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# THE RATE OF INCREASE OF MEAN VALUES OF FUNCTIONS IN HARDY SPACES

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**Abstract**

The norm of a function $f$ in the Hardy space $H^p(\mathbb{D})$ is by definition the limit of $\|f_r\|_p$ as $r \to 1$. We show that $d\|f_r\|_p/dr$ grows at most like $o(1/\log r)$ as $r \to 1$.

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## 1. Introduction

Let $f$ be analytic in the unit disc, and let $0 < p < \infty$. Define

$$\|f_r\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.$$

Hardy showed that $\log \|f_r\|_p$ is a convex increasing function of $\log r$ [3]. The *Hardy space* $H^p(\mathbb{D})$ is, by definition, the family of all analytic functions satisfying

$$\|f\|_p = \lim_{r \to 1} \|f_r\|_p < \infty.$$

In this note we show that

$$\frac{d}{dr} \|f_r\|_p = o(1/\log r) \quad \text{as } r \to 1$$

for each $f \in H^p(\mathbb{D})$.

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Fefferman [1, 2] showed that a $2\pi$-periodic function of bounded mean oscillation is of the form $\varphi + \tilde{\psi}$, where $\varphi$ and $\psi$ are two bounded $2\pi$-periodic functions on $\mathbb{R}$, and moreover,

$$\|\varphi + \tilde{\psi}\|_{\mathrm{BMO}} \asymp \|\varphi\|_\infty + \|\psi\|_\infty.$$

Then, Garsia defined a different (somehow simpler) norm $\|\cdot\|_{\mathrm{G}}$ on this space, and showed that

$$\|\varphi + \tilde{\psi}\|_{\mathrm{BMO}} \asymp \|\varphi + \tilde{\psi}\|_{\mathrm{G}} \asymp \|\varphi\|_\infty + \|\psi\|_\infty.$$

In the proof of this fundamental result [4, p. 225], the following lemma plays a key role.

LEMMA 1.1. *Let $f$ be analytic on the disc $\{|z| < R\}$, with $R > 1$. Suppose that $f$ has a simple zero at the origin, and that it has no other zeros in $\{0 < |z| < R\}$. Then*

$$\int_0^{2\pi} |f(e^{i\theta})|\, d\theta = \iint_{|z|<1} \log(1/|z|) \frac{|f'(z)|^2}{|f(z)|}\, dx\, dy.$$

The hypothesis of this lemma makes the proof of the Garcia–Fefferman theorem longer. Let us first slightly generalize this lemma for an arbitrary function of the classical Hardy space $H^p$. Note that, contrary to Lemma 1.1, we do *not* assume in the following lemma that $f$ has a simple zero at the origin, has no other zeros in the unit disc, and is analytic on a neighbourhood of the closed unit disc.

LEMMA 1.2. *Let $f \in H^p(\mathbb{D})$, $0 < p < \infty$, and $f \not\equiv 0$. Suppose that $f(0) = 0$. Then*

$$\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta = p^2 \iint_{|z|<1} \log(1/|z|) |f(z)|^{p-2} |f'(z)|^2\, dx\, dy.$$

PROOF. Let $\rho$ be so small that on the closed disc $\{|z| \le \rho\}$ there are no zeros of $f$, except of course the origin. Let $1 > \varrho > \rho$ be such that on the circle $\{|z| = \varrho\}$ $f$ has no zeros. Then, in the annulus $\{\rho < |z| < \varrho\}$, there are a finite number of zeros of $f$, say $z_1, z_2, \dots, z_n$. Let $\varepsilon$ be so small that all the discs $\{|z - z_k| \le \varepsilon\}$, $1 \le k \le n$, are entirely in the annulus $\{\rho < |z| < \varrho\}$. Finally, let

$$\Omega = \{\rho < |z| < \varrho\} \Big\backslash \bigcup_{k=1}^{n} \{|z - z_k| \le \varepsilon\}.$$

In the following, we shall eventually let $\varepsilon \to 0$, and then $\rho \to 0$, and $\varrho \to 1$ through a sequence $(\varrho_n)_{n\ge1}$, so that there is no zero on the circles $\{|z| = \varrho_n\}$.

The function $W(z) = |f(z)|^p$ is infinitely differentiable in a neighbourhood of $\bar{\Omega}$. Hence, by Green's theorem,

$$\iint_\Omega \log(\varrho/|z|) \nabla^2 W(z)\, dx\, dy = \int_{\partial\Omega} \left( \log(\varrho/|z|) \frac{\partial W}{\partial n} - W \frac{\partial \log(\varrho/|z|)}{\partial n} \right) d\ell.$$

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First of all, a simple calculation shows that

$$\nabla^2 W(z) = p^2 |f(z)|^{p-2} |f'(z)|^2.$$

Hence, the Green formula becomes

$$p^2 \iint_\Omega \log(\varrho/|z|) |f(z)|^{p-2} |f'(z)|^2 \, dx \, dy = I_\varrho - I_\rho - \sum_{k=1}^{n} I_k, \tag{1.1}$$

where the integrals on the right-hand side are explained below. On the boundary $\{|z| = \varrho\}$,

$$\begin{aligned} I_\varrho &= \int_0^{2\pi} \left( \log(1) \frac{\partial W}{\partial r}(\varrho e^{i\theta}) + W(\varrho e^{i\theta}) \frac{1}{\varrho} \right) \varrho \, d\theta \\ &= \int_0^{2\pi} |f(\varrho e^{i\theta})|^p \, d\theta. \end{aligned}$$

By a well-known result in the theory of $H^p$ spaces, we know that

$$I_\varrho = \int_0^{2\pi} |f(\varrho e^{i\theta})|^p \, d\theta \to \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{1.2}$$

as $\varrho \to 1$. On the boundary $\{|z| = \rho\}$,

$$\begin{aligned} I_\rho &= \int_0^{2\pi} \left( \log(\varrho/\rho) \frac{\partial W}{\partial r}(\rho e^{i\theta}) + W(\rho e^{i\theta}) \frac{1}{\rho} \right) \rho \, d\theta \\ &= \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta + \rho \log(\varrho/\rho) \int_0^{2\pi} \frac{\partial |f|^p}{\partial r}(\rho e^{i\theta}) \, d\theta. \end{aligned}$$

Since $f$ is continuous at the origin,

$$\int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \to 2\pi |f(0)|^p = 0$$

as $\rho \to 0$. On the other hand, since

$$\frac{\partial |f|^p}{\partial r} = p(uu_r + vv_r)|f|^{p-2},$$

by the Cauchy–Schwarz inequality,

$$\left| \int_0^{2\pi} \frac{\partial |f|^p}{\partial r}(\rho e^{i\theta}) \, d\theta \right| \le p \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} |f'(\rho e^{i\theta})| \, d\theta.$$

If $f$ has a zero of order $n_0$ at the origin, then $|f(\rho e^{i\theta})| \asymp \rho^{n_0}$ and $|f'(\rho e^{i\theta})| \asymp \rho^{n_0 - 1}$ as $\rho \to 0$. Hence,

$$\int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} |f'(\rho e^{i\theta})| \, d\theta \asymp \rho^{pn_0 - 1},$$

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which gives

$$\left|\rho\,\log(\varrho/\rho)\int_0^{2\pi}\frac{\partial|f|^p}{\partial r}(\rho e^{i\theta})\,d\theta\right|\le C\rho^{pn_0}|\log\rho|.$$

Thus,

$$\rho\,\log(\varrho/\rho)\int_0^{2\pi}\frac{\partial|f|^p}{\partial r}(\rho e^{i\theta})\,d\theta\to 0$$

as $\rho\to 0$. Therefore,

$$I_\rho\to 0 \tag{1.3}$$

as $\rho\to 0$. Finally, on the boundary $\{|z-z_k|=\varepsilon\}$,

$$I_k=\int_0^{2\pi}\bigg(\log(\varrho/|z_k+\varepsilon e^{i\theta}|)\frac{\partial|f|^p}{\partial n}(z_k+\varepsilon e^{i\theta}) - |f(z_k+\varepsilon e^{i\theta})|^p\frac{\partial\log(\varrho/|z|)}{\partial n}(z_k+\varepsilon e^{i\theta})\bigg)\varepsilon\,d\theta.$$

Since

$$\left|\frac{\partial|f|^p}{\partial n}\right|\le|\nabla|f|^p|\le p|f|^{p-1}|f'|,$$

if $f$ has a zero of order $n_k\ge 1$ at the $z_k$ then

$$|I_k|\le C\varepsilon^{pn_k}.$$

Note that the constant $C$ depends on $\varrho$. However, for a fixed $\varrho$,

$$I_k\to 0 \tag{1.4}$$

as $\varepsilon\to 0$. Now let $\varepsilon\to 0$. By the monotone convergence theorem, and by (1.4), the Green formula (1.1) becomes

$$p^2\iint_{\rho<|z|<\varrho}\log(\varrho/|z|)|f(z)|^{p-2}|f'(z)|^2\,dx\,dy=I_\varrho-I_\rho.$$

Then let $\rho\to 0$, and finally let $\varrho\to 1$. Again by the monotone convergence theorem, and by (1.2) and (1.3),

$$p^2\iint_{|z|<1}\log(1/|z|)|f(z)|^{p-2}|f'(z)|^2\,dx\,dy=\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta. \qquad\square$$

## 2. The rate of increase of $\|f_r\|_p$

Using Lemma 1.2, we are able to show that

$$\frac{d}{dr}\|f_r\|_p$$

does not grow arbitrarily fast as $r\to 1$.

THEOREM 2.1. *Let* $f \in H^p(\mathbb{D})$, $0 < p < \infty$. *Then*

$$\frac{d}{dr}\|f_r\|_p = o(1/\log r)$$

*as* $r \to 1$.

PROOF. Without loss of generality, assume that $f(0) = 0$. Let $\varrho$, $\rho$ and $\varepsilon$ be as in the proof of Lemma 1.2. We apply Green's formula again, but this time we use $\log 1/|z|$ instead of $\log \varrho/|z|$. Hence,

$$p^2 \iint_\Omega \log(1/|z|)|f(z)|^{p-2}|f'(z)|^2\,dx\,dy = J_\varrho - J_\rho - \sum_{k=1}^n J_k. \tag{2.1}$$

On the boundary $\{|z| = \varrho\}$,

$$\begin{aligned} J_\varrho &= \int_0^{2\pi}\left(\log(1/\varrho)\frac{\partial W}{\partial r}(\varrho e^{i\theta}) + W(\varrho e^{i\theta})\frac{1}{\varrho}\right)\varrho\,d\theta \\ &= \int_0^{2\pi}|f(\varrho e^{i\theta})|^p\,d\theta - \varrho\log\varrho\int_0^{2\pi}\frac{\partial|f|^p}{\partial r}(\varrho e^{i\theta})\,d\theta \\ &= M_p(\varrho) - \varrho\log\varrho M_p'(\varrho), \end{aligned}$$

where

$$M_p(r) = \|f_r\|_p^p = \int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta.$$

Using the same techniques as in the proof of Lemma 1.2, we can show that

$$J_k \to 0 \tag{2.2}$$

as $\varepsilon \to 0$, and that

$$J_\rho \to 0 \tag{2.3}$$

as $\rho \to 0$. Now, let $\varepsilon \to 0$, and then let $\rho \to 0$. By the monotone convergence theorem, and (2.2)–(2.3), the Green formula (2.1) becomes

$$p^2 \iint_{|z|<\varrho} \log(1/|z|)|f(z)|^{p-2}|f'(z)|^2\,dx\,dy = M_p(\varrho) - \varrho\log\varrho M_p'(\varrho).$$

Then, let $\varrho \to 1$. Again by the monotone convergence theorem, and by Lemma 1.2, we obtain

$$\lim_{\varrho\to 1}\log\varrho M_p'(\varrho) = 0. \qquad \square$$

REMARK. Minor modification of the above calculations yields the formula

$$\varrho M_p'(\varrho) = p^2 \iint_{|z|<\varrho} |f(z)|^{p-2}|f'(z)|^2\,dx\,dy.$$

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