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p-ADIC EIGEN-FUNCTIONS FOR KUBERT DISTRIBUTIONS

NEAL KOBLITZ

1. Introduction. Functions on **R** (or on **R**/**Z**, or **Q**/**Z**, or the interval (0,1)) which satisfy the identity

$$(1.1)\quad f(x) = m^{s-1}\sum_{h=0}^{m-1} f\left(\frac{x+h}{m}\right)$$

for positive integers m and fixed complex s, appear in several branches of mathematics (see [**8**], p. 65-68). They have recently been studied systematically by Kubert [**6**] and Milnor [**12**]. Milnor showed that for each complex s there is a one-dimensional space of even functions and a one-dimensional space of odd functions which satisfy (1.1). These functions can be expressed in terms of either the Hurwitz partial zeta-function or the polylogarithm functions.

My purpose is to prove an analogous theorem for p-adic functions. The p-adic analog is slightly more general; it allows for a Dirichlet character $\chi_0(m)$ in front of m^{s-1} in (1.1). The functions satisfying (1.1) turn out to be p-adic "partial Dirichlet L-functions", functions of two p-adic variables (x, s) and one character variable χ_0, which specialize to partial zeta-functions when χ_0 is trivial and to Kubota-Leopoldt L-functions when $x = 0$.

In the p-adic case one can interpret (1.1) in terms of continuous representations of the group of p-adic units on a function space. This interpretation suggests further questions about the role of the Kubert identities and the corresponding operators in p-adic function theory.

Let M be a submonoid of the multiplicative semigroup of nonzero integers: $M \subset \{\pm 1, \pm 2, \ldots\}$. Let X be a topological ring which is divisible by all $m \in M$; let K be a topological field of characteristic zero; let $\check{X}^*$ denote the group of all quasicharacters, i.e., all continuous homomorphisms from the units X^* to $\mathcal{K}^*$; and let $\mathcal{F}$ be a space of functions f: $X \rightarrow K$ (or f: $X \rightarrow K \cup \{\infty\}$).

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There are two examples of M, X, K, $\check{X}^*$ and $\mathscr{F}$ to have in mind:
(1) M = {all nonzero integers}; $X = \mathbf{R}$; $K = \mathbf{C}$;

$$\check{X}^* \approx (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{C},$$

where

$$\check{X}^* \ni (\epsilon, s): x \mapsto (\operatorname{sgn} x)^\epsilon \cdot |x|^s;$$

$\mathscr{F} = \{f: \mathbf{R} \to \mathbf{C} \cup \{\infty\} \mid f$ is finite and continuous on $(0, 1)$ and periodic of period $1\}$.

(2) M = {all integers prime to a fixed integer $d = d_0 p$};

$$X = \lim_{\leftarrow} (\mathbf{Z}/dp^N\mathbf{Z});$$

$K = \Omega_p$ = p-adic completion of the algebraic closure of the p-adic numbers $\mathbf{Q}_p$;

$$\check{X}^* \approx \{(\chi_0, t) \mid \chi_0 \text{ a character of } (\mathbf{Z}/d\mathbf{Z})^*,$$

$$t \in \Omega_p, |t - 1|_p < 1\};$$

$$\mathscr{F} = \{\text{all continuous f: } X \to \Omega_p \text{ such that}$$

$$f(x + 1) = f(x) \text{ for } x \in X - X^*\}.$$

Now for any $m \in M$ and any $f: X \to K$ (or $f: X \to K \cup \{\infty\}$) we define the operator T_m by

$$(1.2) \quad T_m f(x) = \frac{1}{|m|} \sum f\left(\frac{x + h}{m}\right),$$

where the summation is over $0 \leqq h < m$ if $m > 0$, $0 > h \geqq m$ if $m < 0$. Note that

$$(1.3) \quad T_{-1} f(x) = f(1 - x).$$

Further note that T commutes with multiplication in M, i.e.,

$$T_{m_1 m_2} = T_{m_1} \cdot T_{m_2},$$

so that T gives a representation of M in the space of functions. Then Milnor's result can be re-stated:

PROPOSITION (Milnor [**12**]). *Let M, X, K, $\mathscr{F}$ be as in example* (1). *For any character* $(\epsilon, s) \in \check{X}^*$ *the restriction of* $(\epsilon, s)^{-1}$ *to M has a one-dimensional eigen-space for the action of T, i.e., f is determined up to a constant multiple by the identity*

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$$T_m f = (\text{sgn } m)^{\epsilon} |m|^{-s} f.$$

In the p-adic case, M is dense in the p-adic units X^*, and our Theorem 1 in Section 2 states that the action of T extends continuously to all of X^*. Then we have

THEOREM 2. *Let M, X, K, $\mathscr{F}$ be as in example* (2). *Then any character $\chi \in \check{X}^*$ has a one-dimensional eigen-space for the action of T. It is spanned by the partial L-function corresponding to χ (see Section 3 below).*

2. p-adic interpolation of the distribution operators. Let d_0 be a fixed positive integer prime to p. Let $d = d_0 p$ for $p > 2$, $d = 4d_0$ for $p = 2$. Let

$$X = \lim_{\substack{\leftarrow \\ N}} (\mathbf{Z}/dp^N\mathbf{Z}).$$

The compact-open subset $\{x \in X \mid x \equiv a \bmod dp^N\}$ will be denoted $a + dp^N\mathbf{Z}_p$. Let X^* denote the union of $a + d\mathbf{Z}_p$ over all a prime to d; thus, X^* is the set of invertible elements of the ring X. Further let $\tilde{\mathscr{F}}$ be the vector space of all continuous $f: X \rightarrow \Omega_p$ with the sup-norm. Let

$$M = \mathbf{Z} \cap X^* = \{\text{all integers prime to } d\}.$$

THEOREM 1. (1) *For every $m \in M$, the map $T_m: \tilde{\mathscr{F}} \rightarrow \tilde{\mathscr{F}}$ is a bounded linear map with $\|T_m\| \leqq 1$; in fact, $\|T_m f\| \leqq \|f\|$ for all $f \in \tilde{\mathscr{F}}$.*

(2) $T_{m_1} \cdot T_{m_2} = T_{m_1 m_2}$.

(3) *T_m depends continuously on m.*

(4) *T extends uniquely to a continuous representation of the group X^* on $\tilde{\mathscr{F}}$.*

(5) *T_x is an isometry, i.e., $\|T_x f\| = \|f\|$ for all $x \in X^*$, $f \in \tilde{\mathscr{F}}$.*

(6) *The subspace $\mathscr{F} = \{f \in \tilde{\mathscr{F}} \mid f(x+1) = f(x)$ for $x \in X - X^*\}$ is T-invariant.*

Proof. (1) and (2) are obvious from the definition (1.2). (6) is also clear, since

$$T_m f(x+1) - T_m f(x) = |m|^{-1}\left(f\left(\frac{x}{m} + 1\right) - f\left(\frac{x}{m}\right)\right).$$

If we prove (3), then (4) will follow, because M is dense in X^* (that T is a homomorphism follows from (2) and the denseness of M in X^*). (5) will then also follow, since (1) implies

$$\|f\| = \|T_{1/x}(T_x f)\| \leqq \|T_x f\| \leqq \|f\|.$$

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So it remains to prove (3).

Let $f: X \to \Omega_p$ be a continuous function bounded by b: $|f(x)|_p \leqq b$ for $x \in X$. Choose N_1 so that

(1.4) $\quad x - y \in dp^{N_1}\mathbf{Z}_p \Rightarrow |f(x) - f(y)|_p < \epsilon.$

Let $m, m' \in M$, and suppose that $m' = m + kdp^N$, $k > 0$. First suppose that m and m' are positive. Then

$$T_{m'}f(x) - T_m f(x) = A + B + C,$$

where

$$A = \left(\frac{1}{m'} - \frac{1}{m}\right) \Sigma f\left(\frac{x + h}{m'}\right),$$

summation over h as in (1.2) for $T_{m'}$;

$$B = \frac{1}{m} \Sigma \left(f\left(\frac{x + h}{m'}\right) - f\left(\frac{x + h}{m}\right)\right),$$

summation over h as in (1.2) for $T_{m'}$;

$$C = \frac{1}{m} \Sigma f\left(\frac{x + h}{m}\right),$$

summation over $m \leqq h < M + kdp^N$.

If m and m' are both negative, then

$$T_{m'}f(x) - T_m f(x) = -A - B + C;$$

and if m is negative and m' is positive, then one checks that

$$T_{m'}f(x) - T_m f(x) = A + B + C.$$

So it suffices to estimate $|A|_p$, $|B|_p$, $|C|_p$. Clearly, $|A|_p \leqq p^{-N}b$. Since

$$\left|\frac{x + h}{m'} - \frac{x + h}{m}\right|_p \leqq |m' - m|_p,$$

we have $|B|_p < \epsilon$ if $N \geqq N_1$, by (1.4). Finally, let $N = N_1 + N_2$, and in C divide the sum into sums over all h of the form

$$a + jkdp^{N_1}, \quad j = 0, \dots, p^{N_2} - 1.$$

That is,

$$C = \frac{1}{m} \sum_{a=m}^{m + kdp^{N_1} - 1} S_a, \quad \text{where}$$

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$$S_a = \sum_{j=0}^{p^{N_2}-1} f\left(\frac{x+a}{m} + \frac{jkd}{m} p^{N_1}\right).$$

Since

$$\left| f\left(\frac{x+a}{m} + \frac{jkd}{m} p^{N_1}\right) - f\left(\frac{x+a}{m}\right) \right|_p < \epsilon$$

for all j by (1.4), we have

$$|S_a|_p \leqq \epsilon + \left| \sum_{j=0}^{p^{N_2}-1} f\left(\frac{x+a}{m}\right) \right|_p$$

$$= \epsilon + \left| p^N f\left(\frac{x+a}{m}\right) \right|_p \leqq \epsilon + p^{-N_2} b.$$

Hence also

$$|C|_p \leqq \epsilon + p^{-N_2} b.$$

This completes the proof of the theorem.

3. Eigen-spaces for the distribution operator. Let X be as in Section 2. We have a natural projection $\pi: X \to \mathbf{Z}_p$ defined by:

$$\pi(a + dp^N\mathbf{Z}_p) = a + p^N \mathbf{Z}_p \quad (\text{“forget mod } d \text{ information”}).$$

We shall always consider functions on $\mathbf{Z}_p$ to be functions on X by means of this projection. We also have two maps $\omega: X \to \mathbf{Z}_p$ and $< >: X \to \mathbf{Z}_p$ given by

$$\omega(x) = \lim \pi(x)^{p^N}$$

(i.e., the Teichmüller representative of $\pi(x)$) and

$$<x> = \pi(x)\, \omega^{-1}(x)$$

(where we take $\omega^{-1}(x) = 0$ if $\omega(x) = 0$).

Fix a topological generator γ for $1 + p\mathbf{Z}_p = \gamma^{\mathbf{Z}_p}$. The correspondence between quasicharacters $\chi \in \check{X}^*$ and pairs (χ_0, t) (where χ_0 is a character of $(\mathbf{Z}/d\mathbf{Z})^*$, $t \in \Omega_p$, $|t - 1|_p < 1$) is as follows:

$$\chi(x) = \chi_0(x \bmod d)\, t^\alpha \quad \text{if } <x> = \gamma^\alpha.$$

If t is within the disc of radius $p^{-1/(p-1)}$ around 1, then we can write $t = \gamma^s$ for some s, $|s|_p < p^{(p-2)/(p-1)}$, and in that case

$$\chi(x) = \chi_0(x \bmod d) <x>^s.$$

Definition. Let $\chi \in \check{X}^*$, $x \in X$. If χ is not locally constant (i.e., the parameter t is not a root of unity), then the *p-adic partial L-function* is defined to be

$$(3.1)\quad f_\chi(x) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{0\leqq j<dp^N} \chi(x+j)$$

$$= \lim_{N\to\infty} \frac{1}{dp^N} \sum_{\substack{0\leqq j<dp^N\\ x+j\in X^*}} \chi_0(x+j)\, t^{\left(\frac{\log_p(x+j)}{\log_p \gamma}\right)}$$

(here $\log_p$ is the Iwasawa p-adic logarithm, see [**3**], p. 36-40 or [**5**], p. 17-18, which we regard as a function on X^* via the projection π: $X \to \mathbf{Z}_p$). For example, if χ is of the form $\chi_0 < >^s$, then

$$(3.2)\quad f_\chi(x) = f_{\chi_0}(x, s) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{0\leqq j<dp^N} \chi_0(x+j) <x+j>^s.$$

If χ is locally constant but not the trivial character, then we define

$$(3.3)\quad f_\chi(x) = \lim_{N\to\infty} \frac{1}{dp^N} \sum_{0\leqq j<dp^N} \chi(x+j) \log_p(x+j).$$

Finally, if χ is the trivial character (i.e., the characteristic function of X^*), then we set

$$f_{\chi_{\text{triv}}}(x) = 1.$$

LEMMA. *In all cases the limit $f_\chi(x)$ exists and is locally analytic in x and t (or in s for χ of the form $\chi_0 < >^s$).*

The proof is routine (see [**1**] or [**5**], p. 48). For every χ we also have $f_\chi \in \mathscr{F}$, since $f_\chi(x+1) = f_\chi(x)$ if $x \in X - X^*$.

For trivial χ_0 the function $f_{\chi_0}(x, s)$ is essentially the p-adic partial zeta-function (see, e.g., [**9**], p. 148). On the other hand, for arbitrary χ_0 but $x = 0$ we obtain the Kubota-Leopoldt-Iwasawa p-adic Dirichlet L-function ([**7**], [**3**]).

Note that in the definition of $f_\chi(x)$, we can replace dp^N by mdp^N for any $m = 1, 2, \ldots$.

We easily compute that for $m \in X^*$ a positive integer and χ not locally constant,

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$$T_m f_\chi(x) = \lim_{N\to\infty} \frac{1}{mdp^N} \sum_{0\leqq j<dp^N} \sum_{h=0}^{m-1} \chi\left(\frac{x+h}{m} + j\right)$$

$$= \chi^{-1}(m) \lim_{N\to\infty} \frac{1}{mdp^N} \sum_{0\leqq j<mdp^N} \chi(x+j)$$

$$= \chi^{-1}(m) f_\chi(x).$$

By continuity,

$$T_a f_\chi = \chi^{-1}(a) f_\chi \quad \text{for all } a \in X^*.$$

A similar computation gives

$$T_a f_\chi = \chi^{-1}(a) f_\chi$$

for χ locally constant and nontrivial (here one uses the fact that the sum of $\chi(x+j)$ over $0 \leqq j < dp^N$ is zero for N large). Finally, we obviously have

$$T_a 1 = 1 = \chi_{\text{triv}}^{-1}(a) \cdot 1.$$

THEOREM 2. *The subspace of* $\mathcal{F} = \{$*continuous* $f: X \to \Omega_p$ *such that* $f(x+1) = f(x)$ *for* $x \in X - X^*\}$ *on which* T *acts by* χ^{-1} *is one-dimensional and is spanned by* f_χ.

Proof. We just saw that $T_a f = \chi^{-1}(a) f$, $a \in X^*$. We must show uniqueness. In the process we shall see how to arrive at the definition of f_χ starting from the distribution identity.

Let $f: X \to \Omega_p$ be any continuous function such that $T_a f = \chi^{-1}(a) f$ and

(3.4) $f(x+1) = f(x)$ for $x \in X - X^*$.

Define a continuous function $g: X \to \Omega_p$ by setting

(3.5) $g(x) = x\chi^{-1}(x)(f(x+1) - f(x))$, $x \in X$

(characters $\chi \in \check{X}^*$ are always assumed to extend by 0 to $X - X^*$). Then for $m \in X^*$ a positive integer,

$$g(x) = x\chi^{-1}(x)(\chi(m)T_m f(x+1) - \chi(m)T_m f(x))$$

$$= \frac{x}{m}\chi^{-1}\left(\frac{x}{m}\right)\left(\sum_{h=0}^{m-1} f\left(\frac{x+1+h}{m}\right) - \sum_{h=0}^{m-1} f\left(\frac{x+h}{m}\right)\right)$$

$$= g(x/m).$$

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Since the m are dense in X^*, this means that $g(x)$ is constant on X^*. By (3.4), $g(x) = 0$ on $X - X^*$. Then by (3.5) we have

$$(3.6)\quad f(x+1) - f(x) = \text{const}\,\frac{\chi(x)}{x} \quad \text{for } x \in X.$$

First suppose that χ is not locally constant. If χ corresponds to the pair (χ_0, t), i.e.,

$$\chi(x) = \chi_0(x)\, t^{\log_p \langle x\rangle/\log_p \gamma},$$

then taking d/dx gives ($x \in X^*$)

$$\begin{aligned}
\chi'(x) &= \lim_{\epsilon\to 0} \chi_0(x)\,\frac{1}{\epsilon}\left(t^{\log_p\langle x+\epsilon\rangle/\log_p\gamma} - t^{\log_p\langle x\rangle/\log_p\gamma}\right)\\
&= \chi_0(x)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\, t^{\log_p\langle x\rangle/\log_p\gamma}\left(t^{\log_p(1+\epsilon/x)/\log_p\gamma} - 1\right)\\
&= \chi(x)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(t^{\epsilon/x\log_p\gamma} - 1\right)\\
&= \frac{\log_p t}{\log_p\gamma}\,\frac{\chi(x)}{x}.
\end{aligned}$$

Since t is not a root of unity, $\log_p t \neq 0$. Hence, multiplying f by a suitable constant, we may assume that

$$(3.7)\quad f(x+1) - f(x) = \chi'(x).$$

Now in [**1**] Diamond shows how to construct a function f with this property, namely, one gets precisely the function f_χ in (3.1). f is uniquely determined by (3.7) up to an additive constant, because

$$f(m) = f(0) + \sum_{0\leqq i<m} f(i+1) - f(i),$$

and the positive integers are dense in X. But if we had $f(x) = f_\chi(x) + C$, then since both f and f_χ transform by χ^{-1} under T, so would C. But $T_a C = C$; hence, $C = 0$.

Now suppose that χ is locally constant and nontrivial. By (3.6), we have

$$f(x+1) - f(x) = \frac{d}{dx}\,\text{const}\,\chi(x)\log_p x.$$

Then the same argument as before shows us that f is a constant multiple of (3.3). Finally, if $\chi = \chi_{\text{triv}}$, then we similarly find that f is of the form

$$a + b \lim_{N \to \infty} \frac{1}{dp^N} \sum_{\substack{0 \leqq j < dp^N \\ x + j \in X^*}} \log_p (x + j),$$

where a and b are constants. Constant functions transform by χ_{triv} under T, but the second function does not; hence, $b = 0$.

This concludes the proof of Theorem 2.

Remarks. 1. If one removes the condition (3.4), one can obtain more eigen-functions of the form (3.1) by extending χ in different ways (not necessarily by 0) to the multiplicative cosets of X^* in $X - X^*$. Namely, if $aX^* \subset X - X^*$ is such a coset and c is a constant, one can extend χ to aX^* by setting

$$\chi(ax) = c\chi(x) \quad \text{for } x \in X^*.$$

Then (3.1) still is a χ^{-1}-eigen-function of T.

2. To check that a function is a χ^{-1}-eigen-function it suffices to verify the identity

$$f(x) = \chi(m)\, T_m f(x)$$

for a set of integers m which generate a dense subset of X^*, i.e., it suffices if the m generate $(\mathbf{Z}/dp\mathbf{Z})^*$.

4. Examples. 1. The p-adic gamma-function Γ_p: $\mathbf{Z}_p \to \mathbf{Z}_p$ (see [**13**], or [**5**], p. 40-42) satisfies the Gauss multiplication formula ([**5**], p. 42):

$$(4.1) \quad \Gamma_p(x) = \frac{\prod_{h=0}^{m-1} \Gamma_p\left(\frac{x+h}{m}\right)}{\prod_{h=1}^{m-1} \Gamma_p\left(\frac{h}{m}\right)} m^{x_0 - 1} (m^{(p-1)})^{x_1},$$

where x is written as $x = x_0 + px_1$, $x \in \{1, 2, \ldots, p\}$. Taking the logarithm and then the second derivative gives (here we denote $\psi_p(x) = d/dx \log_p \Gamma_p(x)$)

$$\psi_p'(x) = \frac{1}{m} T_m \psi_p'(x).$$

If we apply Theorem 2 with $X = \mathbf{Z}_p$ and $\chi(x) = 1/x$, we find that

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$\psi_p'(x)$ is a multiple of

$$\lim_{N\to\infty} p^{-N} \sum_{\substack{0\leqq j<p^N \\ p\nmid x+j}} \frac{1}{x+j}.$$

In fact, Γ_p is related to Diamond's function ([**1**])

$$(4.2)\quad G_p(x) = \lim_{N\to\infty} p^{-N} \sum_{0\leqq j<p^N} (x+j)(\log_p(x+j)-1), \quad x \in \mathbf{Z}_p,$$

by the equality (see [**5**], p. 49)

$$(4.3)\quad \log_p\Gamma_p(x) = \sum_{\substack{0\leqq i<p, \\ p\nmid x+i}} G_p\left(\frac{x+i}{p}\right),$$

which, if we differentiate twice, gives

$$\psi_p'(x) = \frac{1}{p^2} \lim_{N\to\infty} p^{-N} \sum_{\substack{i+pj=0 \\ p\nmid x+i}}^{p^{N+1}-1} \frac{1}{\frac{x+i}{p}+j}$$

$$= \lim_{N\to\infty} p^{-N} \sum_{\substack{0\leqq j<p^N \\ p\nmid x+j}} \frac{1}{x+j}.$$

2. If χ_0 is a nontrivial character of $(\mathbf{Z}/d_0\mathbf{Z})^*$, one can define the twisted ψ_p-function on $X = \varprojlim (\mathbf{Z}/d_0 p^N\mathbf{Z})$ as

$$\psi_{p,\chi_0}(x) = \frac{d}{dx} \sum_{0\leqq i<d_0} \chi_0(x+i) \log_p\Gamma_p\left(\frac{X+i}{d_0}\right)$$

$$= \frac{1}{d_0} \sum_{0\leqq i<d_0} \chi_0(x+i)\, \psi_p\left(\frac{x+i}{d_0}\right).$$

(Recall that a function on $\mathbf{Z}_p$, such as Γ_p, is regarded as a function on X via the projection π: $X \to \mathbf{Z}_p$.) We have

$$T_m\, \psi_{p,\chi_0}(x) = \frac{d}{dx} \sum_{\substack{0\leqq i<d_0 \\ 0\leqq h<m}} \chi_0\left(\frac{x+h+im}{m}\right) \log_p\Gamma_p\left(\frac{x+h+im}{md_0}\right)$$

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$$= \chi_0^{-1}(m) \frac{d}{dx} \sum_{0 \leqq j < d_0} \chi_0(x + j) \sum_{0 \leqq k < m} \log_p \Gamma_p \left(\frac{x + j}{md_0} + \frac{k}{m} \right)$$

(letting $h + im = j + kd_0$)

$$= \chi_0^{-1}(m) \frac{d}{dx} \sum_{0 \leqq j < d_0} \chi_0(x + j) \log_p \Gamma_p \left(\frac{x + j}{d_0} \right)$$

(using (4.1) together with the vanishing of $\Sigma \chi_0(i)$ for χ_0 nontrivial)

$$= \chi_0^{-1}(m) \psi_{p,\chi_0}(x).$$

Hence, ψ_{p,χ_0} is in the χ_0^{-1}-eigen-space of T. And in fact, using (4.2) and (4.3), we see that (regarding χ_0 modulo d, where $d = d_0 p$ or $4d_0$, so that now $\chi_0(p) = 0$):

$$\psi_{p,\chi_0}(x) = \frac{d}{dx} \sum_{0 \leqq i < d} \chi_0(x + i) G_p\left(\frac{x + i}{d} \right)$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leqq j < dp^N} \chi_0(x + j) \log_p (x + j).$$

3. The successive derivatives of these two examples give further eigen-functions of T. For example, for $k = 2, 3, \ldots$

$$\psi_{p,\chi_0}^{(k-1)}(x) = \frac{d^k}{dx^k} \sum_{0 \leqq i < d_0} \chi_0(x + i) \log_p \Gamma_p \left(\frac{x + i}{d_0} \right)$$

$$= (-1)^k (k - 2)! \lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leqq j < dp^N} \frac{\chi_0(x + j)}{(x + j)^{k-1}}$$

$$(4.4) \qquad = (-1)^k (k - 2)! f_{\chi_0 \omega^{1-k}}(x, 1 - k)$$

in the notation of (3.2).

4. If we evaluate the function $f_{\chi_0}(x, s)$ in (3.2) at $x = 0$, we obtain the Kubota-Leopoldt p-adic L-function $-sL_p(1-s, \chi_0)$ (see §3 of [**3**], or [**5**], p. 47-48). Setting $s = 1 - k$ and comparing with (4.4), we find that

$$L_p(k, \chi_0 \omega^{1-k}) = \frac{1}{k - 1} f_{\chi_0 \omega^{1-k}}(0, 1 - k)$$

$$= \frac{(-1)^k}{(k - 1)!} \psi_{p,\chi_0}^{(k-1)}(0),$$

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which is a familiar expression ([**2**], [**4**]) for the value of L-functions at positive integers.

Question. Is there a simple characterization of functions in $\mathscr{F}$ which can be expressed as sums or integrals of f_χ? For example, let μ be a measure on a compact subset of $\check{X}^*$, such as $\chi \cdot < >^{\mathbf{Z}_p}$ for a fixed χ. That is, μ is a bounded, finitely additive Ω_p-valued function on compact-open subsets. Then consider the function

$$g(x) = \int f_{\chi \cdot < >^s}(x)\, d\mu(s).$$

The distribution operators T_a, $a \in X^*$, act as follows:

$$T_a g(x) = \chi^{-1}(a) \int <a>^{-s} f_{\chi \cdot < >^s}(x)\, d\mu(s).$$

Alternately, since $f_{\chi \cdot < >^s}(x)$ is locally analytic in s, we can relax the requirement that μ be bounded, and consider Manin-Višik's boundedly increasing [**10**] and h-admissible [**14**] measures, i.e., μ for which

$$p^{-hN} |\mu(a + p^N \mathbf{Z}_p)|_p \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Perhaps, the properties of the transform $g(x)$ can be characterized in terms of μ and the compact domain of integration in $\check{X}^*$, as Višik has done for the p-adic Stieltjes transform (see [**15**], or Appendix to [**5**]).

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University of Washington,
Seattle, Washington