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Isomorphisms of finite Cayley graphs

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A fundamental problem in graph theory is the so-called isomorphism problem, that is, to decide whether two given graphs are isomorphic. In this thesis we investigate the isomorphism problem for finite Cayley graphs.

For a finite group G and a subset S of $G\backslash\{1\}$, the *Cayley digraph* $\mathrm{Cay}(G,S)$ of G with respect to S is defined as the directed graph with vertex set G and arc set $\{(a,b) \mid a,b \in G,\ ba^{-1} \in S\}$. If $S = S^{-1} := \{s^{-1} \mid s \in S\}$ then $\mathrm{Cay}(G,S)$ may be viewed as an undirected graph and is called the *Cayley graph* of G with respect to S. It easily follows that $\mathrm{Cay}(G,S)$ has *valency* $|S|$ and that $\mathrm{Cay}(G,S)$ is connected if and only if $\langle S\rangle = G$.

The group G acting by right multiplication (that is, $g : x \to xg$) is a subgroup of the automorphism group of $\mathrm{Cay}(G,S)$ and acts regularly on vertices. If σ is an automorphism of G, then σ induces an isomorphism from $\mathrm{Cay}(G,S)$ to $\mathrm{Cay}(G,S^\sigma)$. A Cayley (di)graph $\mathrm{Cay}(G,S)$ is called a *CI-graph* if, for any Cayley (di)graph $\mathrm{Cay}(G,T)$ for G, whenever $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ we have $S = T^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$. (CI stands for *Cayley Invariant.*)

One long-standing open problem about Cayley graphs is to determine the groups G (or the types of Cayley graphs for a given group G) for which all Cayley graphs for G are CI-graphs. The investigation of this problem was begun with a conjecture posed by Ádám [**1**] in 1967 that all cyclic groups had this property. This was disproved by Elspas and Turner [**4**] in 1970, and since then, it has received considerable attention in the literature. In this thesis we investigate the problem for general groups under various conditions.

A finite group is called an m-DCI-group (m-CI-group) if all Cayley digraphs (graphs respectively) of G of valency at most m are CI-graphs. One of the main topics of the thesis is to study m-(D)CI-groups. It is clear from the definition that an m(D)CI-group is automatically an i-(D)CI-group for every $i \leqslant m$. Further, a group G is a 1-DCI-group if and only if all elements of G of the same order are conjugate under

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Aut (G), and if G is a 2-CI-group then any two elements a, b of G of the same order are fused (namely $a^\alpha = b$ for some $\alpha \in \mathrm{Aut}(G)$) or inverse-fused (namely $a = (b^{-1})^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$). We call a group with the latter property an *FIF-group*. First we prove that a finite nonabelian simple group is an FIF-group if and only if it is $\mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(3,4), \mathrm{Sz}(8), \mathrm{M}_{11}$ or M_{23}, and give a good description of general FIF-groups, which are dependent on the classification of finite simple groups. (These results have been written as publications [**14, 15**]). Then we apply the description of FIF-groups to obtain explicit lists which contain m-DCI-groups $(m \geqslant 2)$ and m-CI-groups $(m \geqslant 4)$. Further we prove that a finite nonabelian simple group is a 2-CI-group if and only if it is A_5 or PSL(2,8), and however only A_5 is a 2-DCI-group or a 3-CI-group. Moreover we construct a 29-valency Cayley graph of A_5 which is not a CI-graph so that A_5 is not a 29-CI-group. Combining these results, we prove that all (D)CI-groups are soluble, and obtain a description of (D)CI-groups which completes and improves results of Babai and Frankl in [**2, 3**] dating from 1978. (These results have been written as publications [**11, 16, 17**].)

In contrast to the study of m-(D)CI-groups, the second topic in the thesis is to investigate finite groups which have a weaker property. A group G is said to have the m-(D)CI *property* if all Cayley (di)graphs of G of valency m are CI-graphs. First we answer the question of whether the m-(D)CI property implies the i-(D)CI property for $1 \leqslant i < m$ by constructing, for infinitely many values of m, a family of Frobenius groups which have the m-(D)CI property but not the i-(D)CI property for any $i < m$. Then we prove that the 2-CI property implies the 1-CI property and, on the other hand we show that this is not so for the 2-DCI property and we give a complete classification of the finite groups with the 2-DCI property but not the 1-DCI property. Further, we make a general investigation of the structure of Sylow subgroups of groups with the m-(D)CI property. Finally, although it is very hard to characterize general groups with the m-(D)CI property, we obtain a reasonably complete classification of cyclic groups with the m-DCI property, and prove that every Sylow subgoup of Abelian groups with the m-DCI property is a homocyclic group, namely a direct product of cyclic groups of the same order. (These results have been written as publications [**5, 6, 7, 12, 18**].)

Thirdly, we study the isomorphism problem for connected Cayley graphs. A finite group G is called a *connected m-(D)CI-group* if all connected Cayley (di)graphs of G of valency at most m are CI-graphs. Let G be a finite group and let p be the smallest prime divisor of $|G|$. First we show that G is a connected $(p-1)$-DCI-group but is not necessarily a connected p-DCI-group, the latter of which provides a negative answer to two conjectures posed by Xu [**19**]. Then we prove that an Abelian group G is a connected p-DCI-group but not necessarily a connected $(p+1)$-DCI-group, and we give a complete classification of Abelian $(p+1)$-DCI-groups. As a corollary, we obtain a complete classification of Abelian $(p+1)$-DCI-groups, the case where $p = 2$ of which gives several earlier results. Further we show that the $\mathrm{PSL}(2, q)$ are connected 2-DCI-groups. Finally we prove that all connected symmetric cubic Cayley graphs of simple groups are CI-graphs. As a by-product, we show that, if G is a nonabelian simple group and Γ is a symmetric cubic Cayley graph for G, then with a few possible exceptions G is normal in the full automorphism group $\mathrm{Aut}\,\Gamma$. (These results have been written as publications [**8, 9, 10, 13**].)

Also, related to the results obtained in the thesis, a number of interesting research problems arise and remain to be solved.

References

[1] A. Ádám, 'Research problem 2-10', *J. Combin. Theory* **2** (1967), 309.

[2] L. Babai and P. Frankl, 'Isomorphisms of Cayley graphs I', *Colloq. Math. Soc. János Bolyai* **18** (1978), 35–52.

[3] L. Babai and P. Frankl, 'Isomorphisms of Cayley graphs II', *Acta Math. Hungar.* **34** (1979), 177–183.

[4] B. Elspas and J. Turner, 'Graphs with circulant adjacency matrices', *J. Combin. Theory* **9** (1970), 297–307.

[5] C.H. Li, 'The finite groups with the 2-DCI property', *Comm. Algebra* **24** (1996), 1749–1757.

[6] C.H. Li, 'Finite Abelian group with the m-DCI property', *Ars Combin.* (to appear).

[7] C.H. Li, 'The cyclic group with the m-DCI property', *European J. Combin.* (to appear).

[8] C.H. Li, 'On isomophisms of connected Cayley graphs', *Discrete Math.* (to appear).

[9] C.H. Li, 'Isomorphisms of connected Cayley digraphs', *Graphs Combin.* (to appear).

[10] C.H. Li, 'On isomorphisms of connected Caylet graphs, II', (submitted).

[11] C.H. Li, 'Finite CI-groups are solvable', (submitted).

[12] C.H. Li, 'On finited groups with the Cayley isomorphism property, II', (submitted).

[13] C.H. Li, 'The automorphism group of symmetric cubic graphs', (preprint).

[14] C.H. Li and C.E. Praeger, 'The finite simple groups with at most two fusion classes of every order', *Comm. Algebra* **24** (1996), 3681–3704.

[15] C.H. Li and C.E. Praeger, 'On finite groups in which any two elements of the same order are fused or inverse-fused', *Comm. Algebra* (to appear).

[16] C.H. Li and C.E. Praeger, 'On the isomorphism problem for finite Cayley graphs of bounded valency', (preprint, 1997).

[17] C.H. Li, C.E. Praeger and M.Y. Xu, 'Isomorphisms of finite Cayley diagraphs of bounded valency', (submitted).

[18] C.H. Li, C.E. Praeger and M.Y. Xu, 'On finite groups with the Cayley isomorphism property', (submitted).

[19] M.Y. Xu, 'Some work on vertex-transitive graphs by Chinese mathematicians', in *Group theory in China* (Science Press/Kluwer Academic Publishers, Beijing/New York, 1996), pp. 224–254.

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