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SUPPLEMENT TO CLASSIFICATION OF THREEFOLD DIVISORIAL CONTRACTIONS

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Abstract. Every threefold divisorial contraction to a non-Gorenstein point is a weighted blowup.

This supplement finishes the explicit description of a threefold divisorial contraction whose exceptional divisor contracts to a non-Gorenstein point. Contractions to a quotient singularity were treated by Kawamata [8]. The author's study (see [7]), based on the singular Riemann-Roch formula, provided the classification except for the case of small discrepancies. On the other hand, Hayakawa (see [1], [2], [3]) classified those with discrepancy at most 1 by the fact that there exists only a finite number of divisors with such discrepancies over a fixed singularity. The only case left was when it is a contraction to a $cD/2$ point with discrepancy 2. We demonstrate its classification in Theorem 2 by the method in [7]. It turns out that every contraction is a weighted blowup.

THEOREM 1. *Let $f\colon Y \to X$ be a threefold divisorial contraction whose exceptional divisor E contracts to a non-Gorenstein point P. Then f is a weighted blowup of the singularity $P \in X$ embedded into a cyclic quotient of a smooth fivefold.*

Our method of classification is to study the structure of the bigraded ring $\bigoplus_{i,j} f_*\mathcal{O}_Y(iK_Y + jE)/f_*\mathcal{O}_Y(iK_Y + jE - E)$. We find local coordinates at P to meet this structure, and we verify that f should be a certain weighted blowup. The choice of local coordinates is restricted by the action of the cyclic group, which makes easier the classification in the non-Gorenstein case. We do not know if this method is sufficient to settle all the remaining Gorenstein cases in [4], [5], and [6] with discrepancy at most 4.

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By a threefold divisorial contraction to a point, we mean a projective morphism $f\colon (Y \supset E) \to (X \ni P)$ between terminal threefolds such that $-K_Y$ is f-ample and the exceptional locus E is a prime divisor contracting to a point P. We will treat f on the germ at P in the complex analytic category. According to [7, Theorems 1.2, 1.3], the only case left is

$$\text{type e1 with } P = \mathrm{c}D/2, \text{ discrepancy } a/n = 4/2$$

in [7, Table 3]. We will prove the following theorem.

Theorem 2. *Suppose that f is a divisorial contraction of type* e1 *to a $cD/2$ point with discrepancy* 2*. Then f is the weighted blowup with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = ((r+1)/2, (r-1)/2, 2, 1, r)$ *with $r \geq 7$, $r \equiv \pm 1$ mod 8 for a suitable identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + x_4x_5 + p(x_2, x_3, x_4) = 0 \\ x_2^2 + q(x_1, x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}/\frac{1}{2}(1,1,1,0,0),$$

such that p is of weighted order more than r and q is weighted homogeneous of weight $r-1$ for the weights distributed above.

The proof is along the argument in [7, Section 7]. Henceforth, $f\colon (Y \supset E) \to (X \ni P)$ is a divisorial contraction of type e1 to a $\mathrm{c}D/2$ point with discrepancy 2. By [7, Table 3], Y has only one singular point, say, Q, at which E is not Cartier. Q is a quotient singularity of type $(1/2r)(1, -1, r+4)$ with $r \geq 7$, $r \equiv \pm 1$ mod 8.

We set vector spaces $V_i = V_i^0 \oplus V_i^1$ with

$$\begin{aligned} V_i^0 &:= f_*\mathcal{O}_Y(-iE)/f_*\mathcal{O}_Y\bigl(-(i+1)E\bigr), \\ V_i^1 &:= f_*\mathcal{O}_Y\bigl(K_Y - (i+2)E\bigr)/f_*\mathcal{O}_Y\bigl(K_Y - (i+3)E\bigr). \end{aligned}$$

They are zero for negative i, and we have the bigraded ring $\bigoplus V_i$ by a local isomorphism $\mathcal{O}_X(2K_X) \simeq \mathcal{O}_X$. To study its structure in the lower-degree part, we first compute the dimensions of V_i^j in terms of the finite sets

$$N_i := \Bigl\{(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}^5_{\geq 0} \Bigm| \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + l_4 + rl_5 = i, l_1, l_2 \leq 1\Bigr\}.$$

N_i is decomposed into $N_i^0 \sqcup N_i^1$ with $N_i^j := \{(l_1, l_2, l_3, l_4, l_5) \in N_i \mid l_1 + l_2 + l_3 \equiv j \text{ mod } 2\}$.

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LEMMA 3. *We have* $\dim V_i^j = \# N_i^j$.

Proof. We follow the notation in [7]. We have $(r_Q, b_Q, v_Q) = (2r, r+4, 2)$ and $E^3 = 1/r$ by [7, Tables 2, 3]. By $\dim V_i^j = d(j, -i-2j)$ for $i \geq -2$ in [7, (2.8)], the equality in [7, (2.6)] for $(j, -i-2j)$ implies that for $i \geq 0$,

$$\dim V_i^j - \dim V_{i-2}^{1-j} = \frac{2i+1}{r} + B_{2r}(2i+rj+2) - B_{2r}(2i+rj).$$

Here $B_{2r}(k) = (\overline{k} \cdot \overline{2r-k})/2r$, and $^-$denotes the residue modulo $2r$. On the other hand, by $N_i^j = (N_{i-2}^{1-j} + (0,0,1,0,0)) \sqcup \{(l_1, l_2, 0, l_4, l_5) \in N_i^j\}$,

$$\begin{aligned} &\# N_i^j - \# N_{i-2}^{1-j} \\ &\quad = \begin{cases} \#\{(0,0,0,l_4,l_5) \in N_i^0\} + \#\{(1,1,0,l_4,l_5) \in N_i^0\} & \text{for } j = 0, \\ \#\{(0,1,0,l_4,l_5) \in N_i^1\} + \#\{(1,0,0,l_4,l_5) \in N_i^1\} & \text{for } j = 1. \end{cases} \end{aligned}$$

The lemma follows by verifying the coincidence of their right-hand sides. ∎

We will find bases of V_i starting with an arbitrary identification

$$P \in X \simeq o \in (\phi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}/\frac{1}{2}(1,1,1,0). \tag{1}$$

For a semi-invariant function h, $\operatorname{ord}_E h$ denotes the order of h along E.

LEMMA 4.

(i) *We have* $\operatorname{ord}_E x_4 = 1$ *and* $\operatorname{ord}_E x_i \geq 2$ *for* $i = 1, 2, 3$. *There exists some* k *with* $\operatorname{ord}_E x_k = 2$. *We set* $x_k = x_3$ *by permutation.*

(ii) *For* $i < (r-1)/2$, *the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4,0) \in N_i$ *form a basis of* V_i. *In particular, for* $k = 1, 2$, $\operatorname{ord}_E \bar{x}_k \geq (r-1)/2$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$, *with summation over* $(0,0,l_3,l_4,0) \in \bigcup_{i<(r-1)/2} N_i^1$.

(iii) *There exists some* k *with* $\operatorname{ord}_E \bar{x}_k = (r-1)/2$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4,0) \in N_{(r-1)/2}$ *form a basis of* $V_{(r-1)/2}$. *We set* $\bar{x}_k = \bar{x}_2$ *by permutation; then* $\operatorname{ord}_E \hat{x}_1 \geq (r+1)/2$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$, *with summation over* $(0,l_2,l_3,l_4,0) \in N_{(r-1)/2}^1$.

(iv) *We have* $\operatorname{ord}_E \hat{x}_1 = (r+1)/2$. *For* $i < r-1$, *the monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4,0) \in N_i$ *form a basis of* V_i.

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(v) *Set* $\tilde{N}_i := \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} | ((r+1)/2)l_1 + ((r-1)/2)l_2 + 2l_3 + l_4 + rl_5 = i\}$, *and set* $\tilde{N}^0_i := \{(l_1,l_2,l_3,l_4,l_5) \in \tilde{N}_i \mid l_1+l_2+l_3 \text{ even}\}$. *The monomials* $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4,0) \in \tilde{N}^0_{r-1}$ *have one nontrivial relation, say,* ψ, *in* V^0_{r-1}. *The natural exact sequence below is exact.*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0)\in \tilde{N}_{r-1}} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_{r-1} \to 0.$$

(vi) *We have* $\operatorname{ord}_E \psi = r$. *The natural exact sequence below is exact.*

$$0 \to \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in \tilde{N}_r} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5} \to V_r \to 0.$$

Proof. We follow the proof of [7, Lemma 7.2], using the computation of Lemma 3. Claim (i) above follows from $\dim V^0_1 = 1$, $\dim V^1_1 = 0$, and $\dim V^1_2 = 1$. Then V^0_4 is spanned by x_3^2 and x_4^4, which should form a basis of V^0_4 by $\dim V^0_4 = 2$. Now (ii)–(v) follow from the same argument as in [7, Lemma 7.2]. We obtain the sequence in (vi) also, which is exact possibly except for the middle. Its exactness is verified by comparing dimensions. ∎

Corollary 5. *We distribute weights* $\operatorname{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4) = ((r+1)/2, (r-1)/2, 2, 1)$ *to the coordinates* $\hat{x}_1,\bar{x}_2,x_3,x_4$ *obtained in Lemma* 4. *Then* ϕ *in* (1) *is of form*

$$\phi = cx_4\psi + \phi_{>r}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c \in \mathbb{C}$ *and a function* $\phi_{>r}$ *of weighted order more than* r, *where* ψ *is as in Lemma* 4(v).

Proof. Decompose $\phi = \phi_{\leq r} + \phi_{>r}$ into the part $\phi_{\leq r}$ of weighted order at most r and $\phi_{>r}$ of weighted order more than r. Then $\operatorname{ord}_E \phi_{\leq r} = \operatorname{ord}_E \phi_{>r} > r$, so $\phi_{\leq r}$ is mapped to zero by the natural homomorphism

$$\bigoplus_{(l_1,l_2,l_3,l_4,0)\in \bigcup_{i\leq r}\tilde{N}^0_i} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to \mathcal{O}_X/f_*\mathcal{O}_Y\big(-(r+1)E\big),$$

whose kernel is $\mathbb{C}x_4\psi$ by Lemma 4(iv)–(vi). ∎

We will derive an expression of the germ $P \in X$ in Theorem 2. By [9, Remark 23.1], the $cD/2$ point $P \in X$ has an identification in (1) with ϕ either of

(A) $$\phi = x_1^2 + x_2x_3x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^{\gamma}$$

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or of

$$\text{(B)} \qquad \phi = x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^{2\alpha-1} + g(x_3^2, x_4),$$

with $\alpha, \beta \geq 2$, $\gamma \geq 3$, $\lambda \in \mathbb{C}$, and $g \in (x_3^4, x_3^2 x_4^2, x_4^3)$. As its general elephant has type D_k with $k \geq 2r$ by [7, Lemma 5.2(i)], we have

$$\text{(2)} \qquad \gamma \geq r \quad \text{in (A)}, \qquad \operatorname{ord} g(0, x_4) \geq r \quad \text{in (B)}.$$

LEMMA 6. *Case* (A) *does not happen.*

Proof. By Lemma 4(i), $\operatorname{ord}_E x_4 = 1$, $\operatorname{ord}_E x_i \geq 2$ for $i = 1, 2, 3$, and some $\operatorname{ord}_E x_i = 2$. We have $\operatorname{ord}_E x_1 \geq 3$ by the relation $-x_1^2 = x_2 x_3 x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^\gamma$ and (2). Thus, we may set $\operatorname{ord}_E x_3 = 2$ by permutation and construct $\bar{x}_1, \bar{x}_2$ as in Lemma 4(ii).

Let $W_{(r-1)/2}$ be the subspace of $V_{(r-1)/2}$ spanned by the monomials in x_3, x_4. If $\bar{x}_1 \notin W_{(r-1)/2}$, the triple $(\bar{x}_1, x_3, x_4)$ plays the role of $(\bar{x}_2, x_3, x_4)$ in Lemma 4(iii). We construct $\hat{x}_2$ as in Lemma 4(iii) to obtain a quartuple $(\hat{x}_2, \bar{x}_1, x_3, x_4)$, and distribute $\operatorname{wt}(\hat{x}_2, \bar{x}_1, x_3, x_4) = ((r+1)/2, (r-1)/2, 2, 1)$ as in Corollary 5. Set $\bar{x}_1 = x_1 + p_1(x_3, x_4)$ and $\hat{x}_2 = x_2 + p_2(\bar{x}_1, x_3, x_4)$, and rewrite ϕ as

$$\phi = (\bar{x}_1 - p_1)^2 + (\hat{x}_2 - p_2) x_3 x_4 + (\hat{x}_2 - p_2)^{2\alpha} + x_3^{2\beta} + x_4^\gamma.$$

Here, ϕ has the term $\bar{x}_1^2$ of weight $r-1$, which contradicts Corollary 5.

Hence, $\bar{x}_1 \in W_{(r-1)/2}$, and we obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + p_1(x_3, x_4)$, $\bar{x}_2 = x_2 + p_2(x_3, x_4)$ as in Lemma 4. Distribute $\operatorname{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = ((r+1)/2, (r-1)/2, 2, 1)$, and rewrite ϕ as

$$\phi = (\hat{x}_1 - p_1)^2 + (\bar{x}_2 - p_2) x_3 x_4 + (\bar{x}_2 - p_2)^{2\alpha} + x_3^{2\beta} + x_4^\gamma.$$

So ϕ has the term $\bar{x}_2 x_3 x_4$ of weight $(r+5)/2$, whence $(r+5)/2 \geq r$ by Corollary 5, a contradiction to $r \geq 7$. ∎

LEMMA 7. *The germ $P \in X$ has an expression in Theorem 2, with q not of form $(x_3 s(x_3^2, x_4))^2$, such that each $\operatorname{ord}_E x_i$ coincides with $\operatorname{wt} x_i$ distributed in Theorem 2.*

Proof. We have case (B) by Lemma 6. We have $\operatorname{ord}_E x_4 = 1$ and $\operatorname{ord}_E x_1 \geq 3$ as in (A), so $\operatorname{ord}_E x_2 \geq 3$ and $\operatorname{ord}_E x_3 = 2$. We construct $\bar{x}_1, \bar{x}_2$ as in Lemma 4(ii). By the same reason as in the proof of Lemma 6, we obtain

$\bar{x}_1 \in W_{(r-1)/2}$ and a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + p_1(x_3, x_4)$, $\bar{x}_2 = x_2 + p_2(x_3, x_4)$. Distribute $\mathrm{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = ((r+1)/2, (r-1)/2, 2, 1)$, and rewrite ϕ as

$$\phi = (\hat{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)^2 x_4 + \lambda(\bar{x}_2 - p_2)x_3^{2\alpha-1} + g(x_3^2, x_4).$$

So ϕ has the term $\bar{x}_2^2 x_4$ of weight r and should be of form

$$\phi = \big(\bar{x}_2^2 + h(\hat{x}_1, \bar{x}_2, x_3, x_4)\big)x_4 + \phi_{>r}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

as in Corollary 5 with $\psi = \bar{x}_2^2 + h(\hat{x}_1, \bar{x}_2, x_3, x_4)$. In particular, $p_2 = 0$, as otherwise $p_2\bar{x}_2x_4$ would be of weighted order less than r, and one can write

$$\phi = \hat{x}_1^2 + x_4\psi + p(\bar{x}_2, x_3, x_4), \qquad \psi = \bar{x}_2^2 + q(\hat{x}_1, x_3, x_4),$$

where p is of weighted order more than r and q is weighted homogeneous of weight $r-1$. A desired expression is derived by setting $x_5 := -\psi$ and replacing x_4 with $-x_4$. Thus q is not of form $(x_3 s(x_3^2, x_4))^2$ by Lemma 4(iii) and $\mathrm{ord}_E(\bar{x}_2^2 + q) = r$. ∎

Take an expression of the germ $P \in X$ in Theorem 2 by Lemma 7. We apply the extension of [7, Lemma 6.1] to the case when X is embedded into a cyclic quotient of $\mathbb{C}^5$. Let $g\colon Z \to X$ be the weighted blowup with $\mathrm{wt}\, x_i = \mathrm{ord}_E\, x_i$. By direct calculation, we verify the assumptions of [7, Lemma 6.1] and that Z is smooth outside the strict transform of $x_1x_2x_3x_4x_5 = 0$. We need the condition $q \neq (x_3 s)^2$ to check that the restriction $\bar{F} \cap Z$ of the exceptional locus in the ambient space defines an irreducible reduced 2-cycle on Z. Therefore, f should coincide with g by [7, Lemma 6.1], and Theorem 2 is completed.

Remark 8. Using $H \cap E \simeq \mathbb{P}^1$ in the proof of [7, Theorem 5.4], one can show that

(i) if $r \equiv 1 \bmod 8$, $x_2 x_3^{(r+3)/4}$ appears in p and $x_3^{(r-1)/2}$ appears in q;
(ii) if $r \equiv 7 \bmod 8$, $x_3^{(r+1)/2}$ appears in p and $x_1 x_3^{(r-3)/4}$ appears in q.

Theorem 1 follows from [1], [2], [3], [7], [8], and Theorem 2.

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