BULL. AUSTRAL. MATH. SOC. 47H09, 46E30
VOL. 72 (2005) [299–315]

PROPER 1-BALL CONTRACTIVE RETRACTIONS IN BANACH SPACES OF MEASURABLE FUNCTIONS

D. CAPONETTI, A. TROMBETTA AND G. TROMBETTA

In this paper we consider the Wośko problem of evaluating, in an infinite-dimensional Banach space X, the infimum of all $k \geqslant 1$ for which there exists a k-ball contractive retraction of the unit ball onto its boundary. We prove that in some classical Banach spaces the best possible value 1 is attained. Moreover we give estimates of the lower H-measure of noncompactness of the retractions we construct.

1. INTRODUCTION

Let X be an infinite-dimensional Banach space with unit closed ball $B(X)$ and unit sphere $S(X)$. It is well known that, in this setting, there is a *retraction* of $B(X)$ onto $S(X)$, that is, a continuous mapping $R : B(X) \to S(X)$ with $Rx = x$ for all $x \in S(X)$. In [**4**] Benyamini and Sternfeld, following Nowak ([**13**]), proved that such a retraction can be chosen among Lipschitz mappings. The problem of evaluating the infimum $k_0(X)$ of the Lipschitz constants of such retractions is of considerable interest in the literature. A general result states that in any Banach space X, $3 \leqslant k_0(X) \leqslant k_0$ (see [**8, 10**]), where k_0 is a universal constant. In special spaces more precise estimates have been obtained by means of constructions which depend on each space. We refer the reader to [**9, 10**] for a collection of results on this problem and related ones.

A similar problem can be considered by replacing Lipschitz retractions by k-ball contractive retractions. Let us recall that for a bounded $A \subset X$, the *Hausdorff measure* (briefly *H-measure*) *of noncompactness* $\gamma(A)$ is the infimum of all $\varepsilon > 0$ such that A has a finite ε-net in X. The following properties of γ hold, for bounded $A, B \subset X$:

$\gamma(A) = 0$ if and only if A is precompact;

$\gamma(\overline{co}A) = \gamma(A)$ where $\overline{co}A$ denotes the closed convex hull of A;

$\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;

$\gamma(A + B) \leqslant \gamma(A) + \gamma(B)$;

$\gamma(\lambda A) = |\lambda|\gamma(A)$, for all $\lambda \in \mathbb{R}$.

Received 5th May, 2005

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

A continuous mapping $T : \mathrm{dom}(T) \subset X \to X$ is called *k-ball contractive* if there is $k \geqslant 0$ such that $\gamma(TA) \leqslant k\gamma(A)$ for each bounded $A \subset \mathrm{dom}(T)$.

In [**20**] Wośko has proved that in the space $X = C([0,1])$ for any $\varepsilon > 0$ there exists a $(1+\varepsilon)$-ball contractive retraction of $B(X)$ onto $S(X)$. Moreover he has posed the question of estimating the characteristic:

$$W(X) = \inf\{k \geqslant 1 : \text{there is a } k\text{-ball contractive retraction } R : B(X) \to S(X)\}$$

for special classical Banach spaces, and also the question whether or not there is a Banach space in which $W(X)$ is a minimum. As Wośko has pointed out a 1-ball contractive retraction cannot be a Lipschitz mapping. In [**19**] it was shown that $W(X) \leqslant 6$ for any Banach space, reaching the value 4 and 3 depending on the geometry of the space X. Results in other Banach spaces can be found in [**6, 12, 16, 17**]. Recently, in [**1**, Theorem 4] it has been proved that if the Banach space X has a monotone norm, then for any $\varepsilon > 0$ there exists a $(1+\varepsilon)$-ball contractive retraction of $B(X)$ onto $S(X)$. For a continuous mapping $T : \mathrm{dom}(T) \subset X \to X$ we also consider the following quantitative characteristic which is of interest in nonlinear analysis:

$$\omega(T) = \sup\{k \geqslant 0 : \ \gamma(TA) \geqslant k\gamma(A) \text{ for every bounded } A \subset \mathrm{dom}(T)\},$$

called the *lower H-measure of noncompactness* of T. This characteristic is closed related to properness. In fact, from $\omega(T) > 0$ it follows that T is a *proper* mapping, that is, $T^{-1}K$ is compact for each compact subset K of X.

Aim of this paper is to estimate $W(X)$ in some classical Banach spaces of real valued measurable functions on $[0,1]$ and also to give estimates of the lower H-measure of noncompactness of the retractions we construct. In Section 3 we consider special Banach spaces in which, by means of a suitable compact mapping $P_X : B(X) \to X$, we give an explicit formula of a k-ball contractive retraction with positive lower H-measure of noncompactness. In the sections which follow we give examples of Banach spaces X in which $W(X) = 1$. In Orlicz (Section 4) and Lorentz spaces (Section 5) we obtain that the value $W(X) = 1$ is actually a minimum. Moreover in Lebesgue and Lorentz spaces we show that a 1-ball contractive retraction R can be chosen in such a way that $\omega(R) = 1$. As a consequence in the Lebesgue and Lorentz spaces we have the existence of 1-ball contractive fixed point free mappings $F : B(X) \to B(X)$ with $\omega(F) = 1$.

2. Preliminaries.

Let Σ be the σ-algebra of all Lebesgue measurable subsets of $[0,1]$ equipped with the Lebesgue measure μ, and write *almost everywhere* for μ-almost everywhere. Let $\mathcal{M}_0 := \mathcal{M}_0([0,1], \Sigma, \mu)$ denote the space of all classes of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ and $\mathcal{M}_0^+$ its positive cone. We recall the definition of Banach function space, we refer to the book of Bennett–Sharpley [**3**] for the main results of this theory.

Definition 2.1. *A mapping $\rho : \mathcal{M}_0^+ \to [0,\infty]$ is called a Banach function norm if, for all f, g, f_n $(n = 1, 2, \ldots)$ in $\mathcal{M}_0^+$, for all constants $\lambda \geqslant 0$ and for all $E \in \Sigma$, the following properties hold:*

$\rho(f) = 0$ *if and only if* $f = 0$ *almost everywhere in* $[0,1]$;

$\rho(\lambda f) = \lambda\rho(f)$;

$\rho(f+g) \leqslant \rho(f) + \rho(g)$;

$g \leqslant f$ *almost everywhere* $\Rightarrow \rho(g) \leqslant \rho(f)$;

$f_n \uparrow f$ *almost everywhere* $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

$\rho(\chi_{[0,1]}) < \infty$;

$\int_E f(t)\,dt \leqslant C_E\rho(f)$, *for some constant* $0 < C_E < \infty$ *independent of* f.

Definition 2.2. *If ρ is a Banach function norm, the Banach space*

$$Y = \left\{ f \in \mathcal{M}_0 : \rho(|f|) < \infty \right\}$$

is a Banach function space, endowed with the norm $\|f\| = \rho(|f|)$.

Throughout this section Y is a Banach function space.

Definition 2.3. *A function $f \in Y$ is said to have absolutely continuous norm if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\chi_D\| < \varepsilon$ for every $D \in \Sigma$ with $\mu(D) < \delta$.*

Note that, as the underlying space $[0,1]$ has finite measure, by virtue of [**18**, Lemma 3.3.2], the above definition is equivalent to [**3**, Definition 3.1]. We set

$$Y^0 = \{f \in Y : \ f \text{ has absolutely continuous norm}\}.$$

If $Y^0 = Y$, then the space Y is said to have absolutely continuous norm. We denote by W the set of all simple functions of $\mathcal{M}_0$. We recall that W is a subset of Y and we denote by $\overline{W}^{\|\cdot\|}$ the closure of W in Y.

The next lemma collects some results we need (see [**3**, Theorems 3.8, 3.11 and 3.13]).

Lemma 2.4. *The following statements hold:*

(i) *The space Y^0 is an order ideal of Y, that is, it is a closed linear subspace of Y with the property:*

$$f \in Y^0 \text{ and } |g| \leqslant |f| \text{ almost everywhere} \Rightarrow g \in Y^0. \tag{1}$$

(ii) *The subspace $\overline{W}^{\|\cdot\|}$ is an order ideal of Y and $Y^0 \subset \overline{W}^{\|\cdot\|} \subset Y$.*

(iii) *The subspaces Y^0 and $\overline{W}^{\|\cdot\|}$ coincide if and only if the characteristic function $\chi_{[0,1]}$ has absolutely continuous norm. In particular, $Y^0 = \overline{W}^{\|\cdot\|} = Y$ whenever Y has absolutely continuous norm.*

We recall the following useful characterisation of convergent sequences in Y^0.

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Lemma 2.5. ([2, p. 41]) *A sequence $\{f_n\}$ converges to f in Y^0 if and only if $\{f_n\}$ converges to f in measure and the family $\{f_n : n \in N\}$ has uniformly absolutely continuous norm, that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_n \|f_n\chi_D\| < \varepsilon$ for every $D \in \Sigma$ with $\mu(D) < \delta$.*

Let $C([0,1])$ denote the Banach space of all real and continuous functions on $[0,1]$ endowed with the sup norm $\|\cdot\|_\infty$. By a standard argument (see for example [15, Theorem 3.14]) it can be shown the following lemma.

Lemma 2.6. *Assume $Y^0 = \overline{W}^{\|\cdot\|}$, then $C([0,1])$ is dense in Y^0.*

3. Proper k-ball contractive retractions: abstract results.

Let X denote the Banach space of all functions of absolutely continuous norm of a Banach function space Y. We still denote by W the subset of Y of all simple functions. For $f \in X$ and $a \in [1,2]$, we set

$$f_a(t) = \begin{cases} f(at) & \text{if} \quad t \in \left[0, \frac{1}{a}\right] \\ 0 & \text{if} \quad t \in \left(\frac{1}{a}, 1\right]. \end{cases}$$

Throughout this section we assume that the Banach space X satisfies the following properties:

(P1) $X = \overline{W}^{\|\cdot\|}$;

(P2) there is a continuous decreasing function $\alpha : [1,2] \to \mathbb{R}$ with $\alpha(1) = 1$ and $\alpha(2) > 0$ such that

$$\alpha(a)\|f\| \leqslant \|f_a\| \leqslant \|f\|, \tag{2}$$

for every $f \in X$ and $a \in [1,2]$. Then it is easy to check that $f_a \in X$.

Now for any continuous function $g \in X$ we set $A_g = \{g_a : a \in [1,2]\}$. We need the following two lemmas, the proofs of which are straightforward.

Lemma 3.1. *Let $g \in X$ be continuous. Then the set A_g is compact.*

Proof: Let $g \in X$ be continuous. For any $a \in [1,2]$, we have $|g_a| \leqslant \|g\|_\infty \chi_{[0,1]}$ then (1) implies

$$\|g_a\| \leqslant \|g\|_\infty \|\chi_{[0,1]}\|.$$

From the last inequality it follows that A_g has uniformly absolutely continuous norm. Let now $\{g_{a_n}\}$ be a sequence of elements of A_g. Choose a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which is convergent, say to a. It is easy to check that $g_{a_{n_k}} \to g_a$ almost everywhere in $[0,1]$, so that $g_{a_{n_k}} \to g_a$ in measure. By Lemma 2.5, the thesis follows. ∎

Lemma 3.2. *Let $g \in X$ be continuous and $a_n \to a$ $\left(a_n \in [1,2]\right)$. Then $\|g_{a_n} - g_a\| \to 0$.*

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

PROOF: Let $g \in X$ be continuous and $a_n \to a$ $\big(a_n \in [1,2]\big)$. Given $\varepsilon > 0$, as A_g has uniformly absolutely continuous norm, there exists $\delta > 0$ such that $\|g_a \chi_D\| < \varepsilon$ and $\|g_{a_n} \chi_D\| < \varepsilon$ for all $n \in N$ whenever $D \in \Sigma$ and $\mu(D) < \delta$. Find an index ν such that for all $n \geqslant \nu$ we have $1/a_n \in (1/a - \delta/2, 1/a + \delta/2)$ and $|g(a_n t) - g(at)| \leqslant \varepsilon$ for all $t \in [0,1]$ with $t \leqslant 1/a - \delta/2$. Then $\sup\limits_{[0,1/a-\delta/2]} |g_{a_n}(t) - g_a(t)| \leqslant \varepsilon$ and so

$$\big\|(g_{a_n} - g_a)\chi_{[0,1/a-\delta/2]}\big\| \leqslant \varepsilon \big\|\chi_{[0,1]}\big\|.$$

Hence for every $n \geqslant \nu$ we have

$$\begin{aligned}\|g_{a_n} - g_a\| &\leqslant \big\|(g_{a_n} - g_a)\chi_{[0,1/a-\delta/2]}\big\| + \big\|(g_{a_n} - g_a)\chi_{[1/a-\delta/2,1/a+\delta/2]}\big\| \\ &\leqslant \varepsilon\big\|\chi_{[0,1]}\big\| + 2\varepsilon,\end{aligned}$$

and the thesis follows. ∎

REMARK 3.3. If $a_n \to a$ $\big(a_n \in [1,2]\big)$ by the same argument of Lemma 3.2 we have

$$\|\chi_{(1/a_n,1]} - \chi_{(1/a,1]}\| \to 0.$$

We now define a mapping $Q : B(X) \to B(X)$ and establish the properties of Q we need. The explicit formula of a retraction R, of which we can estimate the H-measure of noncompactness (that is, the infimum of all $k \geqslant 1$ for which R is a k-ball contractive retraction) and the lower H-measure of noncompactness, will depend on a suitable compact mapping $P_X : B(X) \to X$ satisfying the hypotheses of the subsequent Theorem 3.6. To define $Q : B(X) \to B(X)$ we set

$$(Qf)(t) = f_{2/(1+\|f\|)}(t), \quad \text{for all } \ t \in [0,1]. \tag{3}$$

We clearly have $Qf = f$ for all $f \in S(X)$.

PROPOSITION 3.4. *The mapping Q is continuous.*

PROOF: Let $\{f_n\}$ be a sequence of elements of $B(X)$ such that $\|f_n - f\| \to 0$. Let $\varepsilon > 0$. By Lemma 2.6 there is a continuous $g \in B(X)$ such that $\|f - g\| \leqslant \varepsilon$. Choose and index ν such that for all $n \geqslant \nu$ we have $\|f - f_n\| \leqslant \varepsilon$, by Lemma 3.2 we may also assume $\big\|g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)}\big\| \leqslant \varepsilon$. Using the last inequality and the right hand side of (2) we get, for all $n \geqslant \nu$

$$\begin{aligned}\|Qf_n - Qf\| &\leqslant \big\|(f_n)_{2/(1+\|f_n\|)} - f_{2/(1+\|f_n\|)}\big\| + \big\|f_{2/(1+\|f_n\|)} - g_{2/(1+\|f_n\|)}\big\| \\ &\qquad + \big\|g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)}\big\| + \big\|g_{2/(1+\|f\|)} - f_{2/(1+\|f\|)}\big\| \\ &= \big\|(f_n - f)_{2/(1+\|f_n\|)}\big\| + \big\|(f - g)_{2/(1+\|f_n\|)}\big\| \\ &\qquad + \big\|g_{2/(1+\|f_n\|)} - g_{2/(1+\|f\|)}\big\| + \big\|(g - f)_{2/(1+\|f\|)}\big\| \leqslant 4\varepsilon,\end{aligned}$$

which gives the thesis. ∎

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

PROPOSITION 3.5. *Let $A \subset B(X)$. Then*

$$\alpha(2)\gamma(A) \leqslant \gamma(QA) \leqslant \gamma(A).$$

PROOF: Let $A \subset B(X)$. We prove the right inequality. Let $\beta > \gamma(A)$. By Lemma 2.6, $C([0,1])$ is dense in X, thus there exists a β-net $\{\varphi_1, \dots, \varphi_p\}$ for A in $C([0,1])$. By Lemma 3.1 the set $\bigcup_{i=1}^{p} A_{\varphi_i}$ is compact, hence given $\delta > 0$ we can choose a δ-net $\{\psi_1, \dots, \psi_q\}$ for $\bigcup_{i=1}^{p} A_{\varphi_i}$ in X. We now show that $\{\psi_1, \dots, \psi_q\}$ is a $(\beta + \delta)$-net for QA in X.

Let $g \in QA$ and let $f \in A$ be such that $Qf = g$. Fix $i \in \{1, \dots, p\}$ such that $\|f - \varphi_i\| \leqslant \beta$. Since $(\varphi_i)_{2/(1+\|f\|)} \in A_{\varphi_i}$ we can find $j \in \{1, \dots, q\}$ such that

$$\left\|(\varphi_i)_{2/(1+\|f\|)} - \psi_j\right\| \leqslant \delta.$$

Then

$$\begin{aligned}\|Qf - \psi_j\| &\leqslant \left\|f_{2/(1+\|f\|)} - (\varphi_i)_{2/(1+\|f\|)}\right\| + \left\|(\varphi_i)_{2/(1+\|f\|)} - \psi_j\right\| \\ &\leqslant \|f - \varphi_i\| + \delta \leqslant \beta + \delta.\end{aligned}$$

Therefore $\gamma(QA) \leqslant \beta + \delta$, so $\gamma(QA) \leqslant \gamma(A)$.

We now prove the left inequality. Let $\eta > \gamma(QA)$. As $C([0,1])$ is dense in X, there exists an η-net $\{\lambda_1, \dots, \lambda_n\}$ for QA in $C([0,1])$. For $i = 1 \dots, n$, set $(\lambda_i)^b(t) = \lambda_i(bt)$ for $t \in [0,1]$ and $b \in [1/2, 1]$. Since each $(\lambda_i)^b$ is a continuous mapping, the set $\bigcup_{i=1}^{n}\{(\lambda_i)^b : b \in [1/2,1]\}$ is compact with respect to the $\|\cdot\|_\infty$ norm and hence is compact in X. Hence for any $\delta > 0$ we can choose a δ-net $\{\xi_1, \dots, \xi_m\}$ for $\bigcup_{i=1}^{n}\{(\lambda_i)^b : b \in [1/2,1]\}$ in X. We now show that $\{\xi_1, \dots, \xi_m\}$ is an $(\eta/\alpha(2) + \delta)$-net for A in X.

Let $f \in A$. Fix $i \in \{1, \dots, n\}$ such that $\|Qf - \lambda_i\| \leqslant \eta$. Since

$$(\lambda_i)^{(1+\|f\|)/2} \in \{(\lambda_i)^b : b \in [1/2,1]\}$$

we can find $j \in \{1, \dots, m\}$ such that $\left\|(\lambda_i)^{(1+\|f\|)/2} - \xi_j\right\| \leqslant \delta$. Then

$$\begin{aligned}\|f - \xi_j\| &\leqslant \left\|f - (\lambda_i)^{(1+\|f\|)/2}\right\| + \left\|(\lambda_i)^{(1+\|f\|)/2} - \xi_j\right\| \\ &\leqslant \frac{1}{\alpha(2)}\left\|f_{2/(1+\|f\|)} - \left((\lambda_i)^{(1+\|f\|)/2}\right)_{2/(1+\|f\|)}\right\| + \delta \\ &\leqslant \frac{1}{\alpha(2)}\|Qf - \lambda_i\| + \delta \leqslant \frac{\eta}{\alpha(2)} + \delta.\end{aligned}$$

Therefore $\gamma(A) \leqslant \eta/\alpha(2) + \delta$, so $\alpha(2)\gamma(A) \leqslant \gamma(QA)$. ∎

THEOREM 3.6. *Let $P_X : B(X) \to X$ be a compact mapping with $P_X f = 0$ for all $f \in S(X)$, and*

$$\|Qf + P_X f\| \geqslant m, \tag{4}$$

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

for some $m \in (0,1]$ and all $f \in B(X)$. Then the mapping $R : B(X) \to S(X)$ defined by

$$Rf = \frac{Qf + P_X f}{\|Qf + P_X f\|}, \tag{5}$$

is a $(1/m)$-ball contractive retraction. Moreover $\omega(R) \geqslant \alpha(2)/l$ whenever $\|Qf+P_X f\| \leqslant l$ for all $f \in B(X)$. In particular, if $\|Qf + P_X f\| = 1$ for all $f \in B(X)$, the retraction R is 1-ball contractive and $\omega(R) \geqslant \alpha(2)$.

PROOF: Clearly the mapping R defined in (5) is a retraction. Let $A \subset B(X)$. Since P_X is compact, it follows from Proposition 3.5 that

$$\alpha(2)\gamma(A) \leqslant \gamma\big((Q + P_X)A\big) \leqslant \gamma(A). \tag{6}$$

Moreover by the definition of R and by (4) we get

$$RA \subset [0, \frac{1}{m}] \cdot (Q + P_X)A.$$

Using the properties of γ, from (6) it follows $\gamma(RA) \leqslant (1/m)\gamma(A)$. Similarly if $\|Qf + P_X f\| \leqslant l$ for all $f \in B(X)$ we have

$$(Q + P_X)A \subset [0, l] \cdot RA.$$

Therefore $(\alpha(2)/l)\gamma(A) \leqslant \gamma(RA)$, and the proof is complete. ∎

Observe that $\|Qf + P_X f\| = 1$ for $f \in S(X)$, so in condition (4) we necessarily have $m \leqslant 1$.

REMARK 3.7. Whenever in a Banach space X we find $\alpha(a)\|f\| = \|f_a\|$, for all $f \in B(X)$ (a stronger condition than (2)) we modify the mapping Q defined in (3) by setting

$$(Qf)(t) = \frac{1}{\alpha(2/(1 + \|f\|))} f_{2/(1+\|f\|)}(t), \quad \text{for all } \ t \in [0,1]. \tag{7}$$

As no confusion can arise we keep denoting this mapping by Q. Then $\|Qf\| = \|f\|$ for all $f \in B(X)$. Clearly Q is still a continuous mapping and, by slight modifications of the previous arguments and of Proposition 3.5, we get $\gamma(QA) = \gamma(A)$. This allow us to obtain a better estimate of the lower H-measure of noncompactness of the retraction R defined as in (5). In fact, under the same hypotheses of Theorem 3.6, we get $\omega(R) \geqslant 1/l$.

COROLLARY 3.8. *The retraction R defined in (5) is a proper mapping.*

4. THE ORLICZ SPACES L_Φ.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing Young's function. Assume that Φ satisfies the Δ_2-condition, that is, there is $c \in [0, \infty)$ such that $\Phi(2x) \leqslant c\Phi(x)$ $(x \geqslant 0)$. For $f \in \mathcal{M}_0$ set

$$M^\Phi(f) = \int_{[0,1]} \Phi\big(|f(t)|\big)\, dt.$$

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Then

$$\rho_\Phi(f) = \inf\left\{u > 0 : M^\Phi\Big(\frac{f}{u}\Big) \leqslant 1\right\} \ (f \in \mathcal{M}_0^+).$$

is a Banach function norm, and the Banach function space

$$L_\Phi := L_\Phi[0,1] = \left\{f \in \mathcal{M}_0 : \rho_\Phi\big(|f|\big) < \infty\right\}$$

is the Orlicz space generated by Φ endowed with the Luxemburg norm $\|f\|_\Phi = \rho_\Phi\big(|f|\big)$. The Orlicz space L_Φ is of absolutely continuous norm (see for example [**14**]). Then by Lemma 2.4 the space L_Φ satisfies property (P1). The following lemma proved in [**12**] shows that (P2) holds in L_Φ.

Lemma 4.1. ([**12**, Lemma 2.3]) *Let $f \in L_\Phi$ and $a \in [1,2]$. Then*

$$\frac{1}{a}\|f\|_\Phi \leqslant \|f_a\|_\Phi \leqslant \|f\|_\Phi.$$

Let $Q : B(L_\Phi) \to B(L_\Phi)$ be defined as in (3) and define $P_\Phi : B(L_\Phi) \to L_\Phi$ by

$$P_\Phi f = \begin{cases} \Phi^{-1}\Big(\dfrac{2}{1-\|f\|_\Phi}\big(1 - M^\Phi(Qf)\big)\Big)\chi_{\left((1+\|f\|_\Phi)/2,1\right]} & \text{if } f \in B(L_\Phi)\setminus S(L_\Phi) \\ 0 & \text{if } f \in S(L_\Phi). \end{cases}$$

Lemma 4.2. *The mapping P_Φ is compact.*

Proof: We prove that $P_\Phi B(L_\Phi)$ is relatively compact and P_Φ is continuous. Let $\{g_n\}$ be a sequence of elements of $P_\Phi B(L_\Phi)$ and $\{f_n\}$ be a sequence of elements of $B(L_\Phi)$ such that $P_\Phi f_n = g_n$, for all n. Since $0 \leqslant \|f_n\|_\Phi \leqslant 1$ and $0 \leqslant M^\Phi(Qf_n) \leqslant \|Qf_n\|_\Phi \leqslant 1$ for all n, we can choose subsequences $\left\{\|f_{n_k}\|_\Phi\right\}$, $\left\{\|Qf_{n_k}\|_\Phi\right\}$ and $\left\{M^\Phi(Qf_{n_k})\right\}$ which converge, say to b, c and c_Φ, respectively.

If $b = 1$ then by Lemma 4.1, $\|Qf_{n_k}\|_\Phi \to 1$ and consequently

$$M^\Phi(Qf_{n_k}) \to 1$$

Since $M^\Phi(P_\Phi f_{n_k}) = 1 - M^\Phi(Qf_{n_k})$ we have $M^\Phi(P_\Phi f_{n_k}) \to 0$ and hence $\|P_\Phi f_{n_k}\|_\Phi \to 0$. This implies that $\{g_{n_k}\}$ converges in norm to the null function. Assume $b < 1$ and write

$$\begin{aligned}\Big\|P_\Phi f_{n_k} - \Phi^{-1}\Big(\frac{2}{1-b}(1-c_\Phi)\Big)\chi_{((1+b)/2,1]}\Big\|_\Phi \\ = \Big\|\Phi^{-1}\Big(\frac{2}{1-\|f_{n_k}\|_\Phi}\big(1 - M^\Phi(Qf_{n_k})\big)\Big)\chi_{((1+\|f_{n_k}\|_\Phi)/2,1]} \\ - \Phi^{-1}\Big(\frac{2}{1-b}(1-c_\Phi)\Big)\chi_{((1+b)/2,1]}\Big\|_\Phi\end{aligned}$$

By Remark 3.3 we have

$$\|\chi_{((1+\|f_{n_k}\|_\Phi)/2,1]} - \chi_{((1+b)/2,1]}\|_\Phi \to 0$$

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

and by the continuity of Φ^{-1} we also have

$$\Phi^{-1}\Big(\frac{2}{1-\|f_{n_k}\|_\Phi}(1-M^\Phi(Qf_{n_k})\Big) \to \Phi^{-1}\Big(\frac{2}{1-b}(1-c_\Phi)\Big).$$

Thus we get

$$\Big\|P_\Phi f_{n_k} - \Phi^{-1}\Big(\frac{2}{1-b}(1-c_\Phi)\Big)\chi_{((1+b/2),1]}\Big\|_\Phi \to 0.$$

We have proved that $P_\Phi B(L_\Phi)$ is relatively compact.

Let now $\{f_n\}$ be a sequence of elements of $B(L_\Phi)$ such that $\|f_n - f\|_\Phi \to 0$, then, as the Δ_2-condition holds, $M^\Phi(f_n) \to M^\Phi(f)$. An argument similar to that of the first part of the proof implies $\|P_\Phi f_n - P_\Phi f\|_\Phi \to 0$. The proof is complete. ∎

Lemma 4.3. *Let $f \in B(L_\Phi)$, then*

$$\|Qf + P_\Phi f\|_\Phi = 1.$$

Proof: Observe that, for any $u > 0$ we have

$$M^\Phi\Big(\frac{Qf + P_\Phi f}{u}\Big) = M^\Phi\Big(\frac{Qf}{u}\Big) + M^\Phi\Big(\frac{P_\Phi f}{u}\Big).$$

Now for $u = 1$ we get

$$M^\Phi(Qf + P_\Phi f) = \int_{((1+\|f\|_\Phi)/2,1]} \Phi\Big(\Phi^{-1}\Big(\frac{2}{1-\|f\|_\Phi}(1-M^\Phi(Qf))\Big)\Big)dt + M^\Phi(Qf)$$
$$= \int_{((1+\|f\|_\Phi/2),1]} \frac{2}{1-\|f\|_\Phi}(1-M^\Phi(Qf))\ dt + M^\Phi(Qf) = 1$$

It follows that $\|Qf + P_\Phi f\|_\Phi \leqslant 1$. On the other hand if $0 < u < 1$

$$M^\Phi\Big(\frac{Qf + P_\Phi f}{u}\Big) > M^\Phi(Qf + P_\Phi f),$$

consequently $\|Qf + P_\Phi f\|_\Phi = 1$. ∎

From Lemmas 4.1, 4.2 and 4.3 and Theorem 3.6 we obtain the following.

Theorem 4.4. *The mapping $R : B(L_\Phi) \to S(L_\Phi)$ defined by*

$$Rf = Qf + P_\Phi f$$

is a 1-ball contractive retraction and $\omega(R) \geqslant 1/2$.

Observe that, if $\Phi(t) = t^p$ where $1 \leqslant p < \infty$, then L_Φ is the Lebesgue space $L_p := L_p[0,1]$, with the standard norm $\|\cdot\|_p$. But in this case an easy computation shows that $(1/a)^{1/p}\|f\|_p = \|f_a\|_p$. Hence, according to Remark 3.7, a stronger result on the characteristic $\omega(R)$ holds. Define $Q : B(L_p) \to B(L_p)$ (as in (7)) by

$$(Qf)(t) = \Big(\frac{2}{1+\|f\|_p}\Big)^{1/p} f_{2/(1+\|f\|_p)}(t), \quad \text{for all } t \in [0,1].$$

Next define $P_p : B(L_p) \to L_p$ by

$$P_p f = \begin{cases} \left(\frac{2}{1-\|f\|_p}\left(1-\|f\|_p^p\right)\right)^{1/p} \chi_{\left((1+\|f\|_p)/2,1\right]} & \text{if } f \in B(L_p) \setminus S(L_p) \\ 0 & \text{if } f \in S(L_p). \end{cases}$$

Then the following theorem holds.

Theorem 4.5. *The mapping $R : B(L_p) \to S(L_p)$ $(1 \leqslant p < \infty)$ defined by*

$$Rf = Qf + P_p f$$

is a 1-ball contractive retraction and $\omega(R) = 1$.

The results obtained in the Lebesgue spaces L_p can be generalised to the weighted spaces. Let ρ be a measurable weighting function. We consider the *weighted Lebesgue space*

$$L_p(\rho) := L_p([0,1], \rho) \ (1 \leqslant p < \infty)$$

which consists of all $f \in \mathcal{M}_0$ such that $\rho^{1/p} f \in L_p$, endowed with the norm

$$\|f\|_{L_p(\rho)} = \left(\int_{[0,1]} \rho(t)\left|f(t)\right|^p dt\right)^{1/p}.$$

The space $L_p(\rho)$ has absolutely continuous norm.
We define a mapping $Q_\rho : B(L_p(\rho)) \to B(L_p(\rho))$ by a slight modification of (7)

$$(Q_\rho f)(t) = \left(\rho_{2/(1+\|f\|_{L_p(\rho)})}(t)/\rho(t)\right)^{1/p} \left(\frac{2}{1+\|f\|_{L_p(\rho)}}\right)^{1/p} f_{2/(1+\|f\|_{L_p(\rho)})}(t) \quad \text{for all } \ t \in [0,1]$$

and define $P_\rho : B(L_p(\rho)) \to L_p(\rho)$ by

$$P_\rho f = \begin{cases} \left(\frac{2}{1-\|f\|_{L_p(\rho)}}\right)^{1/p} \left(\frac{1-\|f\|_{L_p(\rho)}^p}{\rho(t)}\right)^{1/p} \chi_{\left((1+\|f\|_p)/2,1\right]} & \text{if } f \in B(L_p(\rho)) \setminus S(L_p(\rho)) \\ 0 & \text{if } f \in S(L_p(\rho)). \end{cases}$$

Set

$$C([0,1], \rho) = \{g/\rho^{1/p} : g \in C[0,1]\}$$

and

$$W(\rho) = \{s/\rho^{1/p} : s \in W\}.$$

Then $C([0,1],\rho)$ is dense in $L_p([0,1],\rho)$ and $L_p([0,1],\rho) = \overline{W(\rho)}^{\|\cdot\|_{L_p(\rho)}}$. Moreover for a continuous function g, the set $A_g(\rho) = \{g_a/\rho^{1/p} : a \in [1,2]\}$ is compact. Then the same arguments of Section 3 allow us to obtain the following.

Corollary 4.6. *The mapping*

$$R : B\big(L_p(\rho)\big) \to S(L_p(\rho)) \ (1 \leqslant p < \infty)$$

defined by $Rf = Q_\rho f + P_\rho f$ *is a* 1*-ball contractive retraction with* $\omega(R) = 1$.

In this section we have improved the results in the L_p and L_Φ spaces of [**17, 12**], respectively. Though the mapping Q is the same as the one introduced in those papers, here we construct in both cases a different retraction R and, above all, our proofs are based on different ideas and techniques.

5. The Lorentz spaces $L^{p,q}$.

Let f^* denote the decreasing rearrangement of a function $f \in \mathcal{M}_0$, given by

$$f^*(t) = \inf\Big\{s \geqslant 0 : \mu\{|f(x)| > s\} \leqslant t\Big\}$$

The Lorentz space $L^{p,q} := L^{p,q}\big([0,1]\big)$ $(1 \leqslant q \leqslant p < \infty)$ consists of all $f \in \mathcal{M}_0$ for which the quantity

$$\|f\|_{p,q} = \Big(\frac{q}{p}\int_{[0,1]} \big(t^{1/p} f^*(t)\big)^q \,\frac{dt}{t}\Big)^{1/q}$$

is finite. As the Lorentz space $L^{p,q}$ is reflexive (see for example [**14**]) from [**3**, Corollary 4.4] it follows that it has absolutely continuous norm. Hence by Lemma 2.4 the space $L^{p,q}$ satisfies property (P1).

Lemma 5.1. *Let* $f \in L^{p,q}$ *and* $a \in [1,2]$, *then*

$$\Big(\frac{1}{a}\Big)^{1/p} \|f\|_{p,q} = \|f_a\|_{p,q}.$$

Proof: Let $f \in L^{p,q}$. We observe that we have $(f_a)^* = (f^*)_a$. Then the lemma follows by a direct computation of $\|f_a\|^q_{p,q}$. Indeed we have

$$\begin{aligned}
\|f_a\|^q_{p,q} &= \frac{q}{p}\int_{[0,1]} t^{(q/p)-1}\big((f_a)^*(t)\big)^q \,dt = \frac{q}{p}\int_{[0,1]} t^{(q/p)-1}\big((f^*)_a(t)\big)^q \,dt \\
&= \frac{q}{p}\int_{[0,1/a]} t^{(q/p)-1}\big(f^*(at)\big)^q \,dt \\
&= \Big(\frac{1}{a}\Big)^{q/p} \frac{q}{p}\int_{[0,1]} t^{(q/p)-1}\big(f^*(t)\big)^q \,dt = \Big(\frac{1}{a}\Big)^{(q/p)} \|f\|^q_{p,q},
\end{aligned}$$

hence the thesis. ∎

In view of Lemma 5.1 and Remark 3.7 we define $Q : B(L^{p,q}) \to B(L^{p,q})$ (as in (7)) by

$$(Qf)(t) = \Big(\frac{2}{1 + \|f\|_{p,q}}\Big)^{1/p} f_{2/(1+\|f\|_{p,q})}(t), \quad \text{for all } \ t \in [0,1].$$

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Next define $P_{p,q} : B(L^{p,q}) \to L^{p,q}$

$$P_{p,q}f = \begin{cases} \left(\frac{2}{1-\|f\|_{p,q}}\right)^{1/p} \left(1 - \|f\|^q_{p,q}\right)^{1/q} \chi_{\left((1+\|f\|_{p,q})/2,1\right]} & \text{if } f \in B(L^{p,q}) \setminus S(L^{p,q}) \\ 0 & \text{if } f \in S(L^{p,q}). \end{cases}$$

We have that the mapping $P_{p,q}$ is compact and $\|Qf + P_{p,q}f\|_{p,q} = 1$ for all $f \in B(L^{p,q})$. Hence by Theorem 3.6 and Remark 3.7 we obtain the following.

THEOREM 5.2. *The mapping*

$$R : B(L^{p,q}) \to S(L^{p,q}) \ (1 \leqslant q \leqslant p < \infty)$$

defined by

$$Rf = Qf + P_{p,q}f$$

is a 1-ball contractive retraction and $\omega(R) = 1$.

The questions whether or not $W(X) = 1$ in any infinite-dimensional Banach space X, and eventually if this value is always a minimum remain open.

We conclude this section with some remarks on fixed point free self-mappings of the unit ball $B(X)$. In [**1**, Theorem 3] the following theorem has been proved.

THEOREM 5.3. *Let X be an infinite-dimensional Banach space and $\varepsilon > 0$. Then there exists a fixed point free 1-ball contraction $F : B(X) \to B(X)$ with $\omega(F) \geqslant 1 - \varepsilon$.*

We have that, in some Banach spaces, the best value $\omega(F) = 1$ can be attained by a fixed point free 1-ball contraction $F : B(X) \to B(X)$. Indeed if $R : B(X) \to S(X)$ is a k-ball contractive retraction, then $F = -R : B(X) \to B(X)$ is a fixed point free k-ball contraction. As a consequence of Corollary 4.6 and Theorem 5.2 we obtain the following.

COROLLARY 5.4. *Let X denote either the weighted Lebesgue space $L_p(\rho)$ ($1 \leqslant p < \infty$) or the Lorentz space $L^{p,q}$ ($1 \leqslant q \leqslant p < \infty$). Then there exists a fixed point free 1-ball contraction $F : B(X) \to B(X)$ with $\omega(F) = 1$.*

6. Banach spaces with $(1+\varepsilon)$-ball contractive retractions.

In this section we consider X to be the space of all functions of absolutely continuous norm of a Banach function space Y, where Y is either the grand L^p space or the Marcinkiewicz spaces M_β. Applying Theorem 3.6 we prove that, in both cases, for any $\varepsilon > 0$ there is a $(1+\varepsilon)$-ball contractive retraction R with positive H-lower measure of noncompactness.

Let $1 < p < \infty$. The grand L^p space, which will be denoted by $L^{p)} := L^{p)}([0,1])$, introduced in [**11**], is defined as the space of all functions $f \in \mathcal{M}_0$ such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{[0,1]} |f(t)|^{p-\varepsilon}\, dt\right)^{1/(p-\varepsilon)} < \infty.$$

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

We denote by $X^{p)}$ the set of all functions in $L^{p)}$ of absolutely continuous norm and by W the subset of $L^{p)}$ of all simple functions.

Lemma 6.1. *The subspace $X^{p)}$ coincides with $\overline{W}^{\|\cdot\|_{p)}}$, and the inclusion $X^{p)} \subset L^{p)}$ is proper.*

Proof: Let $\sigma > 0$ and set $\delta = \big(\sigma/(p-1)\big)^p$. Let $D \in \Sigma$ with $\mu(D) < \delta$. As $\sup\limits_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} = p-1$ and $\sup\limits_{0<\varepsilon<p-1} \mu(D)^{1/(p-\varepsilon)} = \mu(D)^{1/p}$ we have

$$\|\chi_D\|_{p)} = \sup_{0<\varepsilon<p-1} \big(\varepsilon\mu(D)\big)^{1/(p-\varepsilon)} \leqslant (p-1)\mu(D)^{1/p} < \sigma.$$

This shows that $\chi_{[0,1]}$ has absolutely continuous norm, hence by Lemma 2.4 (iii) it follows $X^{p)} = \overline{W}^{\|\cdot\|_{p)}}$. To end the proof it suffices to note that the function $t^{-1/p} \in L^{p)}$ has not absolutely continuous norm. ∎

Lemma 6.2. *Let $f \in X^{p)}$ and $a \in [1,2]$,*

$$\frac{1}{a}\|f\|_{p)} \leqslant \|f_a\|_{p)} \leqslant \|f\|_{p)}.$$

Proof: For any $f \in X^{p)}$ and $a \in [1,2]$ we have

$$\begin{aligned}\|f_a\|_{p)} &= \sup_{0<\varepsilon<p-1} \Big(\varepsilon \int_{[0,\frac{1}{a}]} |f(at)|^{p-\varepsilon}\, dt\Big)^{1/(p-\varepsilon)} \\ &= \sup_{0<\varepsilon<p-1} \Big(\frac{1}{a}\Big)^{1/(p-\varepsilon)} \Big(\varepsilon \int_{[0,1]} |f(t)|^{p-\varepsilon}\, dt\Big)^{1/(p-\varepsilon)} \leqslant \|f\|_{p)}.\end{aligned}$$

On the other hand we find

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} a^{1/(p-\varepsilon)} \Big(\varepsilon \int_{[0,1/a]} \big|f(at)\big|^{p-\varepsilon}\, dt\Big)^{1/(p-\varepsilon)} \leqslant a\|f_a\|_{p)},$$

which completes the proof. ∎

Let $Q : B(X^{p)}) \to B(X^{p)})$ be defined as in (3) and define for every $0 < u < \infty$ the mapping $(P_{p)})_u : B(X^{p)}) \to X^{p)}$ by

$$(P_{p)})_u f = \begin{cases} u \dfrac{1-\|Qf\|_{p)}}{\Big\|\chi_{\left((1+\|f\|_{p)})/2,1\right]}\Big\|_{p)}} \chi_{\left((1+\|f\|_{p)})/2,1\right]} & \text{if } f \in B(X^{p)}) \setminus S(X^{p)}) \\ 0 & \text{if } f \in S(X^{p)}). \end{cases}$$

Lemma 6.3. *For any $0 < u < \infty$, the mapping $(P_{p)})_u$ is compact, and for $f \in B(X^{p)})$*

$$\big\|(P_{p)})_u f\big\|_{p)} = u\big(1 - \|Qf\|_{p)}\big)$$

Proof: The proof that $(P_{p)})_u$ is compact is similar to the proof of Lemma 4.2. A direct calculation gives the norm of $(P_{p)})_u f$. ∎

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Lemma 6.4. *Let $0 < u < \infty$. For any $f \in B(X^{p)})$*

$$\max\{1, u\} \geqslant \left\|Qf + (P_{p)})_u f\right\|_{p)} \geqslant \frac{u}{u+1}$$

Proof: Let $f \in B(X^{p)})$, then

$$\left\|Qf + (P_{p)})_u f\right\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon \int_{[0,1]}\left|(Qf)(t)\right|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]}\left|\left((P_{p)})_u f\right)(t)\right|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)}.$$

Now for any fixed $0 < \varepsilon < p-1$

$$\varepsilon \int_{[0,1]}\left|(Qf)(t)\right|^{p-\varepsilon} dt \leqslant \varepsilon \int_{[0,1]}\left|(Qf)(t)\right|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]}\left|\left((P_{p)})_u f\right)(t)\right|^{p-\varepsilon} dt$$

and passing to the $1/(p-\varepsilon)$-power we have

$$\left(\varepsilon \int_{[0,1]}\left|(Qf)(t)\right|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)} \leqslant \left(\varepsilon \int_{[0,1]}\left|(Qf)(t)\right|^{p-\varepsilon} dt + \varepsilon \int_{[0,1]}\left|\left((P_{p)})_u f\right)(t)\right|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)}$$

Taking the supremum over ε we get $\left\|Qf + (P_{p)})_u f\right\|_{p)} \geqslant \|Qf\|_{p)}$. Analogously we get

$$\left\|Qf + (P_{p)})_u f\right\|_{p)} \geqslant \left\|(P_{p)})_u f\right\|_{p)}.$$

Then

$$\left\|Qf + (P_{p)})_u f\right\|_{p)} \geqslant \max\left\{\|Qf\|_{p)}, u\left(1 - \|Qf\|_{p)}\right)\right\} \geqslant u/u+1.$$

On the other hand it easily follows

$$\left\|Qf + (P_{p)})_u f\right\|_{p)} \leqslant \|Qf\|_{p)} + u\left(1 - \|Qf\|_{p)}\right) \leqslant \max\{1, u\}.$$

□

By Lemmas 6.1 and 6.2, the Banach space $X^{p)}$ satisfies properties (P1) and (P2). Hence by Lemmas 6.3 and 6.4 and Theorem 3.6 we have that the mapping $R_u : B(X_{p)}) \to S(X_{p)})$ defined by

$$R_u f = \frac{Qf + (P_{p)})_u f}{\|Qf + (P_{p)})_u f\|_{p)}}$$

is $(u+1)/u-$ball contractive with $\omega(R_u) \geqslant \min\{1/2, 1/(2u)\}$. As $\lim_{u\to\infty}(u+1)/u = 1$ we obtain the following theorem.

Theorem 6.5. *For any $\varepsilon > 0$ there is a retraction*

$$R : B(X^{p)}) \to S(X^{p)}) \ \left(1 < p < \infty\right)$$

which is $(1+\varepsilon)$-ball contractive with $\omega(R) > 0$.

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Remark 6.6. *The same result of Theorem 6.5 can be proved in the small Lebesgue space $L^{p)'}$ $(1 < p < \infty)$ introduced in [7], in which the norm is defined as*

$$\|f\|_{p)'} = \sup_{g \in L^{p)}} \frac{\int_{[0,1]} f(t)g(t)\,dt}{\|g\|_{p)}}.$$

We recall that the spaces $L^{p)'}$ have absolutely continuous norm, and the spaces $L^{p)}$ are characterised as dual spaces of $L^{p)'}$ (see [5]).

An analogous result holds in the Marcinkiewicz space

$$M_\beta := M_\beta([0,1]) \ (0 < \beta < 1)$$

which consists of all $f \in \mathcal{M}_0$ for which

$$\|f\|_\beta = \sup \frac{1}{\mu(E)^\beta} \int_E |f(t)|\,dt < \infty.$$

where the supremum is taken over all $E \in \Sigma$ with $\mu(E) > 0$. We denote by X_β the set of all functions in M_β of absolutely continuous norm and W the subset of M_β of all simple functions.

Lemma 6.7. *The subspace X_β coincides with $\overline{W}^{\|\cdot\|_\beta}$, and the inclusion $X_\beta \subset M_\beta$ is proper.*

Proof: We prove that for every $D \in \Sigma$

(8) $$\|\chi_D\|_\beta = \mu(D)^{1-\beta}.$$

By definition we have

$$\|\chi_D\|_\beta = \sup \frac{1}{\mu(E)^\beta} \mu(D \cap E).$$

Choose for every $n \in N$ a set $E_n \in \Sigma$ such that

$$\|\chi_D\|_\beta - \frac{1}{n} \leqslant \frac{1}{\mu(E_n)^\beta} \mu(D \cap E_n) \leqslant \|\chi_D\|_\beta.$$

Set $D_n = D \cap E_n$. As $D_n \subset E_n$ we get $1/(\mu(E_n)^\beta) \leqslant 1/(\mu(D_n)^\beta)$. Consequently,

$$\|\chi_D\|_\beta - \frac{1}{n} \leqslant \frac{1}{\mu(E_n)^\beta} \mu(D_n) \leqslant \frac{1}{\mu(D_n)^\beta} \mu(D_n) \leqslant \|\chi_D\|_\beta.$$

As n goes to infinity we get (8). From (8) it obviously follows that $\chi_{[0,1]}$ has absolutely continuous norm, hence (iii) of Lemma 2.4 gives $X_\beta = \overline{W}^{\|\cdot\|_\beta}$. As pointed out in [2] the space M_β has not absolutely continuous norm. ∎

It easy to check that the following lemma holds.

https://doi.org/10.1017/S0004972700035097 Published online by Cambridge University Press

Lemma 6.8. *Let $f \in X_\beta$ and $a \in [1,2]$,*

$$\left(\frac{1}{a}\right)^{1-\beta}\|f\|_\beta \leqslant \|f_a\|_\beta \leqslant \|f\|_\beta.$$

Now let $Q : B(X_\beta) \to B(X_\beta)$ be defined as in (3) and define for every $0 < u < \infty$ the mapping $(P_\beta)_u : B(X_\beta) \to X_\beta$ by

$$(P_\beta)_u f = \begin{cases} u\left(\dfrac{2}{1-\|f\|_\beta}\right)^{1-\beta}(1-\|Qf\|_\beta)\chi_{((1+\|f\|_\beta)/2,1]} & \text{if } f \in B(X_\beta)\setminus S(X_\beta) \\ 0 & \text{if } f \in S(X_\beta). \end{cases}$$

For every $0 < u < \infty$, the mapping $(P_\beta)_u$ is compact and

$$\left\|(P_\beta)_u f\right\|_\beta = u(1-\|Qf\|_\beta).$$

Moreover the following estimates of $\left\|Qf + (P_\beta)_u f\right\|_\beta$ can be derived by an argument similar to that of Lemma 6.4.

Lemma 6.9. *Let $0 < u < \infty$. For any $f \in B(X_\beta)$*

$$\max\{1,u\} \geqslant \left\|Qf + (P_\beta)_u f\right\|_\beta \geqslant \frac{u}{u+1}.$$

By Lemmas 6.7 and 6.8, the Banach space X_β satisfies properties (P1) and (P2). Then by the previous Lemma and Theorem 3.6 we have that the mapping $R_u : B(X_\beta) \to S(X_\beta)$ defined by

$$R_u f = \frac{Qf + (P_\beta)_u f}{\|Qf + (P_\beta)_u f\|_\beta}$$

is $(u+1)/u-$ball contractive with

$$\omega(R_u) \geqslant \min\{1/(2^{1-\beta}u),\ 1/(2^{1-\beta})\}.$$

As $\lim_{u\to\infty}(u+1)/u = 1$ we obtain the following.

Theorem 6.10. *For any $\varepsilon > 0$ there is a retraction $R : B(X_\beta) \to S(X_\beta)$ which is $(1+\varepsilon)$-ball contractive with $\omega(R) > 0$.*

References

[1] J. Appell, N.A. Erkazova, S. Falcon Santana and M. Väth, 'On some Banach space constants arising in nonlinear fixed point and eigenvalue theory', *Fixed Point Theory Appl.* **4** (2004), 317–336.

[2] J. Appell and P.P. Zabrejko, *Nonlinear superposition operators* (Cambridge University Press, Cambridge, 1990).

[3] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Maths **129** (Boston Academic Press, Boston MA, 1988).

[4] Y. Benyamini and Y. Sternfeld, 'Spheres in infinite-dimensional normed spaces are Lipschitz contractible', *Proc. Amer. Math. Soc.* **88** (1983), 439–445.

[5] C. Capone and A. Fiorenza, 'On small Lebesgue spaces', *J. Funct. Spaces Appl.* **3** (2005), 73–89.

[6] D. Caponetti and G. Trombetta, 'On proper k-ball contractive retractions in the Banach space $BC([0,\infty))$', *Nonlinear Func. Anal. Appl.* (to appear).

[7] A. Fiorenza, 'Duality and reflexivity in grand Lebesgue spaces', *Collect. Math.* **51** (2000), 131–148.

[8] K. Goebel, 'On the minimal displacement of points under Lipschitzian mappings', *Pacific J. Math.* **45** (1973), 151–163.

[9] K. Goebel, 'On the problem of retracting balls onto their boundary', *Abstr. Appl. Anal.* **2** (2003), 101–110.

[10] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory* (Cambridge, 1990).

[11] T. Iwaniec and C. Sbordone, 'On the integrability of the Jacobian under minimal hypotheses', *Arch. Rational Mech. Anal.* **119** (1992), 129–143.

[12] G. Lewicki and G. Trombetta, 'Almost contractive retractions in Orlicz spaces', *Bull. Austral. Math. Soc.* **68** (2003), 353–369.

[13] B. Nowak, 'On the Lipschitzian retraction of the unit ball in infinite-dimensional Banach spaces onto its boundary', *Bull. Acad. Polon. Sci.* **27** (1979), 861–864.

[14] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics **146** (Marcel Dekker, Inc., New York, 1991).

[15] W. Rudin, *Real and complex analysis* (McGraw-Hill Book Co., New York, 1987).

[16] G. Trombetta, 'k-set contractive retractions in spaces of continuous functions', *Sci. Math. Jpn.* **59** (2004), 121–128.

[17] A. Trombetta and G. Trombetta, 'On the existence of (γ_p) k-set contractive retractions in $L_p[0,1]$ spaces, $1 \leqslant p < \infty$', *Sci. Math. Jpn.* **56** (2002), 327–335.

[18] M. Väth, *Ideal spaces*, Lect. Notes in Math. **1664** (Springer-Verlag, Berlin, 1997).

[19] M. Väth, 'On the minimal displacement problem of γ-Lipschitz maps and γ-Lipschitz retractions onto the sphere', *Z. Anal. Anwendungen* **21** (2002), 901–914.

[20] J. Wośko, 'An example related to the retraction problem', *Ann. Univ. Mariae Curie-Skłodowska* **45** (1991), 127–130.

Department of Mathematics
University of Palermo
Via Archirafi, 34
90123 Palermo
Italy
e-mail: d.caponetti@math.unipa.it

Department of Mathematics
University of Calabria
87036 Arcavacata di Rende (CS)
Italy
e-mail: aletromb@unical.it

Department of Mathematics
University of Calabria
87036 Arcavacata di Rende (CS)
Italy
e-mail: trombetta@unical.it