimes X$ (see [23, Theorem 1.e.13]).

Let $\mathcal{L}(X)$ denote the algebra of all bounded operators on X. We say that a subset $\mathcal{M}$ of $\mathcal{L}(X)$ is *Rademacher bounded* if there exists a constant $K \geq 0$ such that

$$\Big\| \sum_{k\geq 1} \varepsilon_k T_k(x_k) \Big\|_{\mathrm{Rad}_p(X)} \leq K \Big\| \sum_{k\geq 1} \varepsilon_k x_k \Big\|_{\mathrm{Rad}_p(X)}$$

for any finite family $(x_k)_{k\geq 1}$ of X and for any finite family $(T_k)_{k\geq 1}$ of $\mathcal{M}$. We let $\mathcal{R}_p(\mathcal{M})$ denote the smallest K verifying this property. The above property does not depend on p, but the value of $\mathcal{R}_p(\mathcal{M})$ does.

Let $\omega \in (0, \pi)$, and let A be a closed and densely defined operator on X. We recall that A is a sectorial operator of type ω if the spectrum of A is included in the closure of the sector $\Sigma_\omega = \{z \in \mathbb{C}^* : |\operatorname{Arg}(z)| < \omega\}$, and for any angle $\theta \in (\omega, \pi)$, there is a constant K_θ such that $\|\lambda(\lambda - A)^{-1}\| \leq K_\theta$ for any $\lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}$. If further the set

$$\{\lambda(\lambda - A)^{-1} : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} \subset \mathcal{L}(X)$$

is Rademacher bounded for any $\theta \in (\omega, \pi)$, then we say that A is Rademacher sectorial of type ω.

Recall that $-A$ is the infinitesimal generator of a bounded analytic semigroup on X if and only if A is a sectorial operator of type $< \frac{\pi}{2}$. Then Weis's characterization theorem in [25] asserts that if X is a UMD Banach space, then A has L^p-maximal regularity if and only if A is Rademacher sectorial of type $< \frac{\pi}{2}$. There are two approaches to the "if" part of this statement. The original one [25] was to derive it from a Mikhlin-type theorem for operator valued Fourier multipliers on $L^p(\mathbb{R}; X)$ satisfying certain Rademacher boundedness conditions. Later on, Kalton and Weis [18] found a second proof based on H^∞ functional calculus.

In this paper we introduce an analytic form of maximal regularity, called H^p-maximal regularity. Instead of considering the Cauchy problem (1.1) on $L^p(\mathbb{R}; X)$, we study it on the so-called conjugate Hardy space $H^p_{\mathrm{con}}(\mathbb{R}; X)$. This notion will bring out the role of analytic UMD spaces (AUMD in short). It will be clear from the definition that L^p-maximal regularity implies H^p-maximal regularity. We will show in Section 3 that the converse is false. More precisely we will provide an invertible operator A which is sectorial of type ω for any $\omega > 0$, such that A has H^p-maximal regularity, but A does not have L^p-maximal regularity. Also we will establish an analytic version of Weis's characterization of regularity for an AUMD space X: an operator A on X has H^p-maximal regularity if and only if it is Rademacher sectorial of type $< \frac{\pi}{2}$. As for the classical case, the "if" part of this characterization theorem has two proofs. One is based on H^∞ functional calculus; the other is a consequence of an operator valued Fourier multiplier theorem on X-valued Hardy spaces in the case when X is an AUMD space. This result, which is of independent interest, is established in Section 2. It extends a remarkable scalar valued multiplier theorem due to Blower [7], and it turns out to be an analytic version of Weis's multiplier theorem in [25].

We refer the reader to [8] for some background on UMD Banach spaces, and to [1, 19] for comprehensive information on sectorial operators, H^∞ functional calculus, Rademacher boundedness and abstract L^p-maximal regularity. We record for further use the so-called *contraction principle*. For any $1 \leq p < \infty$, for any finite family $(x_k)_{k\geq 1}$ in an arbitrary Banach space X, and for any bounded family $(\alpha_k)_{k\geq 1}$ of complex numbers, we have

$$\Big\| \sum_{k\geq 1} \alpha_k \varepsilon_k x_k \Big\|_{\mathrm{Rad}_p(X)} \leq 2 \sup_k |\alpha_k| \Big\| \sum_{k\geq 1} \varepsilon_k\, x_k \Big\|_{\mathrm{Rad}_p(X)}. \tag{1.2}$$

2 Operator Valued Multipliers on AUMD Banach Spaces

Let X be a Banach space. For any $f \in L^1(\mathbb{R}; X)$, let $\mathcal{F}(f) = \widehat{f} \colon \mathbb{R} \to X$ be the Fourier transform, defined by

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t) e^{-ist}\, dt, \qquad s \in \mathbb{R}.$$

Then we let $H^1(\mathbb{R}; X)$ be the space of all $f \in L^1(\mathbb{R}; X)$ such that $\widehat{f}(s) = 0$ for any $s \leq 0$.

We let $1 \leq p < \infty$. Unless stated otherwise, this condition will remain in force throughout the paper. We let $H^p(\mathbb{R}; X)$ be the closure in $L^p(\mathbb{R}; X)$ of the subspace $L^p(\mathbb{R}; X) \cap H^1(\mathbb{R}; X)$. In the case when $X = \mathbb{C}$, this space coincides with the classical Hardy space $H^p(\mathbb{R})$ (see [13]). The vector valued Hardy space $H^p(\mathbb{R}; X)$ has several equivalent definitions. First, $H^p(\mathbb{R}; X) \subset L^p(\mathbb{R}; X)$ is the subspace of all functions whose Poisson integral on the upper half plane of $\mathbb{C}$ is analytic. Second, $H^p(\mathbb{R}; X)$ is the closure of $H^p(\mathbb{R}) \otimes X$ in $L^p(\mathbb{R}; X)$. Third, a function $f \in L^p(\mathbb{R}; X)$ belongs to $H^p(\mathbb{R}; X)$ if and only if the scalar valued function $t \mapsto \langle \varphi, f(t) \rangle$ belongs to $H^p(\mathbb{R})$ for any $\varphi \in X^*$. We refer to [22, §4] for more on these spaces.

We aim at defining Fourier multipliers on $H^p(\mathbb{R}; X)$, so we introduce the space $\mathcal{U}$ of all C^∞ functions $f \colon \mathbb{R} \to \mathbb{C}$ belonging to $H^1(\mathbb{R})$ such that $\lim_{|t|\to\infty} |t^2 f(t)| = 0$. By [13, Ch. II; Corollary 3.3], $\mathcal{U}$ is a dense subspace of $H^p(\mathbb{R})$. Thus $\mathcal{U} \otimes X$ is dense in $H^p(\mathbb{R}; X)$.

Let $M \colon \mathbb{R}_+^* \to \mathcal{L}(X)$ be a bounded measurable function. For any $f \in \mathcal{U} \otimes X$, the function $\widehat{f}$ belongs to $L^1(\mathbb{R}) \otimes X$. Hence the vector-valued function $M\widehat{f}$ admits an inverse Fourier transform given by

$$\big[\mathcal{F}^{-1}(M\widehat{f}\,)\big](t) = \frac{1}{2\pi} \int_0^\infty M(s)\big(\widehat{f}(s)\big)\, e^{its}\, ds, \qquad t \in \mathbb{R}.$$

https://doi.org/10.4153/CJM-2007-051-5 Published online by Cambridge University Press

We say that M is a bounded Fourier multiplier on $H^p(\mathbb{R};X)$ if there is a constant $C \geq 0$ such that

$$\left\| \mathcal{F}^{-1}(M\widehat{f})\right\|_p \leq C\|f\|_p, \quad f \in \mathcal{U}\otimes X.$$

In this case, $\mathcal{F}^{-1}(M\widehat{f})$ belongs to $H^p(\mathbb{R};X)$ for any $f \in \mathcal{U}\otimes X$, and the resulting mapping $f \mapsto \mathcal{F}^{-1}(M\widehat{f})$ uniquely extends to a bounded linear operator

$$T_M \colon H^p(\mathbb{R};X) \longrightarrow H^p(\mathbb{R};X).$$

Moreover, the norm of T_M is equal to the smallest possible C in the above inequality. For simplicity, $\|T_M\|$ will also be called the norm of the multiplier M.

We will need a similar notion of Fourier multipliers for periodic functions. Let $\mathbb{T}$ be the unit circle equipped with its normalized Haar measure, which we identify with the interval $[-\pi,\pi)$ equipped with the measure $\frac{dt}{2\pi}$. For any $f \in L^1(\mathbb{T};X)$, the X-valued Fourier coefficients are defined by

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt, \quad k \in \mathbb{Z}.$$

We let $H^p_0(\mathbb{T};X)$ be the subspace of $L^p(\mathbb{T};X)$ of all functions f such that $\widehat{f}(k) = 0$ for any $k \leq 0$. Let $e_k(t) = e^{ikt}$ for any $k \in \mathbb{Z}$ and let $\mathcal{A}$ be the linear span of $\{e_k\,;\, k \geq 1\}$. Then $\mathcal{A}\otimes X$ is a dense subspace of $H^p_0(\mathbb{T};X)$.

Let $(M_k)_{k\geq 1}$ be a bounded sequence of $\mathcal{L}(X)$. We say that $(M_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H^p_0(\mathbb{T};X)$ if there is a constant $C \geq 0$ such that

$$\Big\| \sum_{k\geq 1} M_k\big(\widehat{f}(k)\big) \otimes e_k \Big\|_p \leq C\|f\|_p, \quad f \in \mathcal{A}\otimes X.$$

In this case, there is a unique bounded linear operator $H^p_0(\mathbb{T};X) \to H^p_0(\mathbb{T};X)$ taking any $f = \sum_{k\geq 1}\widehat{f}(k)\otimes e_k$ in $\mathcal{A}\otimes X$ to $\sum_{k\geq 1} M_k\big(\widehat{f}(k)\big)\otimes e_k$, and its norm is equal to the smallest possible C in the above inequality.

The following transfer result was established in [22, Proposition 4.3] for scalar valued multipliers. It is easy to check that its proof works as well for operator valued multipliers.

Lemma 2.1 *Let $M\colon \mathbb{R}^*_+ \to \mathcal{L}(X)$ be a bounded continuous function. Let $1 \leq p < \infty$ and let $C > 0$ be a constant. The following two assertions are equivalent.*

(i) *M is a bounded Fourier multiplier on $H^p(\mathbb{R};X)$ whose norm is less than or equal to C.*

(ii) *For any $\varepsilon > 0$, the sequence $(M(\varepsilon k))_{k\geq 1}$ is a bounded Fourier multiplier on $H^p_0(\mathbb{T};X)$ whose norm is less than or equal to C.*

We shall now observe that Rademacher boundedness is a necessary condition for a bounded Fourier multiplier on either $H^p(\mathbb{R};X)$ or $H^p_0(\mathbb{T};X)$. This is an easy variant of analogous results which were proved for multipliers on $L^p(\mathbb{R};X)$ [11] and on $L^p(\mathbb{T};X)$ [2]. See also [16] for related results.

Proposition 2.2 *Let $1 \leq p < \infty$.*

(i) *Let $(M_k)_{k\geq 1}$ be a bounded sequence of $\mathcal{L}(X)$, and assume that it is a bounded Fourier multiplier on $H^p_0(\mathbb{T};X)$. Then the set $\{M_k : k \geq 1\} \subset \mathcal{L}(X)$ is Rademacher bounded.*

(ii) *Let $M\colon \mathbb{R}^*_+ \to \mathcal{L}(X)$ be a bounded continuous function, and assume that M is a bounded Fourier multiplier on $H^p(\mathbb{R};X)$. Then the set $\{M(t) : t > 0\} \subset \mathcal{L}(X)$ is Rademacher bounded.*

Proof Assume that $(M_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H^p_0(\mathbb{T};X)$ and let $C \geq 0$ be its norm. Then arguing as in the proof of [2, Proposition 1.11] we obtain that $\{M_k : k \geq 1\} \subset \mathcal{L}(X)$ is Rademacher bounded, with $\mathcal{R}_p(\{M_k : k \geq 1\}) \leq 4C$.

Now let $M\colon \mathbb{R}^*_+ \to \mathcal{L}(X)$ be a bounded continuous function which is a bounded Fourier multiplier on $H^p(\mathbb{R};X)$, and let $C \geq 0$ be its norm. According to Lemma 2.1, the sequence $(M(\varepsilon k))_{k\geq 1}$ is a bounded Fourier multiplier on $H^p_0(\mathbb{T};X)$ for any $\varepsilon > 0$, and its norm is $\leq C$. Then by the first part of this proof, the sets

$$\mathcal{M}_\varepsilon = \{M(\varepsilon k) : k \geq 1\}$$

are Rademacher bounded and $\mathcal{R}_p(\mathcal{M}_\varepsilon) \leq 4C$ for any $\varepsilon > 0$. Since M is continuous, this implies that the set $\{M(t) : t > 0\}$ is Rademacher bounded. ■

We review the definition of analytic martingales and AUMD spaces, and some of their properties. We refer the reader to [12] and [8, §§7,8] for proofs and further results (see also [7, 15, 22]). We consider the compact space $\mathbb{T}^{\mathbb{N}}$ equipped with Haar measure. We use the notation $\tau = (t_1, \dots, t_k, \dots)$ for elements of $\mathbb{T}^{\mathbb{N}}$. For any integer $k \geq 1$, let $\mathcal{F}_k$ denote the σ-field generated by the first k variables $t_1, \dots, t_k$. Then let $\mathcal{F}_0$ be the trivial σ-field. An X-valued martingale with respect to the filtration $(\mathcal{F}_k)_{k\geq 1}$ is a sequence of functions $g_k\colon \mathbb{T}^{\mathbb{N}} \to X$, $k \geq 1$ such that for any $k \geq 1$, g_k is $\mathcal{F}_k$-measurable and $g_{k-1} = \mathbb{E}(g_k|\mathcal{F}_{k-1})$, with the convention that $g_0 = 0$. We let $dg_k = g_k - g_{k-1}$ denote the martingale differences for any $k \geq 1$.

Next, we say that the martingale $(g_k)_{k\geq 1}$ is analytic if for any $k \geq 1$ there exists a measurable function $\Phi_k\colon \mathbb{T}^{k-1} \to X$ such that

$$dg_k(\tau) = \Phi_k(t_1, \dots, t_{k-1})e^{it_k}, \quad \tau = (t_1, \dots, t_k, \dots) \in \mathbb{T}^{\mathbb{N}}. \tag{2.1}$$

By definition, X is an AUMD Banach space if there is a constant $K_p > 0$ such that for any X-valued analytic martingale $(g_k)_{k\geq 1}$, for any bounded sequence $(\alpha_k)_{k\geq 1}$ of complex numbers and for any integer $N \geq 1$, we have

$$\Big\| \sum_{k=1}^{N} \alpha_k dg_k \Big\|_p \leq K_p \sup_{k\geq 1} |\alpha_k| \Big\| \sum_{k=1}^{N} dg_k \Big\|_p. \tag{2.2}$$

This property does not depend on $1 \leq p < \infty$, and any UMD Banach space is AUMD. Indeed for any $1 < p < \infty$, X is a UMD Banach space if and only if there is a constant $K_p > 0$ such that (2.2) holds for any X-valued martingale with respect to the filtration $(\mathcal{F}_k)_{k\geq 1}$. Any subspace of an AUMD Banach space is AUMD, and the class of AUMD spaces includes L^1-spaces. Indeed, for any measure space Σ and for any $1 \leq q < \infty$, the space $L^q(\Sigma; X)$ is AUMD provided that X is AUMD.

The following observation will be useful. See [14] for related results.

Lemma 2.3 *Let X be an AUMD Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be a Rademacher bounded set. For any $1 \leq p < \infty$, there exists a constant $C_p > 0$ such that for any X-valued analytic martingale $(g_k)_{k\geq 1}$, and for any finite family $(T_k)_{k\geq 1}$ of $\mathcal{M}$, we have*

$$\Big\| \sum_{k\geq 1} T_k(dg_k) \Big\|_p \leq C_p \Big\| \sum_{k\geq 1} dg_k \Big\|_p.$$

Proof Let $(T_k)_{k\geq 1}$ be a finite family of $\mathcal{M}$. If $(g_k)_{k\geq 1}$ satisfies (2.1), then we have

$$[T_k(dg_k)](\tau) = T_k(\Phi_k(t_1, \dots, t_{k-1}))e^{it_k}$$

for any $\tau = (t_1, \dots, t_k, \dots) \in \mathbb{T}^{\mathbb{N}}$. Hence the $T_k(dg_k)$'s are the differences of an X-valued analytic martingale. Hence (2.2) yields

$$\Big\| \sum_{k\geq 1} T_k(dg_k) \Big\|_p \leq K_p \Big\| \sum_{k\geq 1} \varepsilon_k(\lambda) T_k(dg_k) \Big\|_p$$

for any $\lambda \in \Omega$. Integrating over Ω and applying Fubini's theorem, we deduce that

$$\begin{aligned}
\Big\| \sum_{k\geq 1} T_k(dg_k) \Big\|_p^p &\leq K_p^p \int_{\mathbb{T}^{\mathbb{N}}} \Big\| \sum_{k\geq 1} \varepsilon_k\, T_k(dg_k(\tau)) \Big\|_{\mathrm{Rad}_p(X)}^p d\tau \\
&\leq K_p^p\, \mathcal{R}_p(\mathcal{M})^p \int_{\mathbb{T}^{\mathbb{N}}} \Big\| \sum_{k\geq 1} \varepsilon_k\, dg_k(\tau) \Big\|_{\mathrm{Rad}_p(X)}^p d\tau \\
&\leq K_p^p \mathcal{R}_p(\mathcal{M})^p \int_{\Omega} \Big\| \sum_{k\geq 1} \varepsilon_k(\lambda)\, dg_k \Big\|_p^p d\mathbb{P}(\lambda).
\end{aligned}$$

In turn, (2.2) yields

$$\Big\| \sum_{k\geq 1} \varepsilon_k(\lambda) dg_k \Big\|_p \leq K_p \Big\| \sum_{k\geq 1} dg_k \Big\|_p$$

for any $\lambda \in \Omega$. Hence we finally obtain that

$$\Big\| \sum_{k\geq 1} T_k(dg_k) \Big\|_p \leq K_p^2 \mathcal{R}_p(\mathcal{M}) \Big\| \sum_{k\geq 1} dg_k \Big\|_p. \qquad \blacksquare$$

For any sequence $(M_k)_{k\geq 1}$ in $\mathcal{L}(X)$, we set

$$\Delta M_k = M_k - M_{k-1}, \quad k \geq 2,$$

$$\Delta^2 M_k = \Delta\Delta M_k = M_k - 2M_{k-1} + M_{k-2}, \quad k \geq 3.$$

Differences of this kind are used in classical Mikhlin type theorems.

Theorem 2.4 *Let X be an AUMD Banach space and let $(M_k)_{k\geq 1}$ be a sequence of bounded operators on X. Assume that the sets*

$$\{M_k : k \geq 1\}, \quad \{k\Delta M_k : k \geq 2\}, \quad \text{and} \quad \{k^2\Delta^2 M_k : k \geq 3\} \tag{2.3}$$

are Rademacher bounded. Then $(M_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T}; X)$ for any $1 \leq p < \infty$.

This result is an extension of a remarkable theorem due to Blower [7]. Blower's theorem corresponds to the case when $p = 1$ and the M_k's are scalars. Namely, he shows that if X is an AUMD space, and if $(M_k)_{k\geq 1}$ is a sequence of $\mathbb{C}$ such that the three sets in (2.3) are bounded, then $(M_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^1(\mathbb{T}; X)$. A first observation is that with the same proof, one obtains that in this case $(M_k)_{k\geq 1}$ also is a bounded Fourier multiplier on $H_0^p(\mathbb{T}; X)$ for any $1 \leq p < \infty$. The proof of Theorem 2.4 is an adaptation of Blower's proof. Indeed many of Blower's arguments work as well in the operator-valued setting and we will only indicate the relevant modifications.

Proof of Theorem 2.4 Let $(M_k)_{k\geq 1}$ be as in the statement. We may assume that we have $M_1 = \cdots = M_7 = 0$. Then we consider the power series

$$v(z) = \sum_{k\geq 8} M_k z^k, \quad z \in \mathbb{C}, |z| < 1.$$

We let v_r' and v_r'' be the first and second derivative of the function $(r, t) \mapsto v(re^{it})$ with respect to the first variable, so that

$$v_r'(re^{it}) = \sum_{k\geq 8} kM_k r^{k-1} e^{ikt} \quad \text{and} \quad v_r''(re^{it}) = \sum_{k\geq 8} k(k-1)M_k r^{k-2} e^{ikt} \tag{2.4}$$

for any $0 < r < 1$ and any $t \in \mathbb{R}$. We let

$$P_r(t) = \frac{1 - r^2}{1 + r^2 - 2r\cos(t)}$$

denote the Poisson kernel and we set

$$h(r, t) = \frac{(1 - r^4)^2}{r^7} P_r(t)^{-1}(rv_r''(re^{it}) - 2v'(re^{it})), \quad 0 < r < 1, t \in \mathbb{R}.$$

Below we will show that the set

$$\big\{h(r, t) : 0 < r < 1,\ t \in \mathbb{R}\big\} \subset \mathcal{L}(X) \quad \text{is Rademacher bounded.} \tag{2.5}$$

Now as indicated before this proof, we follow Blower's proof in [7]. A thorough reading of that paper shows that Blower's arguments [7, §§3,4] can be reproduced

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verbatim in our case. Next, using (2.5) in the place of [7, Lemma 4] together with Lemma 2.3, the arguments in [7, §5] also carry over to the operator valued case.

Thus we only need to prove (2.5). We note that for any $0 < r < 1$ and $t \in \mathbb{R}$, we have both

$$P_r(t)^{-1} \leq \frac{1 + r^2 - 2r\cos(t)}{1 - r} \quad \text{and} \quad 1 - r^4 = (1 - r)(1 + r)(1 + r^2) \leq 4(1 - r).$$

Hence letting

$$\widetilde{h}(r, t) = \frac{1 - r}{r^7}\big(1 + r^2 - 2r\cos(t)\big)\left(rv_r''(re^{it}) - 2v'(re^{it})\right),$$

we are reduced to showing that the set

$$\{\widetilde{h}(r, t) : 0 < r < 1,\ t \in \mathbb{R}\} \subset \mathcal{L}(X) \quad \text{is Rademacher bounded.} \tag{2.6}$$

Applying (2.4), we find that

$$\widetilde{h}(r, t) = (1 - r)\big(1 + r^2 - 2r\cos(t)\big)\sum_{k \geq 8} k(k - 3)M_k r^{k-8} e^{ikt}.$$

Then, writing $1 + r^2 - 2r\cos(t) = 1 - re^{it} - re^{-it} + r^2$, we deduce that

$$\begin{aligned}\widetilde{h}(r, t) = (1 - r)\Big[&\sum_k k(k - 3)M_k r^{k-8} e^{ikt} - \sum_k k(k - 3)M_k r^{k-7} e^{i(k+1)t} \\ &- \sum_k k(k - 3)M_k r^{k-7} e^{i(k-1)t} + \sum_k k(k - 3)M_k r^{k-6} e^{ikt}\Big] \\ = (1 - r)\Big[&\sum_k k(k - 3)M_k r^{k-8} e^{ikt} - \sum_k (k - 1)(k - 4)M_{k-1} r^{k-8} e^{ikt} \\ &- \sum_k (k + 1)(k - 2)M_{k+1} r^{k-6} e^{ikt} + \sum_k k(k - 3)M_k r^{k-6} e^{ikt}\Big].\end{aligned}$$

For any $k \geq 8$, let

$$\begin{aligned} A_k &= 2k(k - 3)M_k - (k - 1)(k - 4)M_{k-1} - (k + 1)(k - 2)M_{k+1}, \\ B_k &= (k + 1)(k - 2)M_{k+1} - k(k - 3)M_k. \end{aligned}$$

Then the above decomposition of $\widetilde{h}(r, t)$ can be re-written as

$$\widetilde{h}(r, t) = (1 - r)\sum_{k \geq 8} A_k r^{k-8} e^{ikt} + (1 - r)(1 - r^2)\sum_{k \geq 8} B_k r^{k-8} e^{ikt}.$$

Now observe that there exists a constant $K > 0$ such that for any $0 < r < 1$ and any $t \in \mathbb{R}$ we have

$$\Big|(1-r)\sum_{k\geq 8} r^{k-8}e^{ikt}\Big| \leq K \qquad \text{and} \qquad \Big|(1-r)(1-r^2)\sum_{k\geq 8} kr^{k-8}e^{ikt}\Big| \leq K.$$

Hence it follows from the convexity lemma [10, Lemma 3.2] that (2.6) holds true provided that the two sets

$$\{A_k : k \geq 8\} \quad \text{and} \quad \{k^{-1}B_k : k \geq 8\} \tag{2.7}$$

are Rademacher bounded. For any $k \geq 8$, we have

$$\begin{aligned} A_k &= k^2(2M_k - M_{k-1} - M_{k+1}) + k(-6M_k + 5M_{k-1} + M_{k+1}) + (-4M_{k-1} + 2M_{k+1}) \\ &= -k^2\Delta^2 M_{k+1} - 5k\Delta_k + k\Delta_{k+1} + 2M_{k+1} - 4M_{k-1}. \end{aligned}$$

We also have

$$\begin{aligned} k^{-1}B_k &= k^{-1}\big(k^2(M_{k+1} - M_k) + k(-M_{k+1} + 3M_k) - 2M_{k+1}\big) \\ &= k\Delta M_{k+1} - M_{k+1} + 3M_k - 2k^{-1}M_{k+1}. \end{aligned}$$

These decompositions show that the two sets in (2.7) are Rademacher bounded, which completes the proof. ■

Corollary 2.5 *Let X be an AUMD Banach space and let $M\colon \mathbb{R}_+^* \to \mathcal{L}(X)$ be a C^2-function. Assume that the sets*

$$\{M(t) : t > 0\}, \quad \{tM'(t) : t > 0\}, \quad \textit{and} \quad \{t^2M''(t) : t > 0\} \tag{2.8}$$

are Rademacher bounded. Then M is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$ for any $1 \leq p < \infty$.

Proof Let $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}''$ be the three sets in (2.8). Then for any $\varepsilon > 0$ and $k \geq 1$, let $M_k^\varepsilon = M(\varepsilon k)$, and consider the sets

$$\begin{aligned} \mathcal{M}(\varepsilon) = \{M_k^\varepsilon : k \geq 1\}, \quad \mathcal{M}'(\varepsilon) = \{k\Delta M_k^\varepsilon : k \geq 2\}, \\ \mathcal{M}''(\varepsilon) = \{k^2\Delta^2 M_k^\varepsilon : k \geq 3\}. \end{aligned}$$

Then $\mathcal{M}(\varepsilon)$, $\mathcal{M}'(\varepsilon)$ and $\mathcal{M}''(\varepsilon)$ are subsets of the closed absolute convex hull of $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$, respectively. Hence by [10, Lemma 3.2], these sets are Rademacher bounded, and there is a constant $K > 0$ such that $\mathcal{R}_p(\mathcal{M}(\varepsilon)), \mathcal{R}_p(\mathcal{M}'(\varepsilon)), \mathcal{R}_p(\mathcal{M}''(\varepsilon))$ are all less than or equal to K. Hence by Theorem 2.4 and its proof, the sequences $(M(\varepsilon k))_{k\geq 1}$ are bounded multipliers on $H_0^p(\mathbb{T}; X)$, and their multiplier norms are uniformly bounded. The result therefore follows from Lemma 2.1. ■

The paper [4] contains several characterizations of Hilbert spaces in terms of bounded Fourier multipliers on L^p-spaces. These results have simple analogs for bounded Fourier multipliers on H^p-spaces, as follows.

Proposition 2.6 *Let X be a Banach space and let $1 \leq p < \infty$. The following assertions are equivalent.*

(i) *X is isomorphic to a Hilbert space.*
(ii) *Any sequence $(M_k)_{k\geq 1}$ of $\mathcal{L}(X)$ such that the three sets in* (2.3) *are bounded is a bounded Fourier multiplier on $H^p_0(\mathbb{T}; X)$.*
(iii) *Any C^2-function $M\colon \mathbb{R}^*_+ \to \mathcal{L}(X)$ such that the three sets in* (2.8) *are bounded is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$.*

Proof If X is isomorphic to a Hilbert space, then every bounded subset of $\mathcal{L}(X)$ is Rademacher bounded, hence (i) implies (ii) and (iii) by Theorem 2.4 and Corollary 2.5. The proofs of the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are similar to those of [4, Theorem 1] and [4, Theorem 3], respectively. We skip the details. ■

3 H^p-Maximal Regularity

Let X be a Banach space. For any $1 \leq p < \infty$, we let $H^p_-(\mathbb{R}; X)$ be the space of all f in $L^p(\mathbb{R}; X)$ such that the function $t \mapsto f(-t)$ belongs to $H^p(\mathbb{R}; X)$. It is clear that

$$H^p(\mathbb{R}; X) \cap H^p_-(\mathbb{R}; X) = \{0\}. \tag{3.1}$$

We let $H^p_{\mathrm{con}}(\mathbb{R}; X)$ be the domain of the Hilbert transform $\mathcal{H}$ on $L^p(\mathbb{R}; X)$, equipped with the graph norm $\|f\|_{\mathrm{con}} = \|f\|_p + \|\mathcal{H}(f)\|_p$. Using (3.1), we see that there is a canonical Banach space identification

$$H^p_{\mathrm{con}}(\mathbb{R}; X) \approx H^p(\mathbb{R}; X) \oplus_1 H^p_-(\mathbb{R}; X). \tag{3.2}$$

Indeed the linear map taking any pair $(f_1, f_2) \in H^p(\mathbb{R}; X) \oplus H^p_-(\mathbb{R}; X)$ to the function $f_1 + f_2 \in L^p(\mathbb{R}; X)$ is an isomorphism. (The notation $\oplus_1$ means that the norm on the right-hand side of (3.2) is given by $\|(f_1, f_2)\| = \|f_1\|_p + \|f\|_p$, but this choice does not play any special role.) One of the equivalent definitions of the UMD property is that for any $1 < p < \infty$, X is UMD if and only if $H^p_{\mathrm{con}}(\mathbb{R}; X) = L^p(\mathbb{R}; X)$ (see [8]). Also, X is UMD if and only if the $H^1_{\mathrm{con}}(\mathbb{R}; X)$ coincides with the so-called atomic H^1-space $H^1_{\mathrm{at}}(\mathbb{R}; X)$ [5].

Let $1 \leq p < \infty$. As in Section 1, we let B_l be the derivation operator on $L^p(\mathbb{R}; X)$. Then we let B_{l+} be the restriction of B_l to $H^p(\mathbb{R}; X)$. More precisely, we let

$$H^{1,p}(\mathbb{R}; X) = W^{1,p}(\mathbb{R}; X) \cap H^p(\mathbb{R}; X).$$

Then we observe that u' belongs to $H^p(\mathbb{R}; X)$ for any $u \in H^{1,p}(\mathbb{R}; X)$ and we define B_{l+} as the derivation operator $u \mapsto u'$ with domain $D(B_{l+}) = H^{1,p}(\mathbb{R}; X)$. Likewise, we let B_{l-} be the restriction of B_l to $H^p_-(\mathbb{R}; X)$, with domain equal to $H^{1,p}_-(\mathbb{R}; X) = W^{1,p}(\mathbb{R}; X) \cap H^p_-(\mathbb{R}; X)$. Next, we define B_h as the derivation operator on $H^p_{\mathrm{con}}(\mathbb{R}; X)$, with domain equal to the space of all $u \in H^p_{\rm con}(\mathbb{R}; X)$ such that $u' \in H^p_{\rm con}(\mathbb{R}; X)$. Through the identification (3.2), B_h is simply the direct sum of B_{l+} and B_{l-}, and

$$D(B_h) = H^{1,p}(\mathbb{R}; X) \oplus H^{1,p}_-(\mathbb{R}; X).$$

Let $-A$ be the infinitesimal generator of a bounded analytic semigroup on X. As before, we let A_l be the tensor extension of A to $L^p(\mathbb{R}; X)$, and we let A_{l+} and A_{l-} be the restrictions of A_l to $H^p(\mathbb{R}; X)$ and $H^p_-(\mathbb{R}; X)$, respectively. Then we let A_h be the tensor extension of A on $H^p_{\rm con}(\mathbb{R}; X)$. Again, this can be regarded as the direct sum of A_{l+} and A_{l-}, with domain $D(A_h) = H^p(\mathbb{R}; D(A)) \oplus H^p_-(\mathbb{R}; D(A))$. Of course, the operators considered above depend on p, although it is not visible on the notation.

Definition 3.1 We say that A has H^p-maximal regularity if there exists a constant $C > 0$ such that

$$\|A_h u\|_p \leq C\|A_h u + B_h u\|_p, \quad u \in D(A_h) \cap D(B_h). \tag{3.3}$$

It should be noticed that the value $p = 1$ is included in this definition.

Remark 3.2 We wish to record several simple facts on H^p-maximal regularity. Consider A as above.

(i) By construction, A_h is a sectorial operator of type $< \frac{\pi}{2}$, B_h is a sectorial operator of type $\frac{\pi}{2}$, and A_h and B_h commute. As is well known, this implies that

$$A_h + B_h \colon D(A_h) \cap D(B_h) \longrightarrow H^p_{\rm con}(\mathbb{R}; X)$$

is densely defined. Furthermore B_h is one-to-one and has dense range, which implies that $A_h + B_h$ is one-to-one and has dense range (see [20, Proposition 2.6]). Then the operator $A_h(A_h + B_h)^{-1}$ is densely defined on $H^p_{\rm con}(\mathbb{R}; X)$ and a reformulation of (3.3) is that A has H^p-maximal regularity if and only if $A_h(A_h + B_h)^{-1}$ is bounded on $H^p_{\rm con}(\mathbb{R}; X)$.
This is the analog of the fact that if $p > 1$, A has L^p-maximal regularity if and only if $A_l(A_l + B_l)^{-1}$ is bounded on $L^p(\mathbb{R}; X)$.

(ii) As in (i), we can consider the operators $A_{l+}(A_{l+}+B_{l+})^{-1}$ and $A_{l-}(A_{l-}+B_{l-})^{-1}$. Clearly they are the restrictions of $A_l(A_l + B_l)^{-1}$ to $H^p(\mathbb{R}; X)$ and $H^p_-(\mathbb{R}; X)$, respectively. Then it follows from our discussion preceding Definition 3.1 and from (i) above that A has H^p-maximal regularity if and only if $A_{l+}(A_{l+} + B_{l+})^{-1}$ and $A_{l-}(A_{l-} + B_{l-})^{-1}$ are both bounded.

(iii) Let $M \colon \mathbb{R}^* \to \mathcal{L}(X)$ be a bounded measurable function. We say that M is a bounded Fourier multiplier on $H^p_{\rm con}(\mathbb{R}; X)$ if the two functions $t \in \mathbb{R}^*_+ \mapsto M(t)$ and $t \in \mathbb{R}^*_+ \mapsto M(-t)$ are bounded Fourier multipliers on $H^p(\mathbb{R}; X)$.
Consider the special function $M_A \colon \mathbb{R}^* \to \mathcal{L}(X)$ defined by

$$M_A(t) = it(it + A)^{-1}, \quad t \in \mathbb{R}^*.$$

It is well known that M_A is a bounded Fourier multiplier on $L^p(\mathbb{R}; X)$ if and only if the operator $A_l(A_l + B_l)^{-1}$ is bounded on $L^p(\mathbb{R}; X)$ (see [25, §4]). Using (ii) above, we deduce that A has H^p-maximal regularity if and only if M_A is a bounded Fourier multiplier on $H^p_{\rm con}(\mathbb{R}; X)$.

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We do not know whether H^p-maximal regularity is independent of p on general Banach spaces. Parts (b) and (c) of the next statement show that this is the case on AUMD Banach spaces.

Theorem 3.3

(i) *Let $-A$ be the infinitesimal generator of a bounded analytic semigroup on X.*

   (a) *If A has L^p-maximal regularity for any $1 < p < \infty$ (equivalently, for some $1 < p < \infty$), then A has H^p-maximal regularity for any $1 \leq p < \infty$.*

   (b) *If A has H^p-maximal regularity for some $1 \leq p < \infty$, then A is Rademacher sectorial of type $< \frac{\pi}{2}$.*

   (c) *Assume that X is AUMD. If A is Rademacher sectorial of type $< \frac{\pi}{2}$, then A has H^p-maximal regularity for any $1 \leq p < \infty$.*

(ii) *There exists an invertible operator A on some AUMD Banach space, which has H^p-maximal regularity for any $1 \leq p < \infty$, although A does not have L^p-maximal regularity.*

Proof (i) Let $1 < p < \infty$ and assume that A has L^p-maximal regularity. Then by Remark 3.2 (ii), A has H^p-maximal regularity for any $1 < p < \infty$. Further, the operator $A_l(A_l + B_l)^{-1} \colon L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$ extends to a bounded operator on the atomic H^1-space $H^1_{at}(\mathbb{R}; X)$, by [16, Theorem 1.4]. By a theorem of Bourgain and Garcia-Cuerva, the norms induced by $L^1(\mathbb{R}; X)$ and $H^1_{at}(\mathbb{R}; X)$ coincide on $H^1(\mathbb{R}) \otimes X$, and hence $H^1(\mathbb{R}; X)$ is (isomorphic to) a subspace of $H^1_{at}(\mathbb{R}; X)$ (see [6, Theorem 1.1]). We deduce that the restriction of $A_l(A_l + B_l)^{-1}$ to $H^1(\mathbb{R}; X)$ is bounded. Likewise, the restriction of $A_l(A_l + B_l)^{-1}$ to $H^1_-(\mathbb{R}; X)$ is bounded. According to Remark 3.2(ii), this means that A has H^1-maximal regularity.

Part (b) is a combination of Remark 3.2 (iii) and Proposition 2.2.

We give two proofs of (c). Assume that X is AUMD, and let $1 \leq p < \infty$. By [22, Theorem 1.2], the derivation operator B_h has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. Hence by [18, Theorem 6.3], A has H^p-maximal regularity provided that it is Rademacher sectorial of type $< \frac{\pi}{2}$.

Another proof follows from Corollary 2.5. Indeed, assume that A is Rademacher sectorial of type $< \frac{\pi}{2}$, and let M_A be given by Remark 3.2 (iii). Arguing as in [25, Example 2.9], it is not hard to show that M_A satisfies the assumptions of Corollary 2.5. Hence if X is AUMD, the function M_A is a bounded Fourier multiplier on $H^p_{\text{con}}(\mathbb{R}; X)$, and hence A has H^p-maximal regularity.

(ii) Throughout we assume that Y is an AUMD Banach space, that Y has an unconditional basis and a finite cotype, and that Y is not UMD. The space $Y = \ell^1$ fulfills these geometric conditions. We let D be the derivation operator on $L^2(\mathbb{R}; Y)$, with domain equal to the Sobolev space $W^{1,p}(\mathbb{R}; Y)$. According to a forthcoming paper by Kalton and Weis,[1] this operator is not Rademacher sectorial. In fact the set

$$\{s(s + D)^{-1} : s \in \mathbb{R},\ s \geq 1\} \subset \mathcal{L}(L^2(\mathbb{R}; Y)) \tag{3.4}$$

[1] *Euclidean structures,* in preparation.

https://doi.org/10.4153/CJM-2007-051-5 Published online by Cambridge University Press

is not Rademacher bounded. Indeed, assume that it is Rademacher bounded and let K be its Rademacher constant. For any $\mu > 0$, let T_μ be the operator on $L^2(\mathbb{R}; Y)$ defined by $(T_\mu u)(t) = \mu^{\frac{1}{2}} u(\mu t)$. Then T_μ is a surjective isometry and $T_\mu^{-1} D T_\mu = \mu D$. Hence the set $\mathcal{M}_\mu = \{s(s+\mu D)^{-1} : s \in \mathbb{R},\ s \geq 1\}$ is Rademacher bounded and $\mathcal{R}_2(\mathcal{M}_\mu) \leq K$ for any $\mu > 0$. This readily implies that the set $\{s(s+D)^{-1} : s > 0\}$ is Rademacher bounded, a contradiction.

Let $X = \mathrm{Rad}_2(Y)$. We recall that

$$\mathrm{Rad}_2(Y) = \Big\{ \sum_{k=1}^{\infty} \varepsilon_k x_k : \text{ the series } \sum_{k=1}^{\infty} \varepsilon_k x_k \text{ converges in } L^2(\Omega; X) \Big\}.$$

Following an idea from [3], we let $(\lambda_k)_{k \geq 1}$ be a dense sequence of the interval $[1, \infty)$, and we define an operator A on X as follows. We set

$$D(A) = \Big\{ \sum_{k=1}^{\infty} \varepsilon_k x_k \in \mathrm{Rad}_2(Y) : \text{ the series } \sum_{k=1}^{\infty} \lambda_k \varepsilon_k x_k \text{ converges in } L^2(\Omega; X) \Big\},$$

and we let

$$A\Big(\sum_{k=1}^{\infty} \varepsilon_k x_k \Big) = \sum_{k=1}^{\infty} \lambda_k \varepsilon_k x_k$$

for any $\sum_{k=1}^{\infty} \varepsilon_k x_k \in D(A)$. It is clear that A is closed and densely defined.

Using (1.2), it is easy to see that A is sectorial of type ω for any $\omega > 0$, and that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$. Indeed for any $f \in H^\infty(\Sigma_\theta)$, we have

$$f(A)\Big(\sum_{k=1}^{\infty} \varepsilon_k x_k \Big) = \sum_{k=1}^{\infty} f(\lambda_k)\, \varepsilon_k x_k$$

for any $\sum_{k=1}^{\infty} \varepsilon_k x_k$ in $D(A)$, and $\|f(A)\| \leq 2 \|f\|_{\infty,\theta}$, where $\|f\|_{\infty,\theta} = \sup\{|f(z)| : z \in \Sigma_\theta\}$. Since Y is AUMD, the space $X \subset L^2(\Omega; Y)$ is AUMD as well. It therefore follows from [18, Proposition 3.2 and Theorem 5.3] that A is Rademacher sectorial of type ω for any $\omega > 0$. By part (i)(c) of this theorem, this implies that A has H^p-maximal regularity (for any $1 \leq p < \infty$).

Recall that $\lambda_k \geq 1$ for any $k \geq 1$. Hence using (1.2) again, one obtains that A is invertible with A^{-1} sending any $\sum_{k \geq 1} \varepsilon_k x_k$ in X to $\sum_{k \geq 1} \lambda_k^{-1} \varepsilon_k x_k$.

Suppose now that A has L^2-maximal regularity. As before, we let B_l (resp. A_l) be the derivation operator (resp. the tensor extension of A) on $L^2(\mathbb{R}; X)$. Our hypothesis means that the operator $T = A_l(A_l + B_l)^{-1} \colon L^2(\mathbb{R}; \mathrm{Rad}_2(Y)) \to L^2(\mathbb{R}; \mathrm{Rad}_2(Y))$ is bounded (see Remark 3.2(ii)). By Fubini's theorem, we have an isometric identification $L^2(\mathbb{R}; \mathrm{Rad}_2(Y)) \simeq \mathrm{Rad}_2(L^2(\mathbb{R}; Y))$. Let $(z_k)_{k \geq 1}$ be a finite sequence of $L^2(\mathbb{R}; Y)$. It is clear that under the above identification, we have

$$T\Big(\sum_{k \geq 1} \varepsilon_k z_k \Big) = \sum_{k \geq 1} \varepsilon_k \lambda_k (\lambda_k + D)^{-1} z_k.$$

Hence we have

$$\Big\| \sum_{k\geq 1} \varepsilon_k\, \lambda_k(\lambda_k + D)^{-1} z_k \Big\|_2 \leq \|T\| \, \Big\| \sum_{k\geq 1} \varepsilon_k\, z_k \Big\|_2.$$

This shows that the set $\{\lambda_k(\lambda_k + D)^{-1} : k \geq 1\}$ is Rademacher sectorial. Since the sequence $(\lambda_k)_{k\geq 1}$ is dense sequence in $[1, \infty)$, this implies that the set (3.4) is Rademacher bounded, which is a contradiction. ■

As far as we know, the operator A in Theorem 3.3(ii) is the first example of an operator A which is Rademacher sectorial of type $< \frac{\pi}{2}$ without having L^p-maximal regularity.

Remark 3.4 Let $-A$ be the infinitesimal generator of a bounded analytic semigroup on X and assume that A is invertible. Let $1 \leq p < \infty$ and consider A_h, B_h on $H^p_{\text{con}}(\mathbb{R}; X)$. Then A_h also is invertible, hence $A_h(A_h + B_h)^{-1}$ is bounded if and only if $(A_h + B_h)^{-1}$ is bounded. Thus A has H^p-maximal regularity if and only if for any $f \in H^p_{\text{con}}(\mathbb{R}; X)$, there is a (necessarily unique) $u \in D(A_h) \cap D(B_h)$ such that (1.1) holds true.

Thus Theorem 3.3(ii) provides an operator A with the following property: for any $1 \leq p < \infty$ and for any $f \in H^p_{\text{con}}(\mathbb{R}; X)$, (1.1) can be solved with u' belonging to $H^p_{\text{con}}(\mathbb{R}; X)$, hence *a fortiori* to $L^p(\mathbb{R}; X)$, but for any $1 < p < \infty$, one can find $f \in L^p(\mathbb{R}; X)$ such that (1.1) has no solution with u' belonging to $L^p(\mathbb{R}; X)$.

The results discussed so far in this section have analogs for periodic functions that we shall now indicate. We let $L^p_0(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$ be the space of all $f \in L^p(\mathbb{T}; X)$ such that $\widehat{f}(0) = 0$ and we let $H^p_{0-}(\mathbb{T}; X)$ be the space of all $f \in L^p(\mathbb{T}; X)$ such that $\widehat{f}(k) = 0$ for any $k \geq 0$. Then the conjugate Hardy space $H^p_{\text{con}}(\mathbb{T}; X)$ is defined as the space of all $f \in L^p_0(\mathbb{T}; X)$ whose Hilbert transform belongs to $L^p(\mathbb{T}; X)$ and we have a Banach space identification $H^p_{\text{con}}(\mathbb{T}; X) \approx H^p_0(\mathbb{T}; X) \oplus_1 H^p_{0-}(\mathbb{T}; X)$. We let $\widetilde{B}_l$ be the derivation operator on $L^p_0(\mathbb{T}; X)$. Its domain is the Sobolev space $W^{1,p}_0(\mathbb{T}; X)$ of all $u \in L^p_0(\mathbb{T}; X)$ such that there exists $v \in L^p_0(\mathbb{T}; X)$ verifying $\widehat{v}(k) = ik\widehat{u}(k)$ for any $k \in \mathbb{Z}^*$. This v is necessarily unique and we have $u' = \widetilde{B}_l(u) = v$ in this case (see [2, Lemma 2.1] for more on this).

Let A be a closed and densely defined operator on X. We do not assume that A is sectorial in this context. Then we let $\widetilde{A}_l$ be the tensor extension of A on $L^p_0(\mathbb{T}; X)$, with domain $L^p_0(\mathbb{T}; D(A))$. We consider the following Cauchy problem:

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad 0 \leq t < 2\pi; \\ u(0) &= u(2\pi). \end{aligned} \tag{3.5}$$

We say that A has L^p_{per}-maximal regularity if for any $f \in L^p_0(\mathbb{T}; X)$, there exists a (necessarily unique) $u \in D(\widetilde{A}_l) \cap D(\widetilde{B}_l)$ verifying (3.5). According to [2, Theorem 2.3] and its proof, this holds true if and only if $i\mathbb{Z}^*$ is included in the resolvent set of A, and the sequence $(ik(ik + A)^{-1})_{k\in\mathbb{Z}^*}$ is a bounded Fourier multiplier on $L^p_0(\mathbb{T}; X)$.

Note that our notion of $L^p_{\rm per}$-maximal regularity is slightly different from the one considered in [2, Section 2]. Indeed an operator A has the periodic L^p-maximal regularity considered in that paper if and only if A is invertible and has $L^p_{\rm per}$-maximal regularity in our sense.

It is unclear whether $L^p_{\rm per}$-maximal regularity is independent of p in general (although [3, Theorem 2.3] shows that it is the case when X is a UMD Banach space).

Now we consider the derivation operator $\widetilde{B_h}$ and the tensor extension $\widetilde{A_h}$ on $H^p_{\rm con}(\mathbb{T};X)$. Namely, $\widetilde{B_h}$ is the direct sum of the restrictions of $\widetilde{B_l}$ to $H^p_0(\mathbb{T};X)$ and $H^p_{0-}(\mathbb{T};X)$, respectively. The definition of $\widetilde{A_h}$ is similar. Then we say that A has $H^p_{\rm per}$-maximal regularity if for any $f \in H^p_{\rm con}(\mathbb{T};X)$, there exists a (necessarily unique) $u \in D(\widetilde{A_h}) \cap D(\widetilde{B_h})$ verifying (3.5). Arguing as in [2, §2], one obtains that A has $H^p_{\rm per}$-maximal regularity if and only if $i\mathbb{Z}^*$ is included in the resolvent set of A, and the sequence $(ik(ik+A)^{-1})_{k\in\mathbb{Z}^*}$ is a bounded Fourier multiplier on $H^p_{\rm con}(\mathbb{T};X)$. Equivalently, the two sequences $l(ik(ik+A)^{-1})_{k\geq 1}$ and $(ik(ik-A)^{-1})_{k\geq 1}$ are bounded Fourier multipliers on $H^p_0(\mathbb{T};X)$.

Then we have the following analog of Theorem 3.3.

Theorem 3.5

(i) *Let A be a closed and densely defined operator on X. Assume that $i\mathbb{Z}^*$ is included in the resolvent set of A and let $\mathcal{M} = \{k(ik+A)^{-1} : k \in \mathbb{Z}^*\} \subset \mathcal{L}(X)$.*

  (a) *Let $1 < p < \infty$. If A has $L^p_{\rm per}$-maximal regularity, then A has $H^p_{\rm per}$-maximal regularity.*

  (b) *If A has $H^p_{\rm per}$-maximal regularity for some $1 \leq p < \infty$, then $\mathcal{M}$ is Rademacher bounded.*

  (c) *Assume that X is AUMD. If the set $\mathcal{M}$ is Rademacher bounded, then A has $H^p_{\rm per}$-maximal regularity for any $1 \leq p < \infty$.*

(ii) *There exists an operator A on some AUMD Banach space, which has $H^p_{\rm per}$-maximal regularity for any $1 \leq p < \infty$, although it has $L^p_{\rm per}$-maximal regularity for no $1 < p < \infty$.*

Proof Arguing as in the proof of Theorem 3.3, part (i) follows from Proposition 2.2 and Theorem 2.4. We skip the details.

To prove (ii), consider the operator A given by Theorem 3.3(ii). This operator is Rademacher sectorial of type $< \frac{\pi}{2}$, hence the set $\mathcal{M}$ is Rademacher bounded. Thus A has $H^p_{\rm per}$-maximal regularity for any $1 \leq p < \infty$. Now assume that A has $L^p_{\rm per}$-maximal regularity for some $1 < p < \infty$. Since $-A$ generates a bounded analytic semigroup and A is invertible, it follows from [9, §3] that A also has L^p-maximal regularity, a contradiction. ■

Acknowledgements The authors wish to thank the referee for many valuable comments.

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https://doi.org/10.4153/CJM-2007-051-5 Published online by Cambridge University Press