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BI-LEVEL PROGRAMMING APPROACH TO OPTIMAL STRATEGY FOR VENDOR-MANAGED INVENTORY PROBLEMS UNDER RANDOM DEMAND

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Abstract

We present an extension of vendor-managed inventory (VMI) problems by considering advertising and pricing policies. Unlike the results available in the literature, the demand is supposed to depend on the retail price and advertising investment policies of the manufacturer and retailers, and is a random variable. Thus, the constructed optimization model for VMI supply chain management is a stochastic bi-level programming problem, where the manufacturer is the upper level decision-maker and the retailers are the lower-level ones. By the expectation method, we first convert the stochastic model into a deterministic mathematical program with complementarity constraints (MPCC). Then, using the partially smoothing technique, the MPCC is transformed into a series of standard smooth optimization subproblems. An algorithm based on gradient information is developed to solve the original model. A sensitivity analysis has been employed to reveal the managerial implications of the constructed model and algorithm: (1) the market parameters of the model generate significant effects on the decision-making of the manufacturer and the retailers, (2) in the VMI mode, much attention should be paid to the holding and shortage costs in the decision-making.

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1. Introduction

Vendor-managed inventory (VMI) is a family of business modes, where the buyer (retailer) provides information on the market to the vendor (manufacturer) of products, while the vendor manages not only its own inventory of finished products but also the raw material inventory. Additionally, the manufacturer often compensates its retailers for the holding and shortage costs caused by the variation in the replenishment cycle, and each retailer is only responsible for the inventory cost if the products have

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been sold. In practice, the VMI model has been successful through its applications by many big box retailers such as Wal-Mart and Procter.

Recently, VMI has attracted much attention from the academic community and entrepreneurs. The advantages of implementing a VMI include improvement of inventory requirements and customer services. Hohmann and Zelewski [11] demonstrated that the use of VMI leads to a dramatic reduction of the bullwhip effect in supply chains. Other benefits of VMI include improvement in forecasting accuracy and reduction in transportation cost [22]. In the past six decades, many results available in the literature were based on the deterministic demand, such as the famous economic order quantity (EOQ) model. However, due to information asymmetry and market uncertainty, customer demand is often time-varying, especially when faced with variants of selling policies. For example, Kiesmuller and Broekmeulen [15] studied a VMI problem in a stochastic multi-products system. Lee and Ren [16] constructed a VMI model to determine the optimal replenishment policy in a single-vendor and a single-supplier VMI system with stochastic demand. Mateen et al. [18] discussed how a vendor and multiple retailers interact in a VMI system under stochastic demand. More results in the research of stochastic models can be found in the literature [3, 8, 9, 13, 26].

Taking into consideration the structural properties of VMI systems, Yu [28] investigated how the vendor increases profit by using the Stackelberg game. Zhou [32] established a Stackelberg game model, assuming that the manufacturer dominates the supply chain, and studied the quantity discount pricing policies with stochastic and asymmetric demand. In many existing results, the Stackelberg game has been used to describe the relationship between the manufacturer and the retailers (see, for example, [6, 19–21, 24, 27]). However, in these VMI models, the advertising investment policies are not regarded as decision variables, and the demand is assumed not to depend on the price and advertisement policies.

Note that Subramanyam and Kumaraswamy [23] have developed a deterministic EOQ model which considers the impacts of advertising budget and price variations on demand. Goyal and Gunasekaran [10] extended the model to treat the perishable goods in a multi-stage supply chain. However, in their research, the advertising investments were only treated as input parameters, that is, the exogenous variables of models. Yu [29] was the first to incorporate the advertisement and price into the VMI model as the decision variables, but the constructed Stackelberg equilibrium model must be simple enough that the presented backward induction procedure can solve the model. In the case where the demand is price-and-advertisement-dependent as well as being a continuous random variable, no efficient algorithm (other than heuristic algorithms) has been found to solve the constructed VMI models.

Aiming at the construction of a more practicable VMI model, we intend to present an optimization model for maximizing the profits of the VMI system with one manufacturer (vendor) and multiple retailers. As done in [28, 32], the relationship between the manufacturer and the retailers is primarily in the framework of the Stackelberg game [5, 12], where the manufacturer is the leader and the retailers are the followers. However, compared with the existing results, all of the distribution quantities, advertising investments, and prices of products will be incorporated into the model as endogenous variables in this paper. Additionally, in accordance with the practical marketing environment, we assume that the demand is price-and-advertisement-dependent and is a continuous random variable when the price and advertising investments are fixed. Owing to uncertainty of demand, it is necessary to take into account the shortage loss and holding cost in the constructed VMI model.

Since the constructed model in this paper is a stochastic nonlinear bi-level programming problem, we first derive its deterministic equivalent formulation by the expectation method [4, 25]. Then, unlike with the existing results, we reformulate the bi-level programming problem as a mathematical program with complementarity constraints (MPCC) in virtue of the optimality conditions of the lower-level problem [17]. Finally, based on the partially smoothing technique [5], the MPCC is transformed into a series of standard smooth optimization subproblems.

Note that the profit model derived from the expectation method is involved in computing integrals containing the unknown decision variables. Thus, any popular optimization software or existing powerful algorithms such as the sequential quadratic programming (SQP) method in the standard optimization theory cannot be directly used to solve the constructed model. This complexity prompts another focus of this paper on the development of an efficient algorithm to solve the model. Compared with the heuristic algorithms available in the literature, our algorithm will search for the optimal solution based on the analytical properties of the model. In particular, the gradient information of the objective function and constraints will be explored to generate better approximate solutions in a deterministic and efficient fashion.

In summary, using the model and algorithm in the paper, we attempt to answer the following questions by numerical simulation.

(1) How does one efficiently determine the optimal retail price and advertising investment in the VMI system under continuous random demand?
(2) What are the influences of market parameters on the optimal retail price, the advertising investments and the profits of manufacturers and retailers?
(3) What are the effects of randomness in the demand on the maximum profits of the manufacturer and the retailers?
(4) What are the impacts of holding cost and shortage cost on the optimal retail price and the profits of all the players?

The rest of the paper is organized as follows. The next section is devoted to formulation of the bi-level programming model for the VMI problems. Reformulation of the bi-level programming model as an MPCC is presented in Section 3. In Section 4, an efficient algorithm is developed. Preliminary applications and a sensitivity analysis of the model are conducted in Section 5. Some conclusions are drawn in the last section.

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2. Bi-level programming model for VMI problems

In this section, we construct a bi-level programming model for VMI problems.

2.1. Notation and assumptions For convenience, we first present the relevant notation used in this paper.

Indices.

i : the index of retailers.
m : the number of retailers.

Parameters.

c_m : the manufacturing cost for the manufacturer (\$/unit).
c_p : the wholesale price of the finished product (\$/unit).
H_{bi} : the holding cost paid by the manufacturer at the location of retailer i (\$/unit/time).
L_{bi} : the shortage cost paid by the manufacturer to retailer i (\$/unit/time).
I_i : the inventory cost paid by retailer i (\$/unit/time).
T_i : the transportation cost of the finished products shipped from the manufacturer to retailer i (\$/unit).
P : the production capacity of the manufacturer.
S_m : the fixed cost for the manufacturer.
S_{bi} : the fixed cost for retailer i.
TDC : the total direct cost for the manufacturer (\$/time).
TIDC : the total indirect cost for the manufacturer (\$/time).
π_{bi} : the profit of retailer i (\$/time).
π_m : the profit of the manufacturer (\$/time).

Decision variables of retailers.

p_i : the retail price in the market of retailer i (\$/unit).
p : the vector of retail prices, $p = (p_1, p_2, \ldots, p_m)$.
a_i : the advertising investment of retailer i (\$/time).
a : the vector of advertising investments of retailers, $a = (a_1, a_2, \ldots, a_m)$.

Decision variables of the manufacturer.

Q_i : the distribution quantity from the manufacturer to retailer i.
Q : the vector of distribution quantities, $Q = (Q_1, Q_2, \ldots, Q_m)$.
A : the advertising investment of the manufacturer (\$/time).

We make the following assumptions in order to construct a mathematical model, in view of its applicability and computability.

(1) A manufacturer and a number of retailers are included in the VMI system. All of them are risk-neutral to the random demand.

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(2) The manufacturer produces only one type of finished product and distributes it to its retailers at the same wholesale price.
(3) The retailers sell the finished products to the consumers at a different retail prices in their individual regional markets. There is no competition among the retailers. This means that the demands for all the retailers are mutually independent random variables.
(4) Due to the VMI arrangement, the inventory level is managed by the manufacturer, and the manufacturer must be responsible for the retailers' holding costs and shortage costs caused by the variation in the replenishment cycle, as a punishment for the manufacturer.
(5) The demand depends on the retail price and advertising investments from both the manufacturer and the retailer, and is perturbed by the random behaviour of consumers (see, for example, the work by Karray and Martín-Herrán [14]). Mathematically, the demand for retailer i is

$$D_i(p_i, a_i, A, \xi_i) = d_i(p_i, a_i, A)\xi_i, \tag{2.1}$$

where $d_i : R_+ \to R$ is a function with respect to the retail price p_i and the advertising investments a_i and A, and ξ_i describes the random perturbation of markets.

2.2. Demand function In order to describe the profit functions, we assume that the demand in equation (2.1) is specified by

$$\begin{aligned} d_i(p_i, a_i; A) &= \frac{k_i(a_i + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{p_i^{\rho_i}}, \\ \xi_i &\backsim f_i(\xi_i), \end{aligned} \tag{2.2}$$

where $k_i > 0$, $a_0 > 0$, $A_0 > 0$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$ and $\rho_i > 1$ are given constants, and f_i is the density function of the continuous random variable ξ_i with a support set $[0, \infty)$. In practice, A is the advertising investment of the manufacturer, a_0 and A_0 represent the initial advertising investments of the retailer and manufacturer, respectively, a_i is the advertising investment of retailer i, k_i is the market scale of retailer i, α_i is the advertising investment sensitivity coefficient of retailer i, β_i is the advertising investment sensitivity coefficient of the manufacturer, and ρ_i is the retail price sensitivity coefficient of the product in the market of retailer i.

As for the randomness of demand, it is often captured by a random variable with normal distribution [2]. In particular, if $\xi_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, m$. Then, the cumulative distribution function of ξ_i is given by

$$\begin{aligned} F_i(x) = F_i(x; \mu_i, \sigma_i) &= \int_{-\infty}^{x} f_i(\xi_i)\, d\xi_i \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(\xi_i - \mu_i)^2/2\sigma_i^2}\, d\xi_i, \quad i = 1, 2, \ldots, m. \end{aligned}$$

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From the definition of the demand function, it is clear that the demand for products decreases as the retail price increases, and increases as the advertising investments of the manufacturer and retailer i increase. Combined with the random perturbation ξ_i, the expectation of D_i is

$$E[D_i(p_i, a_i; A, \xi_i)] = \frac{k_i(a_i + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{p_i^{\rho_i}}\mu_i,$$

which is useful for maximizing the profits under random demand. In general, we assume that $\mu_i = 1$ (see [7]).

Note that another popular model of demand is additive, given by $D_i = d_i(p_i, a_i; A) + \xi_i$. Compared to the additive demand, the multiplicative demand defined by (2.1) and (2.2) exhibits the desirable property that the variance of the demand depends on the mean of the demand. Zhang et al. [30, 31] have shown that different representations of the random demand can generate serious effects on practical managerial policies.

In this paper, we mainly focus on the random multiplicative demand associated with the retail price and advertising investments, such that some managerial implications are drawn from the constructed model and the developed algorithm under this stochastic market demand.

2.3. Lower-level optimization model for retailers The profit of each retailer is equal to the difference between the total revenue and the total cost. The total revenue is

$$p_i \min\{Q_i, D_i\}.$$

The total cost consists of the products' procurement cost, the inventory cost, the advertising cost a_i and the fixed cost S_{bi}. Specifically, the procurement and inventory costs can be expressed as $c_p Q_i$ and $I_i \min\{Q_i, D_i\}$, respectively. Note that as a VMI partnership, the inventory cost for the retailer is only proportional to the quantity of product which is sold by the retailer since the inventory level is managed by the manufacturer. Thus, the inventory cost of the redundant product caused by product replenishment policies must be compensated to the retailers by the manufacturer. Consequently, the profit of the retailer i is given by

$$\pi_{bi}(p_i, a_i; Q_i, A, \xi_i) = (p_i - I_i)\min\{Q_i, D_i\} - c_p Q_i - a_i - S_{bi}.$$

As the decision variables of the retailer i, the retail price and advertising investment should satisfy the following constraints:

$$p_i \geq c_p + I_i, \quad a_i \geq 0.$$

Thus, the lower-level optimization model for retailer i, $i = 1, 2, \ldots, m$, is given by

$$\begin{aligned} &\text{maximize } pi_{bi}(p_i, a_i; Q_i, A, \xi_i) = (p_i - I_i)\min\{Q_i, D_i\} - c_p Q_i - a_i - S_{bi} \\ &\text{subject to } p_i \geq c_p + I_i, a_i \geq 0. \end{aligned} \tag{2.3}$$

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2.4. Upper-level optimization model for manufacturer The profit of the manufacturer is the difference between the overall revenue and the total costs. The overall revenue is

$$\sum_{i=1}^{m} c_p Q_i.$$

The total costs are divided into two parts: direct and indirect costs. The direct costs (TDC) include the production cost, the transportation cost from the manufacturer to retailers, the advertising investment and the fixed cost. Consequently, the mathematical expression for the total costs is

$$\sum_{i=1}^{m} Q_i(c_m + T_i) + A + S_m.$$

The indirect costs (TIDC) of the manufacturer under VMI are

$$H_{bi}(Q_i - D_i)^+ + L_{bi}(D_i - Q_i)^+,$$

where $(D_i - Q_i)^+ = \max\{D_i - Q_i, 0\}$. Therefore, the profit of the manufacturer is

$$\begin{aligned}\pi_m(Q_i, A; p, a, \xi) &= \sum_{i=1}^{m} c_p Q_i - \text{TDC} - \text{TIDC}\\ &= \sum_{i=1}^{m} (c_p - T_i - c_m) Q_i - A - S_m\\ &\quad - \sum_{i=1}^{m} (H_{bi}(Q_i - D_i)^+ + L_{bi}(D_i - Q_i)^+).\end{aligned}$$

As the decision variables of the manufacturer, the distribution quantities and advertising investment should satisfy the following conditions:

$$\sum_{i=1}^{m} Q_i \le P, \quad A \ge 0. \tag{2.4}$$

Clearly, the conditions in (2.4) are the constraints of production capacity and nonnegativeness of the advertising investment.

Based on the above analysis, we obtain the upper-level optimization model for the manufacturer as follows:

$$\begin{aligned}\text{maximize } \pi_m(Q, A; p, a, \xi) &= \sum_{i=1}^{m} (c_p - T_i - c_m) Q_i - A - S_m\\ &\quad - \sum_{i=1}^{m} (H_{bi}(Q_i - D_i)^+ + L_{bi}(D_i - Q_i)^+) \qquad (2.5)\\ \text{subject to } A \ge 0, \quad & \sum_{i=1}^{m} Q_i \le P.\end{aligned}$$

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2.5. Bi-level programming model for VMI problems In light of models (2.3) and (2.5), we construct a stochastic bi-level programming model for the handled VMI problems as follows:

$$\begin{aligned}
&\text{maximize } \pi_m(Q, A; p, a, \xi) = \sum_{i=1}^{m}(c_p - T_i - c_m)Q_i - A - S_m \\
&\qquad\qquad - \sum_{i=1}^{m}(H_{bi}(Q_i - D_i)^+ + L_{bi}(D_i - Q_i)^+) \\
&\text{subject to } A \geq 0, \quad \sum_{i=1}^{m} Q_i \leq P, \\
&\quad (p_i, a_i) \text{ is the solution of the following optimization problem:} \\
&\quad \max_{p_i,a_i} \ \pi_{bi}(p_i, a_i; Q_i, A, \xi_i) \\
&\qquad\qquad = (p_i - I_i)\min\{Q_i, D_i\} - c_p Q_i - a_i - S_{bi} \\
&\quad \text{subject to} \quad p_i \geq c_p + I_i, a_i \geq 0.
\end{aligned} \tag{2.6}$$

Due to the existence of random demands in (2.6), we first transform (2.6) into a deterministic equivalent formulation by the expectation method. For simplification, we denote $z_i = Q_i/d_i$, where d_i is defined in equation (2.1). Then, the expected profit of the manufacturer is

$$\begin{aligned}
E_m &= E(\pi_m) \\
&= \sum_{i=1}^{m}(c_p - T_i - c_m)Q_i - A - S_m - \sum_{i=1}^{m}\biggl(H_{bi}\int_0^{z_i}(Q_i - D_i)f(\varepsilon_i)\,d\varepsilon_i \\
&\quad + L_{bi}\int_{z_i}^{+\infty}(D_i - Q_i)f(\varepsilon_i)\,d\varepsilon_i\biggr),
\end{aligned}$$

and the expected profit of the retailer i can be written as

$$\begin{aligned}
E_{bi} &= E(\pi_{bi}) \\
&= (p_i - I_i)\biggl(\int_0^{z_i} D_i f(\varepsilon_i)\,d\varepsilon_i + \int_{z_i}^{+\infty} Q_i f(\varepsilon_i)\,d\varepsilon_i\biggr) - c_p Q_i - a_i - S_{bi}.
\end{aligned}$$

Thus, by the expectation method, the deterministic equivalent formulation of the stochastic model (2.6) is given by

$$\begin{aligned}
&\text{minimize } G_m(Q, A; p, a) \\
&\quad = -E(\pi_m) = \sum_{i=1}^{m}(T_i + c_m - c_p)Q_i + A + S_m \\
&\qquad + \sum_{i=1}^{m}\biggl(H_{bi}\int_0^{z_i}(Q_i - D_i)f(\varepsilon_i)\,d\varepsilon_i + L_{bi}\int_{z_i}^{+\infty}(D_i - Q_i)f(\varepsilon_i)\,d\varepsilon_i\biggr)
\end{aligned}$$

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$$\text{subject to } A \geq 0, \quad \sum_{i=1}^{m} Q_i \leq P, \tag{2.7}$$

$$\begin{aligned}
&(p_i, a_i) \text{ are the solutions of the following optimization problem:}\\
&\text{minimize } G_{bi}(p_i, a_i; Q_i, A)\\
&\qquad = -E(\pi_{bi}) = (I_i - p_i)\Big(\int_0^{z_i} D_i f(\varepsilon_i)\, d\varepsilon_i + \int_{z_i}^{+\infty} Q_i f(\varepsilon_i)\, d\varepsilon_i\Big)\\
&\qquad +c_p Q_i + a_i + S_{bi}\\
&\text{subject to } p_i \geq c_p + I_i,\ a_i \geq 0, \quad i = 1, 2, \ldots, m.
\end{aligned}$$

Remark 2.1. Unlike an ordinary bi-level programming problem, model (2.5) contains two objective functions with complicated definite integrals which are associated with the unknown decision variables. Consequently, any popular optimization software or the existing powerful algorithms in the standard optimization theory cannot be directly used to solve model (2.5). Thus, an interesting issue is how to develop efficient algorithms, other than heuristic algorithms, to find the equilibrium point of model (2.5).

3. Reformulation of the bi-level programming model

In this section, we will transform the bi-level programming problem into an MPCC based on the gradient information of the lower-level optimization problem.

Let λ_i and γ_i be the Lagrangian multipliers corresponding to the two types of constraints in model (2.3). Then, the Lagrangian function of the lower-level optimization model can be written as

$$\begin{aligned}
L_{bi}(p_i, a_i; \lambda_i, \gamma_i) = (I_i - p_i)\Big(\int_0^{z_i} D_i f(\varepsilon_i)\, d\varepsilon_i + \int_{z_i}^{+\infty} Q_i f(\varepsilon_i)\, d\varepsilon_i\Big)\\
+ c_p Q_i + a_i + S_{bi} - \lambda_i (p_i - c_p - I_i) - \gamma_i a_i.
\end{aligned}$$

To simplify the calculation, denote

$$\begin{cases}
p_i' = p_i - c_p - I_i,\\
a_i' = a_i,\\
d_i' = \dfrac{k_i (a_i' + a_0)^{\alpha_i} (A + A_0)^{\beta_i}}{(p_i' + c_p + I_i)^{\rho_i}},\\
z_i' = \dfrac{Q_i}{d_i'}.
\end{cases}$$

Then, L_{bi} has a more compact form:

$$\begin{aligned}
L_{bi} = -(p_i' + c_p)\Big(\int_0^{z_i'} D_i' f(\varepsilon_i)\, d\varepsilon_i + \int_{z_i'}^{+\infty} Q_i f(\varepsilon_i)\, d\varepsilon_i\Big)\\
+ c_p Q_i + a_i' + S_{bi} - \lambda_i p_i' - \gamma_i a_i'.
\end{aligned} \tag{3.1}$$

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From equation (3.1) it is clear that under suitable constraint qualification conditions any optimal solution of the lower-level optimization model satisfies the following Karush–Kuhn–Tucker (KKT) conditions [5, 17]:

$$\begin{cases} \dfrac{\{(p_i{}' + c_p)(\rho_i - 1) - c_p - I_i\}k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i+1}} \displaystyle\int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i \\ \qquad -Q_i \displaystyle\int_{z_i'}^{+\infty} f(\xi_i)\, d\xi_i - \lambda_i = 0, \\ \dfrac{-(p_i{}' + c_p)\alpha_i k_i(a_i{}' + a_0)^{\alpha_i - 1}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i}} \displaystyle\int_0^{z_i'} \xi_i f(\xi_i)\, d\xi_i + 1 - \gamma_i = 0, \\ \lambda_i \ge 0, \quad p_i{}' \ge 0, \quad \lambda_i p_i{}' = 0, \\ \gamma_i \ge 0, \quad a_i{}' \ge 0, \quad \gamma_i a_i{}' = 0. \end{cases} \tag{3.2}$$

Set

$$\begin{cases} p' = (p_1{}', p_2{}', \ldots, p_m{}'), \\ a' = (a_1{}', a_2{}', \ldots, a_m{}'), \\ Y = \begin{pmatrix} p' \\ a' \end{pmatrix}, \\ f_{1i}(y_i) = \dfrac{((p_i{}' + c_p)(\rho_i - 1) - c_p - I_i)k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i+1}}, \\ \qquad \times \displaystyle\int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i - Q_i \int_{z_i'}^{+\infty} f(\xi_i)\, d\xi_i \\ f_{2i}(y_i) = \dfrac{-(p_i{}' + c_p)\alpha_i k_i(a_i{}' + a_0)^{\alpha_i - 1}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i}} \displaystyle\int_0^{z_i'} \xi_i f(\xi_i)\, d\xi_i + 1, \\ F(y_i) = \begin{pmatrix} f_{1i}(y_i) \\ f_{2i}(y_i) \end{pmatrix}. \end{cases} \tag{3.3}$$

Then, the KKT conditions in (3.2) are formulated as the following standard complementarity problem, which is involved in the unknown upper-level decision variables:

$$Y \ge 0, \quad F(y_i) \ge 0, \quad Y^T F(y_i) = 0.$$

Similar to [5], we replace the above complementarity constraints with the following smooth inequality constraints:

$$Y \ge 0, \quad F(y_i) \ge 0, \quad \Phi_\varepsilon(y_i) \le 0,$$

where

$$\begin{aligned} \Phi_\varepsilon(y_i) &= \begin{pmatrix} \phi_{\varepsilon,1}(y_i) \\ \phi_{\varepsilon,2}(y_i) \end{pmatrix}, \\ \phi_{\varepsilon,1}(y_i) &= \tfrac{1}{2}[p_i{}' + f_{1i}(y_i) - \psi_\varepsilon\{p_i{}' - f_{1i}(y_i)\}], \\ \phi_{\varepsilon,2}(y_i) &= \tfrac{1}{2}[a_i{}' + f_{2i}(y_i) - \psi_\varepsilon\{a_i{}' - f_{2i}(y_i)\}], \\ \psi_\varepsilon(t) &= \frac{2t}{\pi} \arctan\left(\frac{t}{\varepsilon}\right). \end{aligned} \tag{3.4}$$

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Then, the bi-level programming model (2.5) is reformulated as a standard optimization problem:

$$\begin{aligned} &\text{minimize } G_m(x) \\ &\text{subject to } \sum_{i=1}^{m} Q_i \le 0, A \ge 0, \\ &\qquad\qquad Y \ge 0, F(y_i) \ge 0, \Phi_\varepsilon(y_i) \le 0, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} x &= (Q, A, p', a'), \\ G_m(x) &= -E(\pi_m) \\ &= \sum_{i=1}^{m} (T_i + c_m + H_p - c_p) Q_i + A + S_m \\ &\quad + \sum_{i=1}^{m} \Big(H_{bi} \int_0^{z_i'} (Q_i - D_i') f(\varepsilon_i)\, d\varepsilon_i + L_{bi} \int_{z_i'}^{+\infty} (D_i' - Q_i) f(\varepsilon_i)\, d\varepsilon_i \Big). \end{aligned} \tag{3.6}$$

Remark 3.1. Unlike an ordinary smooth nonlinear optimization problem, the objective function and nonlinear constraints of model (3.5) are associated with computing complicated integrals. Note that these contain the unknown decision variables in the case that the demand is a continuous random variable. Thus, the standard optimization techniques or existing software packages, such as Matlab, CPLEX, and Lingo, cannot be directly employed to solve model (3.5). On the other hand, if a heuristic algorithm is applied to solve model (3.5), then there is often a high computational cost due to the random search of iterative points. In addition, for heuristic algorithms, the random approach causes difficulty in establishing the theory of convergence.

Remark 3.2. Since model (3.5) is a difficult nonlinear optimization problem, the next section is devoted to the development of an efficient algorithm to solve (3.5) based on the gradient information of the objective and constraints.

4. Development of gradient-based algorithm

In this section, we intend to develop an efficient algorithm to solve model (3.5). Heuristic algorithms or analytic methods such as the backward induction procedure [29] are the popular methods available in the literature for solving bi-level programming problems. However, due to the complexity of model (3.5), the existing analytic methods are not suitable for solving model (3.5). Although any heuristic algorithm can be modified to solve model (3.5), its numerical efficiency is often not satisfactory, because its analytic properties are not employed to search for an optimal solution of this model. Zhang et al. [30] have demonstrated that the numerical efficiency of algorithms based on gradient information is better than that of heuristic algorithms, as they are used to solve random optimization models arising from global supply chain management problems.

The above-mentioned reasons impel us to develop a gradient-information-based algorithm to solve model (3.5). Specifically, owing to the difficulty in numerically

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solving the nonlinear constrained optimization model (3.5), we first approximate this model by a series of linear programming problems based on the gradient information of the objective function and the constraints. Then, by solving a linearized subproblem, we obtain a search direction at a given approximate solution of model (3.5). Finally, by a suitable line search rule, a step length along the search direction is computed, such that a better approximate solution is obtained.

By direct calculation, we can get the following results.

PROPOSITION 4.1. *Let $f_{1i}(y_i)$ be defined as in (3.3). Then,*

$$\frac{\partial f_{1i}(y_i)}{\partial Q_i} = \frac{((p_i{}' + c_p)\rho_i - c_p)Q_i(p_i{}' + c_p + I_i)^{\rho_i - 1}}{k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i}} f(z_i{}') - \int_{z_i{}'}^{+\infty} f(\xi_i)\, d\xi_i,$$

$$\frac{\partial f_{1i}(y_i)}{\partial A} = \frac{((p_i{}' + c_p)(\rho_i - 1) - c_p - I_i)k_i\beta_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i - 1}}{(p_i{}' + c_p + I_i)^{\rho_i + 1}}$$
$$\times \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i - \frac{((p_i{}' + c_p)(\rho_i - 1) + p_i{}')(p_i{}' + c_p + I_i)^{\rho_i - 1} f(z_i{}')}{k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i + 1}},$$

$$\frac{\partial f_{1i}(y_i)}{\partial p_i{}'} = \frac{(c_p - 2I_i)k_i\beta_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i + 2}} \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i$$
$$- \frac{((p_i{}' + c_p)(\rho_i - 1) + p_i{}')Q_i^2\beta_i(p_i{}' + c_p + I_i)^{\rho_i - 1)}}{k_i\beta_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i + 1}} f(z_i{}'),$$

$$\frac{\partial f_{1i}(y_i)}{\partial a_i{}'} = \frac{((p_i{}' + c_p)(\rho_i - 1) - c_p - I_i)\alpha_i k_i(a_i{}' + a_0)^{\alpha_i - 1}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i + 1}} \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i$$
$$- \frac{((p_i{}' + c_p)(\rho_i - 1) + p_i{}')Q_i^2\alpha_i(p_i{}' + c_p + I_i)^{\rho_i - 1}}{k_i(a_i{}' + a_0)^{\alpha_i + 1}(A + A_0)^{\beta_i}} f(z_i{}').$$

PROPOSITION 4.2. *Let $f_{2i}(y_i)$ be defined as in (3.3). Then,*

$$\frac{\partial f_{2i}(y_i)}{\partial Q_i} = \frac{-(p_i{}' + c_p)\alpha_i Q_i(p_i{}' + c_p + I_i)^{\rho_i}}{k_i(a_i{}' + a_0)^{\alpha_i + 1}(A + A_0)^{\beta_i}} f(z_i{}'),$$

$$\frac{\partial f_{2i}(y_i)}{\partial A} = \frac{(p_i{}' + c_p)Q_i^2\alpha_i\beta_i(p_i{}' + c_p + I_i)^{\rho_i}}{k_i(a_i{}' + a_0)^{\alpha_i + 1}(A + A_0)^{\beta_i + 1}} f(z_i{}')$$
$$- \frac{p_i{}'\alpha_i\beta_i k_i(a_i{}' + a_0)^{\alpha_i - 1}(A + A_0)^{\beta_i - 1}}{(p_i{}' + c_p + I_i)^{\rho_i}} \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i,$$

$$\frac{\partial f_{2i}(y_i)}{\partial p_i{}'} = \frac{((p_i{}' + c_p)\rho_i - p_i{}' - c_p - I_i)\alpha_i k_i(a_i{}' + a_0)^{\alpha_i - 1}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)_i^{\rho}}$$
$$\times \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i - \frac{p_i{}'Q_i^2\alpha_i\rho_i(p_i{}' + c_p + I_i)^{\rho_i - 1}}{k_i(a_i{}' + a_0)^{\alpha_i + 1}(A + A_0)^{\beta_i}} f(z_i{}'),$$

$$\frac{\partial f_{2i}(y_i)}{\partial a_i{}'} = \frac{(p_i{}' + c_p)Q_i^2\alpha_i^2(p_i{}' + c_p + I_i)^{\rho_i}}{k_i(a_i{}' + a_0)^{\alpha_i+2}(A + A_0)^{\beta_i}} f(z_i{}') - \frac{p_i{}'\alpha_i(\alpha_i - 1)k_i(a_i{}' + a_0)^{\alpha_i-2}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i}} \int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i.$$

Proposition 4.3. *Let $\phi_{\varepsilon,1}(y_i)$ and $\phi_{\varepsilon,2}(y_i)$ be defined as in (3.4). Then,*

$$\frac{\partial\phi_{\varepsilon,1}(y_i)}{\partial Q_i} = \frac{1}{2}\frac{\partial f_{1i}}{\partial Q_i)} + \frac{(\partial f_{1i}/\partial Q_i)\arctan((p_i{}' - f_{1i})/\varepsilon)}{\pi} + \frac{\varepsilon(p_i{}' - f_{1i})(\partial f_{1i}/\partial Q_i)}{\pi(\varepsilon^2 + (p_i{}' - f_{1i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,1}(y_i)}{\partial A} = \frac{1}{2}\frac{\partial f_{1i}}{\partial A} + \frac{(\partial f_{1i}/\partial A)\arctan((p_i{}' - f_{1i})/\varepsilon)}{\pi} + \frac{\varepsilon(p_i{}' - f_{1i})(\partial f_{1i}/\partial A)}{\pi(\varepsilon^2 + (p_i{}' - f_{1i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,1}(y_i)}{\partial p_i{}'} = \frac{1}{2}\Big(1 + \frac{\partial f_{1i}}{\partial p_i{}'}\Big) - \Big(1 - \frac{\partial f_{1i}}{\partial p_i{}'}\Big)\frac{\arctan((p_i{}' - f_{1i})/\varepsilon)}{\pi} - \frac{\varepsilon(p_i{}' - f_{1i})(1 - (\partial f_{1i}/\partial p_i{}'))}{\pi(\varepsilon^2 + (p_i{}' - f_{1i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,1}(y_i)}{\partial a_i{}'} = \frac{1}{2}\frac{\partial f_{1i}}{\partial a_i{}'} + \frac{(\partial f_{1i}/\partial a_i{}')\arctan((p_i{}' - f_{1i})/\varepsilon)}{\pi} + \frac{\varepsilon(p_i{}' - f_{1i})(\partial f_{1i}/\partial a_i{}')}{\pi(\varepsilon^2 + (p_i{}' - f_{1i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,2}(y_i)}{\partial Q_i} = \frac{1}{2}\frac{\partial f_{2i}}{\partial Q_i} + \frac{(\partial f_{2i}/\partial Q_i)\arctan((a_i{}' - f_{2i})/\varepsilon)}{\pi} + \frac{\varepsilon(a_i{}' - f_{2i})(\partial f_{2i}/\partial Q_i)}{\pi(\varepsilon^2 + (a_i{}' - f_{2i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,2}(y_i)}{\partial A} = \frac{1}{2}\frac{\partial f_{2i}}{\partial A} + \frac{(\partial f_{2i}/\partial A)\arctan((a_i{}' - f_{2i})/\varepsilon)}{\pi} + \frac{\varepsilon(a_i{}' - f_{2i})(\partial f_{2i}/\partial A)}{\pi(\varepsilon^2 + (a_i{}' - f_{2i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,2}(y_i)}{\partial p_i{}'} = \frac{1}{2}\frac{\partial f_{2i}}{\partial p_i{}'} + \frac{(\partial f_{2i}/\partial p_i{}')\arctan((a_i{}' - f_{2i})/\varepsilon)}{\pi} + \frac{\varepsilon(a_i{}' - f_{2i})(\partial f_{2i}/\partial p_i{}')}{\pi(\varepsilon^2 + (a_i{}' - f_{2i})^2)},$$

$$\frac{\partial\phi_{\varepsilon,2}(y_i)}{\partial a_i{}'} = \frac{1}{2}\Big(1 + \frac{\partial f_{2i}}{\partial a_i{}'}\Big) - \Big(1 - \frac{\partial f_{2i}}{\partial a_i{}'}\Big) \times \frac{\arctan((a_i{}' - f_{2i})/\varepsilon)}{\pi} - \frac{\varepsilon(a_i{}' - f_{2i})(1 - \partial f_{2i}/\partial a_i{}')}{\pi(\varepsilon^2 + (a_i{}' - f_{2i})^2)}.$$

Proposition 4.4. *Let $G_m(x)$ be defined as in (3.6). Then,*

$$\frac{\partial G_m(x)}{\partial Q_i} = c_m + T_i + H_p - c_p + (H_{bi} - I_i)\int_0^{z_i{}'} f(\xi_i)\, d\xi_i - L_{bi}\int_0^{+\infty} f(\xi_i)\, d\xi_i,$$

$$\frac{\partial G_m(x)}{\partial A} = \Big(L_{bi}\int_{z_i{}'}^{+\infty} \xi_i f(\xi_i)\, d\xi_i - (H_{bi} + I_i)\int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i\Big) \times \frac{\beta_i k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i - 1}}{(p_i{}' + c_p + I_i)^{\rho_i}} + 1,$$

$$\frac{\partial G_m(x)}{\partial p_i{}'} = \Big((H_{bi} + I_i)\int_0^{z_i{}'} \xi_i f(\xi_i)\, d\xi_i - L_{bi}\int_{z_i{}'}^{+\infty} \xi_i f(\xi_i)\, d\xi_i\Big) \times \frac{\rho_i k_i(a_i{}' + a_0)^{\alpha_i}(A + A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i} + 1},$$

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$$\frac{\partial G_m(x)}{\partial a_i{}'} = \left(L_{bi}\int_{z_i{}'}^{+\infty} \xi_i f(\xi_i)\,d\xi_i - (H_{bi}+I_i)\int_0^{z_i{}'} \xi_i f(\xi_i)\,d\xi_i\right) \times \frac{\alpha_i k_i (a_i{}' + a_0)^{\alpha_i - 1}(A+A_0)^{\beta_i}}{(p_i{}' + c_p + I_i)^{\rho_i}}.$$

REMARK 4.5. For a given point x_k, Propositions 4.1 to 4.3 are useful for computing the gradients of $f_{1i}(y_i)$, $f_{2i}(y_i)$, $\phi_{\varepsilon,1}(y_i)$ and $\phi_{\varepsilon,2}(y_i)$, which are referred to as $\nabla f_{1i}(y_i)$, $\nabla f_{2i}(y_i)$, $\nabla\phi_{\varepsilon,1}(y_i)$ and $\nabla\phi_{\varepsilon,2}(y_i)$, respectively. Thus, the gradients of $F(y_i)$ and $\Phi_\varepsilon(y_i)$, referred to as $\nabla F(y_i)$ and $\nabla\Phi_\varepsilon(y_i)$ respectively, are obtained. By Proposition 4.4, the gradient of the objective function at x_k can be calculated.

By the above gradient information, we can construct a linear approximate model of (3.5) at a given point x_k. Specifically, denote $d = x - x_k$; then any nonzero solution of the following linear programming problem determines a feasible descent direction of model (3.5) at x_k (see the work by Birge et al. [1]):

$$(DF(x))\quad \begin{aligned} &\text{minimize}\; z\\ &\text{subject to}\; \nabla G_m(x_k)^T d - z \le 0,\\ &\qquad \sum_{i=1}^{m} Q_i^k + d_{Q_i} - z \le P, \quad i = 1,2,\ldots,m,\\ &\qquad A^k + d_A + z \ge 0, \quad Y_k + d_Y + z \ge 0,\\ &\qquad F(y_k) + \nabla F(y_k)^T d + z \ge 0,\\ &\qquad \Phi_\varepsilon(y_k) + \nabla\Phi_\varepsilon(y_k)^T d - z \le 0,\\ &\qquad \|d\|_\infty \le 1, \end{aligned} \tag{4.1}$$

where d_{Q_i}, d_A and d_Y are the components of d corresponding to the variables Q_i, A and Y, respectively. Thus, (4.1) is called the linearized subproblem in solving model (3.5). Determination of a search direction by solving (4.1) is one of the main steps in the following algorithm.

Algorithm 1: Modified Topkis–Veinott algorithm [1, 30]

Step 0 Choose an initial point $x_0 \in D$ and z_0 large enough; $\epsilon_1 > 0$ is a given constant. Set $k := 0$.

Step 1 If $|z_k| < \epsilon_1$, the algorithm stops. Otherwise, go to Step 2.

Step 2 For the given x_k, solve subproblem (4.1). Its solution is referred to as d_k.

Step 3 With d_k, compute $\alpha_k^{\max} = \max\{\alpha | x_k + \alpha d_k \in D\}$. Then, find an optimal step length by solving the following single-variable optimization model:

$$\min_{0\le\alpha\le\alpha_k^{\max}} G_m(x_k + \alpha d_k). \tag{4.2}$$

Denote the optimal solution of (4.2) as α_k.

Step 4 Set $x_{k+1} := x_k + \alpha_k d_k$. Update $k := k+1$. Return to Step 1.

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REMARK 4.6. Since it is often difficult to calculate the optimal step length α_k in Step 3 of Algorithm 1, instead of solving the problem (4.2) we find $\alpha_k = \eta^i \alpha_k^{\max}$ satisfying the following inequality:

$$G_m(x_k + \alpha_k d_k) \le G_m(x_k) + \delta\alpha_k \nabla G_m(x_k)^T d_k,$$

where i is the largest integer such that the above inequality holds. Further, $0 < \eta < 1$ and $0 < \delta < 1$ are given constants.

REMARK 4.7. Algorithm 1 can be regarded as a modified variant of the Topkis–Veinott method in [1] for solving the complicated model (3.5). Unlike more popular SQP-type algorithms for solving smooth nonlinear constrained optimization problems, the Topkis–Veinott method does not require the second-order information of the objective function to generate a search direction. As done in Step 2 of Algorithm 1, we obtain the search direction d_k by solving the linearly approximate model (4.1). Actually, because it is difficult to obtain the second-order information of the objective function, SQP-type algorithms are not suited to solving model (3.5).

Another advantage of Algorithm 1 is that it can globally converge to a Fritz–John point in the case where model (3.5) has no KKT point (see Birge et al. [1]). In other words, Algorithm 1 may work well for solving this model in the cases where SQP-type methods do not.

From Propositions 4.1 to 4.4, it is clear that the complexity of model (3.5) does not eliminate its first-order smoothness. In other words, this model is still a smooth nonlinear optimization problem. Similar to the proofs in [1] for the Topkis–Veinott method, we can prove the following properties of Algorithm 1.

THEOREM 4.8. *Let $\{x_k\}$ be a sequence generated by Algorithm 1 in solving model* (3.5). *Then, any accumulation point of $\{x_k\}$ is a Fritz–John point of this model.*

5. Preliminary applications and sensitivity analysis

In this section, we first conduct a case study on the VMI problems to show the practicability and computability of the presented model. Then, by sensitivity analysis, we intend to investigate the impacts of some key model parameters on decision-making.

In order to illustrate the effectiveness of Algorithm 1, we first apply the proposed model and algorithm in this paper to solve a practical VMI problem in which there are a manufacturer and two retailers. In model (3.5), we choose the values of the model parameters (also see [29]):

$$\begin{cases} m = 2, \quad k_i = 15\,000, \quad \alpha_i = 0.5, \quad \beta_i = 0.5, \quad \rho_i = 1.6, \quad P = 1\,000\,000, \\ H_{bi} = 32, \quad L_{bi} = 60, \quad c_p = 200, \quad S_m = 100, \quad S_{bi} = 50, \quad a_0 = 650\,000, \\ A_0 = 650\,000, \quad c_m = 20, \quad H_p = 20, \quad T_i = 20, \\ I_i = 30, \quad \mu_i = 1, \quad \sigma_i = 1. \end{cases}$$

The computer code of Algorithm 1 is written using MATLAB 2012b, and run on a personal computer with the operating system Windows 8.1, a 1.8 GHz CPU and

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4.00 GB RAM. As we implement Algorithm 1 to solve (3.5), the total escaped CPU time of Algorithm 1 is 11.221 s, and the Stackelberg equilibrium solution is obtained:

$$x^* = (390\,000.177, 390\,000.177, 533\,367.722, 1091.253,\\ 1091.253, 2\,649\,907.191, 2\,649\,907.191).$$

In other words, the optimal distribution quantities from the manufacturer to retailers 1 and 2 are 390 000.177 and 390 000.177, respectively. The optimal advertising investment of the manufacturer is 533 367.722. The optimal retail prices in the market of retailers 1 and 2 are 1091.253 and 1091.253, respectively. Also, the optimal advertising investments of retailers 1 and 2 are 2 649 907.191 and 2 649 907.191, respectively. The corresponding profits of the manufacturer and retailers are

$$\pi_m(Q_i, A) = 67\,917\,365.441, \quad \pi_{bi}(p_i, a_i) = 66\,039\,676.269,$$

respectively. The above preliminary application has shown that the proposed model and Algorithm 1 in this paper are promising.

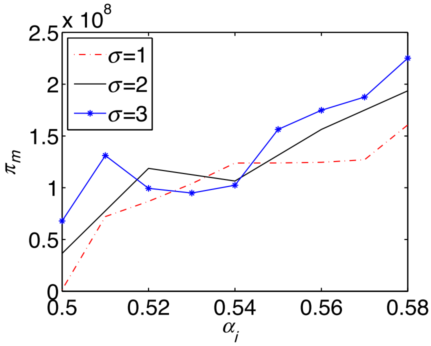
5.1. Sensitivity analysis of market parameters In this sensitivity analysis, we attempt to answer the following questions by changing the values of the model parameters.

(1) What are the impacts of the market parameters, such as α_i and ρ_i, on the optimal decisions and the profits?
(2) What are the impacts of the holding cost paid by the manufacturer at the location of retailer i on the optimal decisions and on the maximal profits?
(3) What are the impacts of the shortage cost paid by the manufacturer at the location of retailer i on the optimal decisions and on the maximal profits?

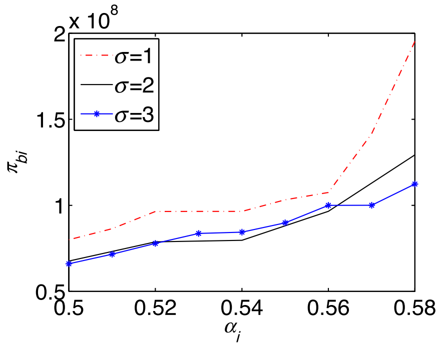
We first study the impacts of the market parameters. To begin with, let us consider the elasticity coefficient of advertising investment α_i. In Figure 1, we plot the dependence relation between α_i and the profits, that is, between the demand and the advertising investment of the manufacturer, respectively. We take into consideration the uncertainty of the demand by setting three different standard deviations. As a result, three curves in Figures 1(a) to 1(c) are obtained, which correspond to the three standard deviations $\sigma = 1, 2, 3$, respectively.

From Figure 1, the following is clear.

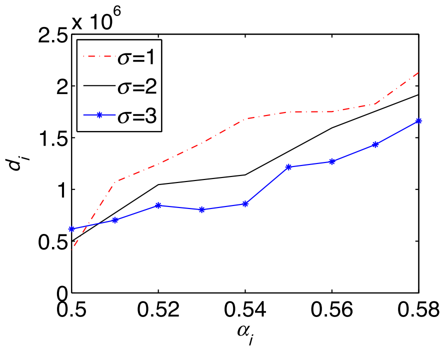
(1) The profits of both the manufacturer and the retailers dramatically rise with the increasing elasticity coefficient of the advertising investment. Different from the results for the manufacturer's profit, for given α_i small enough ($\alpha_i \leq 0.56$), the retailer's profit is not sensitive to the uncertainty of the demand (the standard deviation). However, if α_i is large enough ($\alpha_i > 0.56$), the uncertainty of the demand generates a serious impact on the retailer's profit. Actually, the retailer's profit becomes greater for decreasing standard deviation σ. The smaller the standard deviation, the greater the retailer's profit (see Figures 1(a) and 1(b)).



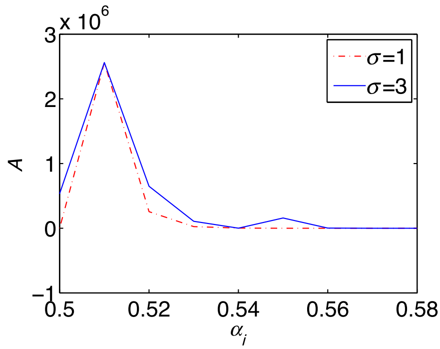
(a) Impact on manufacturer's profit



(b) Impact on retailer's profit



(c) Impact on demand



(d) Impact on manufacturer's advertising investment

FIGURE 1. Effects of advertising investment elasticity.

(2) For the given standard deviation, the demand increases as the elasticity coefficient of the advertising investment increases. It is easily seen that the demand of the smaller standard deviation is greater than that of the larger standard deviation (see Figure 1(c)).
(3) The manufacturer's advertising investment first increases and then decreases as the elasticity coefficient of the advertising investment increases. The process of change is consistent for different standard deviations (see Figure 1(d)).

Next, we study the impact of the retail price's elasticity coefficient ρ_i. The dependence relations between ρ_i and the profits, demand, retail price and retailer's advertising investment are presented in Figure 2. First, Figure 2(a) indicates that the manufacturer's profit goes down with an increment of ρ_i. Clearly, in the case where the standard deviation $\sigma = 3$, the increasing price elasticity results in a linear reduction of the manufacturer's profit.

In contrast, in the case where the standard deviation is 3, the retailer's profit is less sensitive to the change in ρ_i for ρ_i ($\rho_i \leqslant 1.6$) small enough, but reduces sharply if ρ_i is large enough. A similar result arises in the case where the standard deviation is 4 (see Figure 2(b)). Figure 2(c) demonstrates that for the given standard deviation

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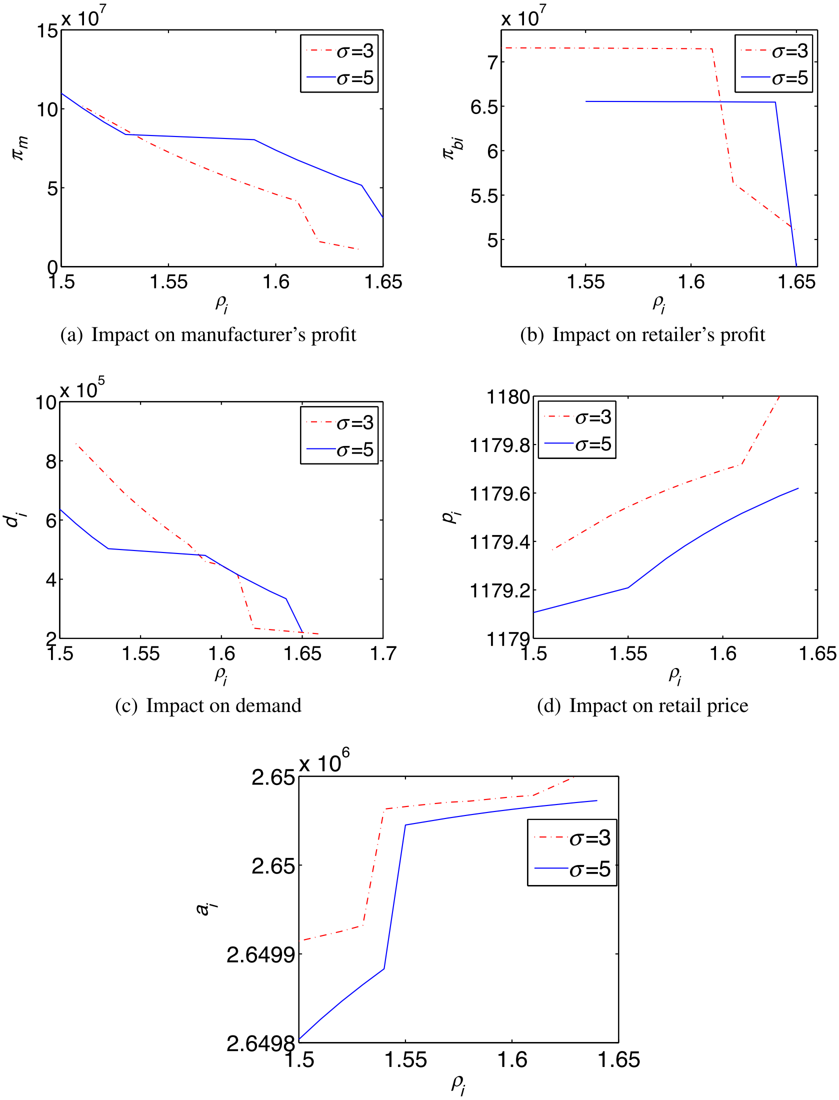

(a) Impact on manufacturer's profit

(b) Impact on retailer's profit

(c) Impact on demand

(d) Impact on retail price

(e) Impact on retailer's advertising investment

FIGURE 2. Effects of retail price's elasticity.

the demand decreases as ρ_i increases. Note that the demand with a smaller standard deviation is greater than that with a greater standard deviation. From Figures 2(d) and 2(e), it follows that:

(1) both, the optimal sales price and the advertising investment of the retailers increase as ρ_i increases;
(2) for a smaller standard deviation, the sales price or the retailer's advertising investment becomes greater.

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5.2. Sensitivity analysis of holding cost We now investigate the influences of unit holding cost on optimal decision-making, which is paid by the manufacturer at the location of the retailer.

We discuss two cases: $\sigma = 3$ and $\sigma = 4$. More intuitively, in the two cases we obtain the relations between the holding cost and the manufacturer's profit in Figures 3(a) and 3(b). The relations between the holding cost and the retailer's profit are plotted in Figures 3(c) and 3(d). In Figures 3(e) and 3(f), we evaluate the impacts of the holding cost on the demand, and the relations between the holding cost and the retail price are presented in Figures 3(g) and 3(h).

In summary, from Figure 3, the following is clear.

(1) The profits, the demand and the retail price have a significant linear correlation with the holding cost H_{bi}. Specifically, the increment of H_{bi} leads to the linear increase of the manufacturer's profit and the demand. However, the opposite results occur for the retailer's profit and the retail price.
(2) The manufacturer's profit, demand and retail price become greater as the standard deviation grows.
(3) The increment of H_{bi} results in the increase of the manufacturer's profit and the decrease of the retailer's profit. Therefore, the value of H_{bi} seriously affects the system equilibrium point.

5.3. Sensitivity analysis of shortage cost In this section, we analyze the effects of the shortage cost paid by the manufacturer at the location of the retailer.

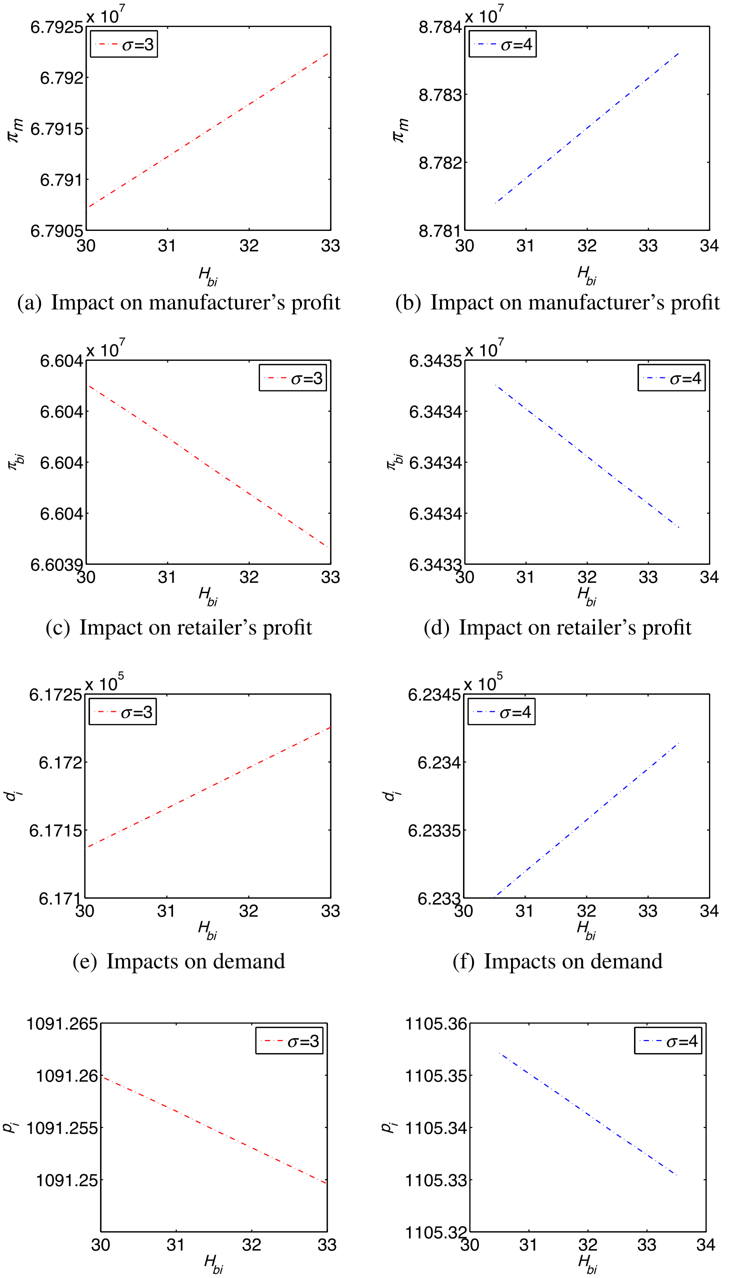
Figure 4 demonstrates the following points.

(1) The manufacturer's profit increases as the shortage cost increases. In addition, the manufacturer's profit for a smaller standard deviation is greater than that for a larger standard deviation. In Figure 4(a), the curve with a standard deviation of 3 is always over that with standard deviation 4.
(2) As the shortage cost L_{bi} rises, both the retailer's profit and the demand also go up (see Figures 4(b) and 4(c)).
(3) The incrementing of L_{bi} contributes to a rise in the retail price. Different from the results of the manufacturer's profit, a smaller standard deviation does not always mean a higher profit. There exists a balance point, at which the two profits are the same.

6. Conclusions and directions for future research

In this paper, we have constructed a stochastic bi-level programming model to formulate VMI problems. The holding cost, shortage cost and randomness of demand have been taken into consideration for optimal decision-making on distribution quantities, advertising investments and pricing in the VMI supply chain.

For the constructed random model, a deterministic equivalent formulation was obtained by the expectation method. Then, we reformulated the bi-level programming problem into a MPCC. Based on a smoothing technique, we transformed the MPCC



(a) Impact on manufacturer's profit (b) Impact on manufacturer's profit

(c) Impact on retailer's profit (d) Impact on retailer's profit

(e) Impacts on demand (f) Impacts on demand

(g) Impacts on retail price (h) Impacts on retail price

FIGURE 3. Sensitivity of holding cost.

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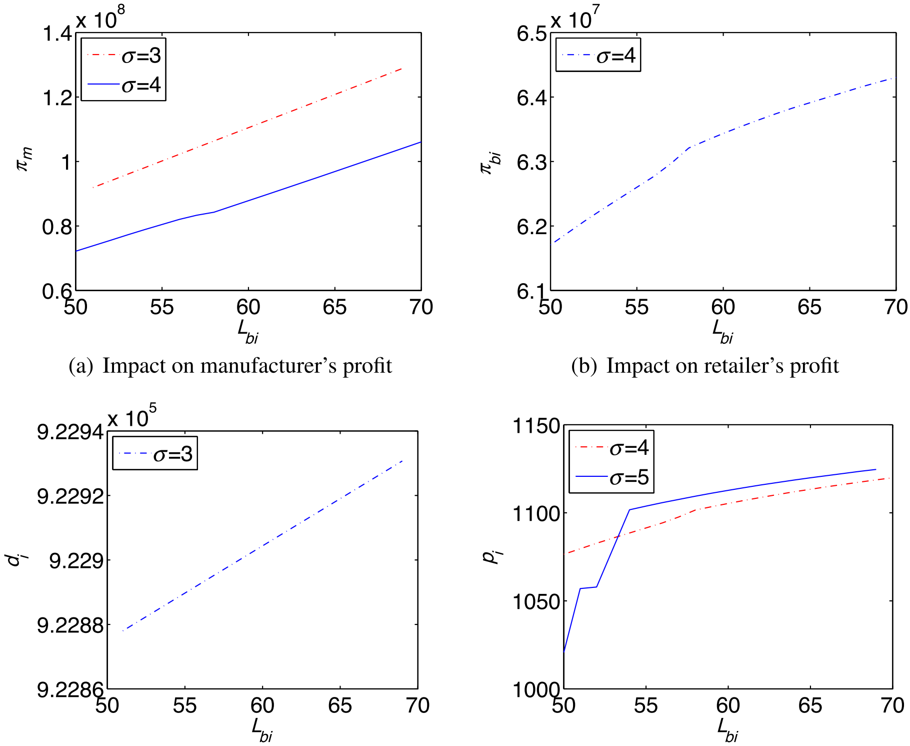

(a) Impact on manufacturer's profit (b) Impact on retailer's profit

(c) Impact on demand (d) Impact on retail price

FIGURE 4. Sensitivity of shortage cost.

into a series of standard constrained optimization subproblems such that an efficient algorithm is developed to solve the original model.

By sensitivity analysis, managerial implications of the model have been obtained in virtue of the model and algorithm.

(1) The parameters of the model related to the market have significant effects on the decision-making performance of the manufacturer and the retailers. The uncertainty of demand generates a greater effect on the retailer's profit than on the manufacturer's profit for the given retailer's elasticity coefficient of advertising investment. The bigger the standard deviation (the uncertainty of demand), the more the advertising investment and the less the profit. Thus, it is helpful to take into consideration the uncertainty of demand in practice, as discussed in this paper.

(2) For the manufacturer, given a fixed standard deviation, an increasing elasticity coefficient of advertising investment leads to increase in demand and a reduction in the advertising investment. As a result, the manufacturer's profit increases.

(3) For the retailer, given a fixed standard deviation, an increasing elasticity coefficient of the retail prices leads to a decreasing demand, an increasing retail price and an increasing advertising investment. But the increased retail price is not enough to compensate for the incrementing of the advertising investment. Consequently, the retailer's profit reduces.
(4) In the VMI model, the holding and shortage costs should be paid much attention in the decision-making. Actually, the manufacturer as the leader must be responsible for retailers' holding costs and shortage costs caused by the variation in the replenishment cycle.

In future, our model can be further extended to multi-product and multi-manufacturer supply chains, instead of a single product and one manufacturer. In addition, as the competition in price and the advertising investment among retailers are incorporated into the extended model, it would be valuable to develop efficient algorithms to solve these complicated models.

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References

[1] J. R. Birge, L. Qi and Z. Wei, "A variant of the Topkis–Veinott method for solving inequality constrained optimization problems", *Appl. Math. Optim.* **41** (2000) 309–330; doi:10.1007/s002459911015.
[2] T. A. Burgin, "Inventory control with normal demand and gamma lead times", *J. Oper. Res. Soc.* **23** (1972) 73–80; doi:10.1057/jors.1972.7.
[3] X. Chen, G. Hao, X. Li and K. F. C. Yiu, "The impact of demand variability and transshipment on vendor's distribution policies under vendor managed inventory strategy", *Int. J. Prod. Econ.* **139** (2012) 42–48; doi:10.1016/j.ijpe.2011.05.005.
[4] X. R. Chen, Y. M. Liu and Z. Wan, "Optimal decision-making for the online and offline retailers under BOPS model", *ANZIAM J.* **58** (2016) 187–208; doi:10.1017/S1446181116000201.
[5] Y. Chen and Z. Wan, "A locally smoothing method for mathematical programs with complementarity constraints", *ANZIAM J.* **56** (2015) 299–315; doi:10.1017/S1446181115000048.
[6] Y. Chu and F. You, "Integrated scheduling and dynamic optimization by Stackelberg game: bilevel model formulation and efficient solution algorithm", *Ind. Eng. Chem. Res.* **53** (2014) 5564–5581; doi:10.1021/ie404272t.
[7] D. K. H. Fong, V. M. Gempesaw and J. K. Ord, "Analysis of a dual sourcing inventory model with normal unit demand and Erlang mixture lead times", *European J. Oper. Res.* **120** (2000) 97–107; doi:10.1016/S0377-2217(98)00394-4.
[8] B. C. Giri, S. Bardhan and T. Maiti, "Coordinating a three-layer supply chain with uncertain demand and random yield", *Int. J. Prod. Res.* **54** (2016) 2499–2518; doi:10.1080/00207543.2015.1119324.
[9] K. Govindan, "The optimal replenishment policy for time-varying stochastic demand under vendor managed inventory", *European J. Oper. Res.* **242** (2015) 402–423; doi:10.1016/j.ejor.2014.09.045.

[10] S. K. Goyal and A. Gunasekaran, "An integrated production-inventory-marketing model for deteriorating items", *Comput. Ind. Eng.* **28** (1995) 755–762; doi:10.1016/0360-8352(95)00016-T.

[11] S. Hohmann and S. Zelewski, "Effects of vendor-managed inventory on the bullwhip effect", *IJISSCM* **4** (2012) 1–17; doi:10.4018/jisscm.2011070101.

[12] S. Huang and Z. Wan, "A new nonmonotone spectral residual method for nonsmooth nonlinear equations", *J. Comput. Appl. Math.* **313** (2017) 82–101; doi:10.1016/j.cam.2016.09.014.

[13] C. H. Huynh and W. Pan, "Operational strategies for supplier and retailer with risk preference under VMI contract", *Int. J. Prod. Econ.* **169** (2015) 413–421; doi:10.1016/j.ijpe.2015.07.026.

[14] S. Karray and G. Martín-Herrán, "A dynamic model for advertising and pricing competition between national and store brands", *European J. Oper. Res.* **193** (2009) 451–467; doi:10.1016/j.ejor.2007.11.043.

[15] G. P. Kiesmller and R. A. C. M. Broekmeulen, "The benefit of VMI strategies in a stochastic multi-product serial two echelon system", *Comput. Oper. Res.* **37** (2011) 406–416; doi:10.1016/j.cor.2009.06.013.

[16] J. Y. Lee and L. Ren, "Vendor-managed inventory in a global environment with exchange rate uncertainty", *Int. J. Prod. Econ.* **130** (2011) 169–174; doi:10.1016/j.ijpe.2010.12.006.

[17] Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical programs with equilibrium constraints* (Cambridge University Press, New York, NY, 1996); doi:10.1017/CBO9780511983658.

[18] A. Mateen, A. K. Chatterjee and S. Mitra, "VMI for single-vendor multi-retailer supply chains under stochastic demand", *Comput. Ind. Eng.* **79** (2015) 95–102; doi:10.1016/j.cie.2014.10.028.

[19] D. Mohammaditabar, S. H. Ghodsypour and A. Hafezalkotob, "A game theoretic analysis in capacity-constrained supplier-selection and cooperation by considering the total supply chain inventory costs", *Int. J. Prod. Econ.* **181** (2015) 87–97; doi:10.1016/j.ijpe.2015.11.016.

[20] N. H. Shah, G. A. Widyadana and H. M. Wee, "Stackelberg game for two-level supply chain with price markdown option", *Int. J. Comput. Math.* **91** (2014) 1054–1060; doi:10.1080/00207160.2013.819973.

[21] A. Sinha, P. Malo, A. Frantsev, A. Frantsev and K. Deb, "Finding optimal strategies in a multi-period multi-leader-follower Stackelberg game using an evolutionary algorithm", *Comput. Oper. Res.* **41** (2014) 374–385; doi:10.1016/j.cor.2013.07.010.

[22] S. H. W. Stanger, "Vendor managed inventory in the blood supply chain in Germany: evidence from multiple case studies", *Strategic Outsourcing: An Int. J.* **6** (2013) 25–47; doi:10.1108/17538291311316054.

[23] S. Subramanyam and S. Kumaraswamy, "EOQ formula under varying marketing policies and conditions", *AIIE Trans.* **13** (1981) 312–314; doi:10.1080/05695558108974567.

[24] Y. C. Tsao, J. C. Lu, N. An, F. Al-Khayyal, R. W. Lu and G. Han, "Retailer shelf-space management with trade allowance: a Stackelberg game between retailer and manufacturers", *Int. J. Prod. Econ.* **148** (2014) 133–144; doi:10.1016/j.ijpe.2013.09.018.

[25] Z. Wan, S. J. Zhang and K. L. Teo, "Polymorphic uncertain nonlinear programming approach for maximizing the capacity of V-belt driving", *Optim. Eng.* **15** (2014) 267–292; doi:10.1007/s11081-012-9205-3.

[26] K. J. Wang, B. Makond and S. Y. Liu, "Location and allocation decisions in a two-echelon supply chain with stochastic demand – a genetic-algorithm based solution", *Expert Syst. Appl.* **38** (2011) 6125–6131; doi:10.1016/j.eswa.2010.11.008.

[27] D. Yang and R. J. Jiao, "Simultaneous configuration of product families and supply chains for mass customization using leader-follower game theory", in: *Industrial Engineering and Engineering Management (IEEM),* Malaysia, 9–12 December 2014, (IEEE, 2014) 707–711.

[28] Y. Yu, F. Chu and H. Chen, "A Stackelberg game and its improvement in a VMI system with a manufacturing vendor", *European J. Oper. Res.* **192** (2009) 929–948; doi:10.1016/j.ejor.2007.10.016.

[29] Y. Yu, G. Q. Huang and L. Liang, "Stackelberg game-theoretic model for optimizing advertising, pricing and inventory policies in vendor managed inventory (VMI) production supply chains", *Comput. Ind. Eng.* **57** (2009) 368–382; doi:10.1016/j.cie.2008.12.003.

[30] X. B. Zhang, S. Huang and Z. Wan, "Optimal pricing and ordering in global supply chain management with constraints under random demand", *Appl. Math. Model.* **40** (2016) 10105–10130; doi:10.1016/j.apm.2016.06.054.
[31] X. B. Zhang, S. Huang and Z. Wan, "Stochastic programming approach to global supply chain management under random additive demand", *Oper. Res. Int. J.* online (2016) 1–32; doi:10.1007/s12351-016-0269-2.
[32] Y. W. Zhou, "A comparison of different quantity discount pricing policies in a two-echelon channel with stochastic and asymmetric demand information", *European J. Oper. Res.* **181** (2007) 686–703; doi:10.1016/j.ejor.2006.08.001.