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The box and Hausdorff dimension of self-affine sets

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Abstract Under a natural assumption the Hausdorff dimension of a measure μ canonically associated with a given self-affine set is computed A simplified proof of Bowen's formula for the box dimension of self-affine sets proved earlier is given A condition for the box dimension and Hausdorff dimension to be equal is proven, and a collection of examples in which this condition can be checked is discussed

1 *Introduction*

In this paper we consider the box dimension (or 'capacity') and Hausdorff dimension of certain self-affine sets which include those studied in [**Be2**] and in § 6 of [**PU**] Bedford [**Be2**] calculated the box dimension of some self-affine connected curves and obtained a formula involving the topological pressure of a certain function This formula is analogous to that of Bowen [**Bo2**] for the Hausdorff dimension of self-similar sets (see also Manning and McCluskey's formula [**MM**] in a slightly different context, and also [**Be1**] for the connection with box dimension) Here we give a simplified proof of the formula for the box dimension of self-affine sets Our approach is based only on the theory of Gibbs states presented in [**Bo1**] and does not involve the more advanced thermodynamic formalism of [**R**] and the 'singularity spectrum' results of [**BR**]

The main part of this paper is stimulated by a conjecture made in the preprint version of [**Be2**] The box dimension formula obtained in [**Be2**] states that the box dimension of a self-affine curve $E\subset\mathbb{R}^2$ is given by the unique real $t+1$ such that the topological pressure $P(tf_W+f_H)=0$ where f_W and f_H are functions measuring the scaling structure of E If μ is the equilibrium state for tf_W+f_H and $\pi_*\mu$ an associated measure on E then we can state the conjecture from [**Be2**] as follows the Hausdorff dimension of E is equal to the box dimension of E if and only if $HD(p_H\pi_*\mu)=1$, where p_H $\mathbb{R}^2\to\mathbb{R}$ is orthogonal projection onto the y-axis For certain self-affine sets which generalize the so-called limit Radamacher functions considered in [**PU**] we calculate the Hausdorff dimension of the measure $\pi_*\mu$ The formula obtained permits (for this class of sets) to give a positive answer for one direction of this conjecture and some positive partial contributions to the other direction In the last section of this paper we describe two classes of self-affine sets for which we are able to verify the assumptions we make to obtain our results

We now recall some of the general notions and results used in this paper

If A is a subset of a metric space (X, ρ) then the box dimension of A is defined as follows Let $N(A, \varepsilon)$ denote the minimum number of balls of radius $\varepsilon > 0$ needed to cover A We set

$$\underline{D}_B(A) = \liminf_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

$$\bar{D}_B(A) = \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon}$$

If $\underline{D}_B(A) = \bar{D}_B(A)$ then the common value is called the box dimension of A and is denoted by $D_B(A)$ For the definition and basic properties of Hausdorff dimension, which we denote by HD, we refer the reader to the book by Falconer [**Fa**] The Hausdorff dimension and box dimension are related by

$$HD(A) \le \underline{D}_B(A) \le \bar{D}_B(A)$$

If μ is a Borel probability measure on X then the Hausdorff dimension of μ is defined as

$$HD(\mu) = \inf\{HD(Y) \;\; Y \subset X \text{ and } \mu(Y) = 1\}$$

In order to estimate the Hausdorff dimension of a Borel probability measure on a Euclidean space we shall rely on the following well known result (see [**PU**])

FROSTMAN'S LEMMA *If for μ-a e x we have*

$$\delta_1 \le \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon} \le \delta_2,$$

where $B(x, \varepsilon)$ is the ball of radius ε around x, then $\delta_1 \le HD(\mu) \le \delta_2$

If $f, g \;\; X \to \mathbb{R}$ are two real-valued functions then we shall say that f and g are boundedly equivalent and write $f \asymp g$ if there is a constant $C \ge 1$ such that

$$C^{-1} g(x) \leqslant f(x) \leqslant Cg(x) \quad \text{for every } x \in X \tag{1 1}$$

Note that $\asymp$ is an equivalence relation and that $af \asymp f$ for every $a > 0$

2 *The construction of self-affine sets*

In this section we recall some of the facts and definitions from [**Be2**] that we need here They are formulated in a slightly more general setting which enables us to deal with disconnected self-affine sets as well At the end of this section we present another proof of the Bowen dimension formula for self-affine sets first proved in [**Be2**] Note that our notation does not coincide everywhere with that of [**Be2**]

We put $I = [0, 1]$ and consider orientation preserving, contractive $C^{1+\varepsilon}$ diffeomorphisms $\varphi_i \;\; I \times \mathbb{R} \to I \times \mathbb{R}$ (for $0 \le i < k$) satisfying the following properties

$\varphi_i(x, y) = (\psi_i(x), \tau_i(x, y)), 0 \le i < k$ for some differentiable functions $\psi_i \;\; I \to I$

and τ_ι $I\times\mathbb{R}\to\mathbb{R}$ such that

$$A=\sup\{\psi_\iota'(x)\quad x\in I, 0\le\iota<k\}<1$$

and

$$B=\sup\left\{\frac{\partial}{\partial y}\tau_\iota(x,y)\quad (x,y)\in I\times\mathbb{R}, 0\le\iota<k\right\}<1, \tag{2 1}$$

the fixed points of φ_0 and φ_{k-1} are $(0, y_0)$ and $(1, y_{k-1})$ respectively for some $y_0, y_{k-1}\in\mathbb{R}$, (2 2)

and

$$\psi_\iota(1)=\psi_{\iota+1}(0)\quad \text{for every } 0\le\iota<k-1 \tag{2 3}$$

We also define a and b by

$$a=\inf\{\psi_\iota'(x)\quad x\in I, 0\le\iota<k\}>0$$

and

$$b=\inf\left\{\frac{\partial}{\partial y}\tau_\iota(x,y)\quad (x,y)\in I\times\mathbb{R}, 0\le\iota<k\right\}>0$$

The constants a, b, A and B will be used as defined here throughout the paper

By a result of Hutchinson [**H**] there is a unique compact non-empty set E such that $E=\bigcup_{\iota=0}^{k-1}\varphi_\iota(E)$ Following [**Be2**] we shall call such sets self-affine Let $\Sigma=\prod_1^\infty\{0,\quad, k-1\}$ For $\underline{x}=(x_1,x_2,\quad)\in\Sigma$ we set $\underline{x}(n)=(x_1,\quad,x_n)$, $(n\ge1)$, and write

$$\varphi_{\underline{x}(n)}=\varphi_{x_1}\circ\quad\circ\varphi_{x_n},\quad \psi_{\underline{x}(n)}=\psi_{x_1}\circ\quad\circ\psi_{x_n}$$

and

$$\pi(\underline{x})=\bigcap_{n=1}^\infty\varphi_{\underline{x}(n)}(E)\quad \tilde\pi(\underline{x})=\bigcap_{n=1}^\infty\psi_{\underline{x}(n)}(E)$$

We shall often identify the finite sequence $\underline{x}(n)$ with the subset $\{\underline{y}\in\Sigma\quad \underline{y}(n)=\underline{x}(n)\}\subset\Sigma$

Since φ and ψ are strict contractions and the families

$$\{\varphi_{\underline{x}(n)}(E)\}_{n=1}^\infty,\quad \{\psi_{\underline{x}(n)}(I)\}_{n=1}^\infty$$

are decreasing in n, the sets $\pi(\underline{x})$ and $\tilde\pi(\underline{x})$ are singletons Furthermore $\pi(\underline{x})\in E$ since $\varphi_{\underline{x}(n)}(E)\subset E$ for all $n\ge1$ We have thus constructed maps

$$\pi\quad \Sigma\to E\quad\text{and}\quad \tilde\pi\quad \Sigma\to I,$$

which are continuous since $\operatorname{diam}(\varphi_{\underline{x}(n)}(E))$ and $\operatorname{diam}(\psi_{\underline{x}(n)}(I))$ converge to 0 uniformly (in fact exponentially) fast The two maps are surjective because $E=\bigcup_{\iota=0}^{k-1}\varphi_\iota(E)$ and $I=\bigcup_{\iota=0}^{k-1}\psi_\iota(I)$ The following properties are also easy to see

$$\pi(\iota,x_1,x_2,\quad)=\varphi_\iota(x_1,x_2,\quad)\quad\text{and}\quad\tilde\pi(\iota,x_1,x_2,\quad)=\psi_\iota(x_1,x_2,\quad)$$

for any $\underline{x}\in\Sigma$ (2 4)

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π and $\tilde{\pi}$ are injective except that given $x_1, \quad , x_n$ with $x_n \neq k-1$ the two sequences $(x_1, \quad , x_n, k-1, k-1, \quad)$ and $(x_1, \quad , x_n, 0, 0, \quad)$ have the same image under $\tilde{\pi}$ and may have the same image under π (2 5)

An easy consequence of (2 4) is

$$\varphi_{\underline{x}(n)}(E) = \pi(\underline{x}(n)) \quad \text{and} \quad \psi_{\underline{x}(n)}(I) = \tilde{\pi}(\underline{x}(n)) \tag{2 6}$$

Finally, we define $p_W \; I\times\mathbb{R}\to I$ and $p_H \; I\times\mathbb{R}\to\mathbb{R}$ to be the standard projection maps $(x, y)\mapsto x$ and $(x, y)\mapsto y$ respectively For a subset D of $\mathbb{R}$ we denote the diameter by $|D|$ and the Lebesgue measure by $l(D)$ For $K\subset\mathbb{R}^2$ we put $|K|_W = |p_W(K)|$, $|K|_H = |p_H(K)|$, $l_W(K) = l(p_W(K))$ and $l_H(K) = l(p_H(K))$ (W and H stand for 'width' and 'height' respectively)

As $p_W\varphi_i = \psi_i p_W$, for $0\le i<k$, we obtain

$$p_W\circ\pi = \tilde{\pi} \tag{2 7}$$

The following lemma says that E is almost the graph of a function

Lemma 1 *For all $x\in I$ outside a countable set $Z\subset I$, each line $\{x\}\times\mathbb{R}$ contains exactly one point of E*

Proof Let $X_0 = \{\underline{x}\in\Sigma \; x_n = 0$ for all large $n\}$, $X_{k-1} = \{\underline{x}\in\Sigma \; x_n = k-1$ for all large $n\}$, and let $Z = \tilde{\pi}(X_0\cup X_{k-1})$ Now Z is countable since X_0 and X_{k-1} are countable By (2 5), card $\tilde{\pi}^{-1}(x) = 1$ for every $x\in I\backslash Z$, and so (2 7) implies that card $(E\cap\{x\}\times\mathbb{R}) = 1$ This proves the lemma □

Condition (2 1) implies that

$$D\varphi_i(x, y) = \begin{pmatrix} \psi_i'(x) & 0 \\ \frac{\partial}{\partial x}\tau_i(x, y) & \frac{\partial}{\partial y}\tau_i(x, y) \end{pmatrix} \quad 0\le i<k$$

From now on we shall assume that

$$\lambda = \sup\left\{\frac{\psi_i'(x)}{\frac{\partial}{\partial y}\tau_i(x, y)} \quad (x, y)\in I\times\mathbb{R}, 0\le i<k \quad <1,\right\} \tag{2 8}$$

which means that the maps φ_i have a sharper contraction horizontally than vertically We define two functions which together measure the contraction rates of the φ_i Let $f_W, f_H \; \Sigma\to\mathbb{R}$ be given by

$$f_W(\underline{x}) = \log\psi_{x_1}'(\tilde{\pi}(\sigma x)) \quad f_H(\underline{x}) = \log\frac{\partial}{\partial y}\tau_{x_1}(\pi(\sigma x))$$

(note that these functions are *minus* the corresponding functions in [**Be2**], but that they have the same sign as the corresponding function in [**Bo2**] and [**MM**]) As the functions σ, π, $\tilde{\pi}$, $(\partial/\partial y)\tau_{x_1}$ and ψ_{x_1}' are continuous, both f_W and f_H are continuous It is not difficult to check (see [**Be2**]) that they are actually Holder continuous For $f = f_W$ or f_H and $n\ge 1$ we denote the sum $\sum_{i=0}^{n-1} f(\sigma^i\underline{x})$ by $S_n f(\underline{x})$

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We can reformulate Lemma 3, Lemma 5 and Proposition 8 of [**Be2**] as follows

LEMMA 2 *For every* $n \geq 1$ *and* $\underline{x}, \underline{y} \in \Sigma$ *with* $\underline{x}(n) = \underline{y}(n)$ *we have*

$$\exp S_n f_W(\underline{x}) \simeq \exp S_n f_W(\underline{y}) \quad \text{and} \quad \exp S_n f_H(\underline{x}) \simeq \exp S_n f_H(\underline{y})$$

LEMMA 3 *For every* $n \geq 1$ *and* $\underline{x} \in \Sigma$,

$$|\tilde{\pi}(\underline{x}(n))| = |\psi_{\underline{x}(n)}(I)| = |\varphi_{\underline{x}(n)}(E)|_W \simeq \exp S_n f_W(\underline{x})$$

LEMMA 4 *For every* $n \geq 1$ *and* $\underline{x} \in \Sigma$,

$$|\pi(\underline{x}(n))|_H = |\varphi_{\underline{x}(n)}(E)|_H \simeq \exp S_n f_H(\underline{x})$$

An extra condition is needed for Lemma 4 to hold as stated This is because there is a degenerate case in which E is a differentiable manifold However, this happens if and only if the strong stable manifolds of the maps φ_i all coincide (see [**Be**$\geq$**2**] for more details) and we shall assume that this highly non-generic possibility does not occur We remark also that the proof of (our) Lemma 4 given in [**Be2**] required the existence of sets $C \subset E$ with $|C|_H/|C|_W$ arbitrarily large This is true in the disconnected case since for some i we have $\varphi_{i+1}((0, y_0)) \neq \varphi_i((0, y_{k-1}))$ (using the notation of (2 2))—for otherwise [**Be2**] shows that E is connected Hence the set $C = \{\varphi_{i+1}((0, y_0)), \varphi_i((0, y_{k-1}))\}$ has $|C|_H > 0$, but $|C|_W = 0$ by (2 3) With this remark, the proofs of the above lemmas work in exactly the same way as those in [**Be2**]

The main technical tools we use in this paper are the notions of topological pressure and Gibbs states Topological pressure with respect to $\sigma\ \Sigma \to \Sigma$ is an operator on the space of real-valued continuous function on Σ It satisfies a variational principle relating it to measure theoretic entropy,

$$P(g) = \sup \left\{ h_\mu(\sigma) + \int g \, d\mu \right\} \quad g \in C(\Sigma),$$

where the supremum is taken over σ-invariant Borel probability measures A measure taking the supremum is called an equilibrium state for g When g is Holder continuous there is a unique equilibrium state μ for g, which is a Gibbs measure This means that for all $\underline{x} \in \Sigma$ and $n \geq 0$ we have

$$\mu(\underline{x}(n)) \simeq \exp\{S_n g(\underline{x}) - nP(g)\} \tag{2 9}$$

(see [**Bo1**], 1 4 pp 9-10) More information about pressure and Gibbs states can be found in [**Bo1**] The formula for box dimension of E established in [**Be2**] involves the zero of the function $s \mapsto P(sf_W + f_H)$ Our assumptions that $0 < a, b, A, B < 1$ imply that there is a unique $t \in \mathbb{R}$ with $P(tf_W + f_H) = 0$ Furthermore we have the following bound on the value of t

LEMMA 5 *The unique real number t defined by* $P(tf_W + f_H) = 0$ *satisfies* $0 < t < 1$

Proof We first show that $t > 0$ Let λ be the equilibrium state for the function f_W By (2 9) and Lemma 3 we have

$$\lambda(\underline{x}(n)) \sim \exp(S_n f_W(\underline{x}) - nP(f_W)) \sim |\tilde{\pi}(\underline{x}(n))| \exp(-nP(f_W))$$

for every $n \geq 1$ and $\underline{x} \in \Sigma$ Therefore

$$1 = \sum_{\underline{x}(n)} \lambda(\underline{x}(n)) \sim \sum_{\underline{x}(n)} |\tilde{\pi}(\underline{x}(n))| \exp(-nP(f_W)) = \exp(-nP(f_W))$$

for every $n \geq 1$, which implies that $P(f_W) = 0$ By (2 8) and (2 1) we have $f_W < f_H < 0$, which gives $0 = P(f_W) < P(f_H)$ As the function $s \mapsto P(sf_W + f_H)$ is strictly decreasing we obtain $t > 0$

To see that $t < 1$, let ν be the equilibrium state for the function $f_W + f_H$ As $P(f_W) = 0$, the variational principle for pressure implies that

$$\begin{aligned} P(f_W + f_H) &= h_\nu \sigma + \int f_W \, d\nu + \int f_H \, d\nu \\ &\leq P(f_W) + \int f_H \, d\nu \\ &= \int f_H \, d\nu < 0 \end{aligned}$$

This together with strict monotonicity of $s \mapsto P(sf_W + f_H)$ shows that $t < 1$ □

DEFINITION *We say that E satisfies the Darboux property if the set* $p_H(\varphi_{\underline{x}(n)}(E))$ *is connected for every* $n \geq 1$ *and* $\underline{x} \in \Sigma$

We end this section with another proof of the Bowen formula established in [**Be2**]

THEOREM 6 *If E satisfies the Darboux property then* $D_B(E) = t + 1$

Proof Let μ be the equilibrium state for the function $tf_W + f_H$, and fix $\varepsilon > 0$ Since $0 < a \leq A < 1$ we can find a finite number of points $\underline{x}^1, \ldots, \underline{x}^p \in \Sigma$ and integers $n_1, \ldots, n_p \geq 1$ such that

$$\bigcup_{j=1}^{p} \tilde{\pi}(\underline{x}^j(n_j)) = I,$$

$$\operatorname{int}(\tilde{\pi}(\underline{x}^j(n_j))) \cap \operatorname{int}(\tilde{\pi}(\underline{x}^i(n_i))) = \varnothing \quad \text{for } 1 \leq i \neq j \leq p,$$

and

$$\varepsilon a \leq |\tilde{\pi}(\underline{x}^j(n_j))| \leq \varepsilon \quad \text{for } 1 \leq j \leq p$$

For each j define $q_j = [|\pi(\underline{x}^j(n_j))|_H \cdot |\tilde{\pi}(\underline{x}^j(n_j))|^{-1}] + 1$ Clearly we can find $u_1, \ldots, u_{q_j} \in \mathbb{R}$ (depending on j) such that $\tilde{\pi}(\underline{x}^j(n_j)) \times [u_1, u_{q_j}] \supset \pi(\underline{x}^j(n_j))$ and $u_{i+1} - u_i = |\tilde{\pi}(\underline{x}^j(n_j))|$ for $1 \leq i < q_j$ By Lemmas 3 and 4 we have

$$\begin{aligned} q_j &\asymp |\pi(\underline{x}^j(n_j))|_H \; |\tilde{\pi}(\underline{x}^j(n_j))|^{-1} \\ &\asymp \exp(S_{n_j} f_H(\underline{x}^j)) \; \exp(-S_{n_j} f_W(\underline{x}^j)) \\ &= \exp(S_{n_j}(f_H - f_W)(\underline{x}^j)) \end{aligned}$$

Clearly $N(E, \varepsilon) \leq \sum_{j=1}^{p} q_j$ Furthermore as E satisfies the Darboux property and any ball of radius $\frac{1}{2}\alpha\varepsilon$ can intersect at most 4 rectangles of the form $\tilde{\pi}(\underline{x}^j(n_j)) \times [u_i, u_{i+1}]$, we must have $N(E, \frac{1}{2}a\varepsilon) \geq \frac{1}{4} \sum_{j=1}^{p} q_j$

Since μ is the equilibrium state for $tf_W + f_H$ where $P(tf_W + f_H) = 0$, it follows from (2 9) that for any $\varepsilon > 0$

$$
\begin{aligned}
N(E,\varepsilon) &\simeq \sum_{j=1}^{p} q_j \simeq \sum_{j=1}^{p} \exp\left(S_{n_j}(f_H - f_W)(\underline{x}^j)\right) \\
&= \sum_{j=1}^{p} \exp\left(S_{n_j}(tf_W + f_H)(\underline{x}^j)\right) \exp\left(-S_{n_j}(1+t) f_W(\underline{x}^j)\right) \\
&\simeq \sum_{j=1}^{p} \mu(\underline{x}^j(n_j)) |\tilde{\pi}(\underline{x}^j(n_j))|^{-(1+t)} \\
&\simeq \sum_{j=1}^{p} \tilde{\pi}_* \mu(\tilde{\pi}(\underline{x}^j(n_j))) \varepsilon^{-(1+t)} = \varepsilon^{-(1+t)}
\end{aligned}
$$

Therefore $D_B(E) = t+1$ and the theorem is proved □

REMARK (i) *The Darboux property was not used to obtain the estimate*

$$N(E, \varepsilon) \leq \sum_{j=1}^{p} q_j$$

and so the inequality $D_B(E) \leq t+1$ is true for any self-affine set E

(ii) *The proof of $D_B(E) = t+1$ given in* **[Be2]** *establishes the result via a variational principle This principle also contains information about the number of boxes required to cover certain subsets of E*

3 *The dimension of E and $\pi_*\mu$*

From now on we make the additional assumption that

$$\frac{\partial}{\partial x} \tau_i(x, y) \equiv 0 \quad \text{for } 0 \leq i < k \qquad (*)$$

and will say that E has an invariant weak foliation We are aiming towards conditions under which the Hausdorff dimension of E is equal to $t+1$, and shall do this via a study of the Hausdorff dimension of $\pi_*\mu$ If E has an invariant weak foliation then each φ_i $I \times \mathbb{R} \to I \times \mathbb{R}$ can be expressed in the form

$$\varphi_i(x, y) = (\psi_i(x), \tau_i(y)) \quad \text{where } \tau_i \ \mathbb{R} \to \mathbb{R}$$

Thus horizontal lines are mapped under φ_i to horizontal lines and so if $K \subset \varphi_i(I \times \mathbb{R})$ is a rectangle with sides parallel to the x and y-axes then $\varphi_i^{-1}(K)$ is also such a rectangle—in future all rectangles used in the proofs will be of this form It follows from Lemma 4 that if $\underline{x} \in \Sigma$, $n \geq 1$ and $K \subset I \times \mathbb{R}$ has the property that $p_H(K) \subset p_H \varphi_{\underline{x}(n)}(E)$ then

$$|\varphi_{\underline{x}(n)}^{-1}(K)|_H \simeq |K|_H \exp\left(-S_n f_H(\underline{x})\right)$$

In particular we obtain the following

LEMMA 7 *Let*

$$
\begin{aligned}
K_+(\underline{x}, n) &= \tilde{\pi}(\underline{x}(n)) \times [p_H \pi(\underline{x}), p_H \pi(\underline{x}) + \tfrac{1}{2} |\tilde{\pi}(\underline{x}(n))|], \\
K_-(\underline{x}, n) &= \tilde{\pi}(\underline{x}(n)) \times [p_H \pi(\underline{x}) - \tfrac{1}{2} |\tilde{\pi}(\underline{x}(n))|, p_H \pi(\underline{x})],
\end{aligned}
$$

and $K(\underline{x}, n) = K_+(\underline{x}, n) \cup K_-(\underline{x}, n)$ *Then for any* $\underline{x} \in \Sigma$ *and* $n \geq 1$ *we have*

$$|\varphi^{-1}_{\underline{x}(n)}(K_+(\underline{x}, n))|_H \asymp |\varphi^{-1}_{\underline{x}(n)}(K_-(\underline{x}, n))|_H \asymp |\varphi^{-1}_{\underline{x}(n)}(K(\underline{x}, n))|_H$$
$$\asymp |\tilde{\pi}(\underline{x}(n))| \exp(-S_n f_H(\underline{x}))$$

As before we let μ be the equilibrium state for the function $tf_W + f_H$, and we now take ν to be the Borel probability measure on Σ given by Ruelle's Perron-Frobenius theorem (see [**Bo1**] 1 7 p 14) for $tf_W + f_H$ The two measures μ and ν are equivalent with continuous never vanishing densities It follows from this theorem that for $K \subset \varphi_i(I \times \mathbb{R})$ $(0 \leq i < k)$ we have

$$\pi_* \nu(\varphi_i^{-1}(K)) = \int_K \exp(-tf_W - f_H)\, d(\pi_* \nu) \geq \pi_* \nu(K) \tag{3 2}$$

We consider π_*, ν here as a measure defined on $I \times \mathbb{R}$ with the property that $\pi_* \nu(I \times \mathbb{R}) \setminus E) = 0$ Therefore, using Lemma 2, we get the following,

LEMMA 8 *For any* $\underline{x} \in \Sigma$, $n \geq 1$ *and* $K \subset \varphi_{\underline{x}(n)}(I \times \mathbb{R})$ *we have*

$$\pi_* \nu(\varphi^{-1}_{\underline{x}(n)}(K)) = \int_K \exp(-S_n(tf_W + f_H))\, d(\pi_* \nu)$$
$$\asymp \pi_* \nu(K) \exp(-S_n(tf_W + f_H)(\underline{x}))$$

In particular, for any $\underline{x} \in \Sigma$ and $n \geq 1$,

$$\tilde{\pi}_* \nu(\psi_{\underline{x}(n)}(I)) = \pi_* \nu(\varphi_{\underline{x}(n)}(E)) \asymp \exp(S_n(tf_W + f_H)(\underline{x})) \tag{3 3}$$

We now obtain a volume lemma for the measure $\pi_* \nu$ (compare to Lemma 8 of [**PU**])

LEMMA 9 *For* ν *and* μ*-almost all* $\underline{x} \in \Sigma$ *we have*

$$\liminf_{n \to \infty} \frac{\log(\pi_* \nu(K(\underline{x}, n)))}{\log|\tilde{\pi}(\underline{x}(n))|} = HD(\pi_* \nu)$$

Proof Let

$$L(\underline{x}) = \liminf_{n \to \infty} \frac{\log(\pi_* \nu(K(\underline{x}, n)))}{\log|\tilde{\pi}(\underline{x}(n))|}$$

If $\underline{y} = \sigma(\underline{x})$ then $K(\underline{y}, n-1) \supset \varphi_{x_1}^{-1}(K(\underline{x}, n))$ and we get by (3 2) that $\pi_* \nu(K(\underline{y}, n-1)) \geq \pi_* \nu(K(\underline{x}, n))$ Hence

$$\frac{\log \pi_* \nu(K(\underline{y}, n-1))}{\log|\tilde{\pi}\underline{y}(n-1)|} \leq \frac{\log \pi_* \nu(K(\underline{x}, n))}{\log|\tilde{\pi}\underline{y}(n-1)|} \leq \frac{\log \pi_* \nu(K(\underline{x}, n))}{\log|\tilde{\pi}\underline{x}(n)| - \log a}$$

Letting $n \to \infty$ we get $L(\sigma\underline{x}) \leq L(\underline{x})$ As σ $\Sigma \to \Sigma$ is ergodic with respect to μ, this implies that $L(\underline{x})$ is constant μ-almost everywhere and equivalently ν-almost everywhere We denote this μ-almost sure value by L Since $B(\pi(\underline{x}), 2|\tilde{\pi}(\underline{x}(n))|) \supset K(\underline{x}, n)$, we obtain

$$\liminf_{n \to \infty} \frac{\log \pi_* \nu B(\pi(\underline{x}), 2|\tilde{\pi}(\underline{x}(n))|)}{\log|\tilde{\pi}(\underline{x}(n))|} \leq L(\underline{x})$$

The Frostman Lemma (stated in the introduction) now gives $HD(\pi, \nu) \leq L$

The calculation of the other inequality is more complicated The main technical problem we face in obtaining the lower bound on $HD(\pi_* \nu)$ is that we do not have a good description of the density of $\pi_*\nu$ around points of E with more than one corresponding symbol sequence We now make some estimates which will enable us to deal with this problem

First note that, since $P(tf_W+f_H)=0$, by the definition of pressure we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\underline{x}\in B_n}\exp\left(S_n(tf_W+f_H)(\underline{x})\right)=0$$

where for each $n\geq 1$, $B_n\subset\Sigma$ is a maximal set such that if $x, y\in B_n$ and $\underline{x}\neq\underline{y}$ then $\underline{x}(n)\neq\underline{y}(n)$ By (3 3) this is equivalent to the formula

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{\underline{x}(n)}\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}(n)))=0$$

Thus, given $\varepsilon>0$, for large enough n we have $\sum_{\underline{x}(n)}\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}(n)))\leq e^{\frac{1}{2}\varepsilon n}$ In particular

$$\sum_{n=1}^{\infty}\sum_{\underline{x}(n)}e^{-\varepsilon n}\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}(n)))<\infty \tag{3 4}$$

Now fix $\beta>0$, choose some $n\geq 1$ and define $m=1+[(1+\beta)n]$ Corresponding to each interval $\tilde{\pi}(\underline{x}(n))$ there are two endpoints x^1 and x^2 and two sequences such that $\tilde{\pi}(\underline{x}^1)=x^1$, $\tilde{\pi}(\underline{x}^2)=x^2$, and $\underline{x}^1(n)=\underline{x}^2(n)=\underline{x}(n)$ Put

$$A(\beta, n)=\bigcup_{\underline{x}(n)}\left(\tilde{\pi}(\underline{x}^1(m))\cup\tilde{\pi}(\underline{x}^2(m))\right)$$

We can now use (3 3) to estimate $\tilde{\pi}_*\nu(A(\beta, n))$,

$$\begin{aligned}\tilde{\pi}_*\nu(A(\beta, n))&=\sum_{\underline{x}(n)}\left(\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}^1(m)))+\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}^2(m)))\right)\\&\simeq\sum_{\underline{x}(n)}\left(\exp\left(S_m(tf_W+f_H)(\underline{x}^1)\right)+\exp\left(S_m(tf_W+f_H)(\underline{x}^2)\right)\right)\\&\leq\sum_{\underline{x}(n)}\left(\exp\left(S_n(tf_W+f_H)(\underline{x}^1)\right)\right.\\&\quad\left.+\exp\left(S_n(tf_W+f_H)(\underline{x}^2)\right)\right)\ \exp\left((t\log A+\log B)\beta n\right)\\&\simeq 2\exp\left((t\log A+\log B)\beta n\right)\sum_{\underline{x}(n)}\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}(n)))\end{aligned}$$

Using the fact that $\log A$, $\log B<0$, (3 4) implies that

$$\sum_{n=1}^{\infty}\tilde{\pi}_*\nu(A(\beta, n))<\infty \tag{3 5}$$

Now observe that by Lemma 3 we have

$$\begin{aligned}|\tilde{\pi}(\underline{x}^1(m))|&\simeq\exp\left(S_m f_W(\underline{x}^1)\right)\\&\geq\exp\left(S_n f_W(\underline{x}^1)\right)\exp\left(\beta n\log a\right)\\&\simeq\exp\left(\beta n\log a\right)|\tilde{\pi}(\underline{x}^1(n))|\end{aligned}$$

The same estimate is obviously true for $|\tilde{\pi}(\underline{x}^2(m))|$ Thus if $\underline{x}$ is not in $\tilde{\pi}^{-1}(A(\beta, n))$ then

$$K(\underline{x}, n)\supset B(\pi(\underline{x}), ca^{\beta n}|\tilde{\pi}(\underline{x}(n))|),$$

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where $0 < c \leq 1$ is a universal constant Now by (3 5) and the Borel–Cantelli lemma, ν-almost every $\underline{x} \in \Sigma$ does not belong to $\tilde{\pi}^{-1}(A(\beta, n))$ for all large n Therefore

$$L \leq \liminf_{n\to\infty} \frac{\log \tilde{\pi}_* \nu B(\pi(\underline{x}), ca^{\beta n}|\tilde{\pi}(\underline{x}(n))|)}{\log |\tilde{\pi}(\underline{x}(n))|}$$

$$= \liminf_{n\to\infty} \frac{\log c + \beta n \log a + \log |\tilde{\pi}(\underline{x}(n))|}{\log |\tilde{\pi}(\underline{x}(n))|} \frac{\log \tilde{\pi}_* \nu B(\pi(\underline{x}), ca^{\beta n}|\tilde{\pi}(\underline{x}(n))|)}{\log (ca^{\beta n}|\tilde{\pi}(\underline{x}(n))|)}$$

and as

$$1 \leq \frac{\log c + \beta n \log a + \log |\tilde{\pi}(\underline{x}(n))|}{\log |\tilde{\pi}(\underline{x}(n))|} \leq 1 + \beta \frac{\log a}{\log A} + \frac{1}{n} \frac{\log c}{\log A}$$

we deduce from Frostman's lemma that

$$L \leq \left(1 + \beta \frac{\log a}{\log A}\right) HD(\pi_* \nu)$$

Letting $\beta \to 0$ we get $L \leq HD(\pi_* \nu)$ which proves the lemma □

We shall obtain a formula for the dimension of $\pi_* \nu$ in terms of (amongst other things) the dimension of the measure $(p_H \pi)_* \nu$ The following two lemmas are volume lemmas for $(p_H \pi)_* \nu$

Lemma 10 *For μ and ν-almost every $x \in \Sigma$ we have*

$$\liminf_{r\to 0} \frac{\log (p_H \pi)_* \nu(B(p_H \pi(\underline{x}), r))}{\log r} = HD((p_H \pi)_* \nu)$$

Proof Denote the map $p_H \pi$ by π_H and the measure $(p_H \pi)_* \nu$ by $\bar{\nu}$ For every $\underline{x} \in \Sigma$ let

$$L(\underline{x}) = \liminf_{r\to 0} \frac{\log \bar{\nu}(B(\pi_H(\underline{x}), r))}{\log t}$$

By Frostman's lemma it is sufficient to show that $L(\underline{x})$ is ν-almost surely constant Now for every $n \geq 1$ and $\underline{x} \in \Sigma$ we have

$$\varphi_{x_1}^{-1}(\tilde{\pi}(\underline{x}(1)) \times B(\pi_H(\underline{x}), r)) \supset I \times B(\pi_H(\sigma \underline{x}), B^{-1} r) \supset I \times B(\pi_H(\sigma \underline{x}), r)$$

Thus, putting $D = \max (1, \sup \{\exp (-tf_W - f_H)(\underline{x}) \ \underline{x} \in \Sigma\})$ we obtain

$$\begin{aligned} \bar{\nu}(B(\pi_H(\sigma \underline{x}), r)) &= \pi_* \nu(I \times B(\pi_H(\sigma \underline{x}), r)) \\ &\leq \pi_* \nu(\varphi_{x_1}^{-1}(\tilde{\pi}(\underline{x}(1)) \times B(\pi_H(\underline{x}), r))) \\ &\leq D \pi_* \nu((\tilde{\pi}(\underline{x}(1)) \times B(\pi_H(\underline{x}), r))) \\ &\leq D \pi_* \nu(I \times B(\pi_H(\underline{x}), r)) \\ &= D \bar{\nu}(B(\pi_H(\underline{x}), r)) \end{aligned} \tag{3 6}$$

Consequently, for every $0 < r < 1$ we get

$$\frac{\log \bar{\nu}(B(\pi_H(\sigma \underline{x}), r))}{\log r} \geq \frac{\log \bar{\nu}(B(\pi_H(\underline{x}), r))}{\log r} + \frac{\log D}{\log r}$$

Thus, letting $r \to 0$, we obtain $L(\sigma \underline{x}) \geq L(\underline{x})$ Since

$$\int L(\sigma \underline{x}) \, d\mu(\underline{x}) = \int L(\underline{x}) \, d\mu(\underline{x}),$$

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this implies that $L(\sigma \underline{x}) = L(\underline{x})$ for μ-a e $x \in \Sigma$ Therefore, by ergodicity of μ with respect to σ, $L(\underline{x})$ is μ-a s constant and hence also ν-a s constant (as μ and ν are equivalent) □

The next lemma says that the limit around the point $p_H \pi \underline{x}$ in the statement of Lemma 10 can be replaced by a limit taken along a sequence of points $p_H \pi \sigma^n \underline{x}$

LEMMA 11 *If θ_n $\Sigma \to (0, \infty)$ is a sequence of measurable functions such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \theta_n(\underline{x}) = \theta$$

for μ a a $\underline{x} \in \Sigma$ and some $\theta < 0$, then

$$\liminf_{n\to\infty} \frac{\log (p_H\pi)_* \nu(B(p_H \pi(\sigma^n \underline{x}), \theta_n(\underline{x})))}{\log \theta_n(\underline{x})} = HD((p_H \pi)_* \nu),$$

for μ a a $\underline{x} \in \Sigma$

Proof Define π_H and $\bar{\nu}$ as in the last lemma For every $\underline{x} \in \Sigma$ let

$$L(\underline{x}) = \liminf_{n\to\infty} \frac{\log \bar{\nu}(B(\pi_H(\sigma^n \underline{x}), \theta^n(\underline{x})))}{\log \theta_n(\underline{x})}$$

First we show that

$$L(\underline{x}) \geq HD(\bar{\nu}) \quad \text{for } \mu \text{ a e } \underline{x} \in \Sigma \tag{3 7}$$

Let $Z = \{\underline{x} \in \Sigma \ \lim_{n\to\infty} n^{-1} \log \theta_n(\underline{x}) = \theta\}$ By assumption Z has μ-measure 1 From the last lemma and Egorov's Theorem it follows that for every $p < HD(\bar{\nu})$ there exists $r(p) > 0$ and a Borel set $S \subset \Sigma$ such that $\mu(S) > 0$ and

$$\frac{\log \bar{\nu}(B(\pi_H(\underline{x}), r))}{\log r} \geq p \tag{3 8}$$

for every $\underline{x} \in S$ and $0 < r \leq r(p)$ By ergodicity of μ, for μ-almost all $\underline{x} \in \Sigma$ there is an increasing sequence $\{n_j\}$ (depending on $\underline{x}$) such that $\sigma^{n_j}(\underline{x}) \in S$ for $j = 1, 2,$ Now take $j_0 \geq 1$ (again depending on $\underline{x}$) large enough that $\theta_n(\underline{x}) \leq r(p)$ for every $n \geq n_{j_0}$ Then by (3 8) we have

$$\frac{\log \bar{\nu}(B(\pi_H(\sigma^{n_j} \underline{x}), \theta_{n_j}(\underline{x})))}{\log \theta_{n_j}(\underline{x})} \geq p \tag{3 9}$$

for every $j \geq j_0$ Fix now $0 < \varepsilon < -\theta$ and let $l_{-1} \geq 1$ be so large that

$$\theta - \varepsilon \leq \frac{1}{m} \log \theta_m(\underline{x}) \leq \theta + \varepsilon \quad \text{for every } m \geq l_{-1} \tag{3 10}$$

Let $t(m) = [(\theta + \varepsilon)/(\theta - \varepsilon) m] \leq m$ and let $j(m)$ be the largest integer such that $n_{j(m)} \leq t(m)$ Since $\lim_{j\to\infty} n_j = \infty$, there exists $l_0 \geq l_{-1}$ so large that $n_{j(m)} \geq l_{-1}$ and $j(m) \geq j_0$ for $m \geq l_0$ Using (3 10) we therefore have that for every $m \geq l_0$,

$$\log \theta_m(\underline{x}) \leq (\theta + \varepsilon) m \leq (\theta - \varepsilon) t(m) \leq (\theta - \varepsilon) n_{j(m)} \leq \log \theta_{n_{j(m)}}(\underline{x})$$

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Hence $\theta_m(\underline{x}) \le \theta_{n_{j(m)}}(\underline{x})$ and using (3 6) we get for $m \ge l_0$ that

$$\bar{\nu}(B(\pi_H(\sigma^m \underline{x}), \theta_m(\underline{x}))) \le D^{m-n_{j(m)}} \bar{\nu}(B(\pi_H(\sigma^{n_{j(m)}} \underline{x}), \theta_m(\underline{x})))$$
$$\le D^{m-n_{j(m)}} \bar{\nu}(B(\pi_H(\sigma^{n_{j(m)}} \underline{x}), \theta_{n_{j(m)}}(\underline{x})))$$

Therefore for $m \ge l_0$ we have,

$$\frac{\log \bar{\nu}(B(\pi_H(\sigma^m \underline{x}), \theta_m(\underline{x})))}{\log \theta_m(\underline{x})}$$
$$\ge \frac{(m-n_{j(m)}) \log D}{\log \theta_m(\underline{x})} + \frac{\log \theta_{n_{j(m)}}(\underline{x})}{\log \theta_m(\underline{x})} \frac{\log \bar{\nu}(B(\pi_H(\sigma^{n_{j(m)}} \underline{x}), \theta_{n_{j(m)}}(\underline{x})))}{\log \theta_{n_{j(m)}}(\underline{x})}$$
$$= \frac{\frac{(m-n_{j(m)})}{m} \log D}{\frac{1}{m} \log \theta_m(\underline{x})} + \frac{n_{j(m)}}{m} \frac{\frac{1}{n_{j(m)}} \log \theta_{n_{j(m)}}(\underline{x})}{\frac{1}{m} \log \theta_m(\underline{x})} \frac{\log \bar{\nu}(B(\pi_H(\sigma^{n_{j(m)}} \underline{x}), \theta_{n_{j(m)}}(\underline{x})))}{\log \theta_{n_{j(m)}}(\underline{x})} \tag{3 11}$$

By the definition of $t(m)$ and $j(m)$ we have $n_{j(m)+1} \ge \{(\theta+\varepsilon)/(\theta-\varepsilon)\} m$ From the Birkhoff ergodic theorem we deduce that $\lim_{j\to\infty}(n_j/n_{j+1}) = 1$ Therefore for $m \ge l_0$ large enough we have

$$n_{j(m)} \ge (1-\varepsilon) n_{j(m)+1} \ge (1-\varepsilon) \frac{\theta+\varepsilon}{\theta-\varepsilon} m$$

Thus taking the limit of $m \to \infty$ from (3 11) and (3 9) we get,

$$L(\underline{x}) \ge \frac{\varepsilon \frac{\theta+\varepsilon}{\theta-\varepsilon} \log D}{\theta} + (1-\varepsilon) \frac{\theta+\varepsilon}{\theta-\varepsilon} \frac{\theta}{\theta} p$$

Letting $\varepsilon \searrow 0$ we get $L(\underline{x}) \ge p$ In particular, as $\mu(Z) = 1$ we get $L(\underline{x}) \ge p$ for μ-a a $\underline{x} \in \Sigma$ Letting $p \nearrow HD(\bar{\nu})$ we obtain inequality (3 7)

In order to prove the converse inequality, let

$$R = \text{sup ess}\,(L) = \inf\{\sup\{L(\underline{x}) \ \ \underline{x} \in \Sigma\} \ \ \mu(X) = 1\}$$

From Egorov's theorem we have that for every $Q < R$ there exists an integer $l_1 \ge 1$ and a Borel set $Y_1 \subset \Sigma$ such that $\mu(Y_1) > 0$ and

$$\frac{\log \bar{\nu} B(\pi_H(\sigma^n \underline{x}), \theta_n(\underline{x}))}{\log \theta_n(\underline{x})} \ge Q \quad \text{for every } \underline{x} \in Y_1 \text{ and } n \ge l_1 \tag{3 12}$$

Also by the same theorem, for every $0 < \varepsilon < -\theta$ there exists an integer $l_2 \ge 1$ and a Borel set $Y_2 \subset \Sigma$ such that $\mu(Y_2) \ge 1 - \frac{1}{2}\mu(Y_1)$ and

$$(\theta-\varepsilon) n \le \log \theta_n(\underline{x}) \le (\theta+\varepsilon) n \quad \text{for every } \underline{x} \in Y_2 \text{ and } n \ge l_2 \tag{3 13}$$

Now, for any $\underline{x} \in \Sigma$ let $m_k = m_k(\underline{x})$, $k = 1, 2,$ be the increasing sequence of integers $m \ge 0$ for which $\sigma^{-m}(\underline{x}) \cap Y_1 \cap Y_2 \ne \varnothing$ As $\mu(Y_1 \cap Y_2) \ge \frac{1}{2}\mu(Y_1) > 0$ we deduce from the Birkhoff ergodic theorem applied to the Rochlin natural extension of the system

(σ, μ) that for μ-a e $\underline{x} \in \Sigma$, say $\underline{x} \in V$ with $\mu(V)=1$, the sequence $m_k = m_k(\underline{x})$ is infinite and furthermore that

$$\lim_{k\to\infty} \frac{m_k}{m_{k+1}} = 1 \tag{3 14}$$

Consider now any $\underline{x} \in V$, let $m_k = m_k(\underline{x})$ and let $\{\underline{x}_k\}_{k=1}^{\infty}$ be a sequence consisting of elements of $Y_1 \cap Y_2$ such that $\sigma^{m_k}(\underline{x}_k) = \underline{x}$ for every $k = 1, 2,$ For every small enough $0 < r < 1$ let $p = p(r) \geq 1$ be the largest integer such that

$$\theta_{m_p}(\underline{x}_p) \geq r, \tag{3 15}$$

so that in particular,

$$\theta_{m_{p+1}}(\underline{x}_{p+1}) < r \tag{3 16}$$

(a largest such p exists because (3 13) holds for each $\underline{x}_k$) Thus $p(r) \to \infty$ as $r \to 0$ and so there exists $0 < r_0 < 1$ such that

$$m_{p(r)} \geq \max(l_1, l_2) \quad \text{for every } 0 < r \leq r_0 \tag{3 17}$$

Thus for any $r \in (0, r_0]$ it follows from (3 15) that $B(\pi_H(\underline{x}), r) \subset B(\pi_H(\underline{x}), \theta_{m_p}(\underline{x}_p))$ From (3 12) we therefore get

$$\begin{aligned} \frac{\log \bar{\nu}B(\pi_H(\underline{x}), r)}{\log r} &\geq \frac{\log \bar{\nu}B(\pi_H(\underline{x}), \theta_{m_p}(\underline{x}_p))}{\log r} \\ &= \frac{\log \bar{\nu}B(\pi_H \sigma^{m_p}(\underline{x}_p), \theta_m(\underline{x}_p))}{\log \theta_{m_p}(\underline{x}_p)} \; \frac{\log \theta_{m_p}(\underline{x}_p)}{\log r} \\ &\geq Q \, \frac{\log \theta_{m_p}(\underline{x}_p)}{\log r} \end{aligned} \tag{3 18}$$

In view of (3 13) and (3 17) we obtain $\log \theta_{m_p}(\underline{x}_p) \leq (\theta+\varepsilon) m_p$ and $\log \theta_{m_{p+1}}(\underline{x}_{p+1}) \geq (\theta - \varepsilon) m_{p+1}$ Consequently

$$\frac{\log \theta_{m_p}(\underline{x}_p)}{\log \theta_{m_{p+1}}(\underline{x}_{p+1})} \geq \frac{\theta+\varepsilon}{\theta-\varepsilon} \; \frac{m_p}{m_{p+1}}$$

From this, (3 16) and (3 18) we get

$$\frac{\log \bar{\nu}B(\pi_H(\underline{x}), r)}{\log r} \geq Q \; \frac{\theta+\varepsilon}{\theta-\varepsilon} \; \frac{m_p}{m_{p+1}}$$

Hence, letting $r \to 0$, by Lemma 10, and (3 14) we obtain $HD(\bar{\nu}) \geq Q[(\theta+\varepsilon)/(\theta-\varepsilon)]$, and letting first $\varepsilon \to 0$ and then $Q \nearrow R$ we get $HD(\bar{\nu}) \geq R$ This completes the proof □

We shall now prove the main result of the paper (compare with Theorem 7 of [**PU**])

THEOREM 12 *The Hausdorff dimension of* $\pi_*\mu$ *satisfies*

$$HD(\pi_*\mu) = t + \frac{\chi_H}{\chi_W} HD((p_H\pi)_*\mu)\left(1 - \frac{\chi_H}{\chi_W}\right)$$

where $\chi_W = \int f_W \, d\mu$ *and* $\chi_H = \int f_H \, d\mu$

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Proof As μ and ν are equivalent, $HD(\pi_*\mu)=HD(\pi_*\nu)$ and it is enough to prove that

$$HD(\pi_*\nu)=t+\frac{\chi_H}{\chi_W}+HD((p_H\pi)_*\mu)\left(1-\frac{\chi_H}{\chi_W}\right)$$

We keep the notation $\pi_H=p_H\circ\pi$, and $\bar\nu=(p_H\pi)_*\nu$, and we put $\bar\mu=(p_H\pi)_*\mu$ Recall from Lemma 7 that $K(\underline{x},n)$ is a square of side length $|\tilde\pi(\underline{x}(n))|$ about $\pi(\underline{x})$, then by Lemma 7 there exists a constant $0<C\le 1$ such that

$$B(p_H\pi(\sigma^n\underline{x}),C|\tilde\pi(\underline{x}(n))|\exp(-S_nf_H(\underline{x})))\subset p_H\varphi^{-1}_{\underline{x}(n)}K(\underline{x},n)$$
$$\subset B(p_H\pi(\sigma^n\underline{x}),C^{-1}|\tilde\pi(\underline{x}(n))|\exp(-S_nf_H(\underline{x}))) \qquad (3\ 19)$$

for every $\underline{x}\in\Sigma$ and $n\ge 1$ In view of Lemma 8,

$$\pi_*\nu(K(\underline{x},n))\asymp\exp(S_n(tf_W+f_H)(\underline{x}))\pi_*\nu(\varphi^{-1}_{\underline{x}(n)}K(\underline{x},n)) \qquad (3\ 20)$$

Take now

$$\theta^1_n(\underline{x})=C|\tilde\pi(\underline{x}(n))|\exp(-S_nf_H(\underline{x}))$$

and

$$\theta^{(2)}_n(\underline{x})=C^{-1}\cap\tilde\pi(\underline{x}(n))|\exp(-S_nf_H(\underline{x}))$$

for $\underline{x}\in\Sigma$ and $n=1,2,$ As $\varphi^{-1}_{\underline{x}(n)}(K(\underline{x},n))$ is a rectangle which lies across the full width of the unit interval, by (3 19) and (3 20) we get

$$\log\pi_*\nu(K(\underline{x},n))=S_n(tf_W+f_H)(\underline{x})+\log\pi_*\nu(\varphi^{-1}_{\underline{x}(n)}K(\underline{x},n))+X_1$$
$$\ge S_n(tf_W+f_H)(\underline{x})+\log\bar\nu(B(\pi_H(\sigma^n\underline{x}),\theta^{(1)}_n(\underline{x}))+X_1$$

and

$$\log\pi_*\nu(K(\underline{x},n))\le S_n(tf_W+f_H)(\underline{x})+\log\bar\nu(B(\pi_H(\sigma^n\underline{x}),\theta^{(2)}_n(\underline{x}))+X_2,$$

where $X_1,X_2>0$ are uniformly bounded, both from above and from zero, with respect to $\underline{x}\in\Sigma$ and $n\ge 1$ Therefore

$$\frac{\log\pi_*\nu(K(\underline{x},n))}{\log|\tilde\pi(\underline{x}(n))|}$$
$$\le\frac{S_n(tf_W+f_H)(\underline{x})}{\log|\tilde\pi(\underline{x}(n))|}+\frac{\log\theta^{(1)}_n(\underline{x})}{\log|\tilde\pi(\underline{x}(n))|}\frac{\log\bar\nu(B(\pi_H(\sigma^n\underline{x}),\theta^{(1)}_n(\underline{x})))}{\log\theta^{(1)}_n(\underline{x})}+\frac{\log X_1}{\log|\tilde\pi(\underline{x}(n))|}, \qquad (3\ 21)$$

and

$$\frac{\log\pi_*\nu(K(\underline{x},n))}{\log|\tilde\pi(\underline{x}(n))|}$$
$$\ge\frac{S_n(tf_W+f_H)(\underline{x})}{\log|\tilde\pi(\underline{x}(n))|}+\frac{\log\theta^{(2)}_n(\underline{x})}{\log|\tilde\pi(\underline{x}(n))|}\frac{\log\bar\nu(B(\pi_H(\sigma^n\underline{x}),\theta^{(2)}_n(x)))}{\log\theta^{(2)}_n(\underline{x})}+\frac{\log X_2}{\log|\tilde\pi(\underline{x}(n))|} \qquad (3\ 22)$$

Now, by Lemma 3 and the Birkhoff ergodic theorem we know that for μ-a a $\underline{x}\in\Sigma$,

$$\lim_{n\to\infty}\frac{1}{n}\log\theta^{(1)}_n(\underline{x})=\lim_{n\to\infty}\frac{1}{n}\log\theta^{(2)}_n(\underline{x})=\int f_W-f_H\,d\mu<0$$

This property means that we can apply Lemma 11 to each of the sequences $\{\theta_n^{(1)}\}$ and $\{\theta_n^{(2)}\}$ Using this Lemma, taking the limit of $n \to \infty$ in (3 21) and (3 22) gives

$$HD(\pi_* \nu) \le \frac{t\chi_W + \chi_H}{\chi_W} + HD(\bar{\nu}) \frac{\chi_W - \chi_H}{\chi_W}$$

and

$$HD(\pi_* \nu) \ge \frac{t\chi_W + \chi_H}{\chi_W} + HD(\bar{\nu}) \frac{\chi_W - \chi_H}{\chi_W}$$

Rearranging these expressions gives the claimed dimension formula The proof is thus complete □

As μ is an atom free measure, the set E (mod $\pi_*\mu$) can be regarded as the graph of a Borel function f $I \to \mathbb{R}$ defined $\tilde{\pi}_*\mu$ almost everywhere In this context the measure $(p_H\pi)_*\mu$ can be interpreted as the probability distribution of the random variable f $I \to \mathbb{R}$ defined on the probability space $(I, \tilde{\pi}_*\mu)$ The following theorem gives a positive answer to one direction of the conjecture stated in the introduction

THEOREM 13 *If E has a weak invariant foliation* $(*)$ *and* $HD((p_H\pi)_*\mu) = 1$ *then* $HD(E) = D_B(E) = t+1$

Proof From the remark after Theorem 6 we have $D_B(E) \le t+1$, and Theorem 12 gives $HD(E) \ge HD(\pi_*\mu) = t+1$ As $HD(E) \le D_B(E)$, this finishes the proof □

Remark. In the above theorem we do not use the assumption that E satisfies the Darboux property

We can also give a partial positive answer in the second direction of the conjecture stated in the introduction

THEOREM 14 *Suppose that E has a weak invariant foliation* $(*)$ *and that* $HD((p_H\pi)_*\mu) < 1$ *Then* $H_{t+1}(E) = 0$ *where* $H_{t+1}(E)$ *is the* $(t+1)$*-dimensional Hausdorff measure of E*

Proof Theorem 12 implies the existence of a Borel set $F \subset E$ such that $\pi_*\mu(F) = 1$ and $HD(F) < t+1$ Hence, if $Z = E \setminus F$ then $\tilde{\pi}_*\mu(p_W(Z)) = 0$ Since the sequence $\{\{\tilde{\pi}(\underline{x}(n))\}_{\underline{x}}\}_{n=1}^{\infty}$ of partitions (mod $\pi_*\mu$) of I is increasing and generates the Borel σ-algebra on I, for given $\theta, \gamma > 0$ we can find a countable subset $\{\underline{x}^j\}_{j=1}^{\infty}$ of Σ and a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that

$$p_W(Z) \subset \bigcup_{j=1}^{\infty} \tilde{\pi}(\underline{x}^j(n_j)), \quad \sum_{j=1}^{\infty} \tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}^j(n_j))) \le \gamma$$

and

$$\operatorname{diam}(\tilde{\pi}(\underline{x}^j(n_j))) \le \theta \quad \text{for every } j \ge 1$$

By Lemmas 3 and 4, every set $\varphi_{\underline{x}^j(n_j)}(E)$ can be covered by at most const $\exp(S_{n_j}(f_H - f_W)(\underline{x}^j))$ squares with edges of length $|\tilde{\pi}(\underline{x}^j(n_j))|$ We can therefore estimate Hausdorff measure as follows Let

$$H_{t+1}(Z, \theta) = \inf\left\{ \sum_{j=1}^{\infty} (\operatorname{diam} U_j)^{t+1} \ \bigcup_j U_j \supset Z, \operatorname{diam} U_j \le \theta, U_j \text{ open} \right\}$$

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Then

$$H_{t+1}(Z,\theta)\le \text{const}\sum_{j=1}^{\infty}\exp(S_{n_j}(f_H-f_W)(\underline{x}^j))(\exp(S_{n_j}f_W(\underline{x}^j)))^{t+1}$$

$$=\text{const}\sum_{j=1}^{\infty}\exp(S_{n_j}(tf_W+f_H)(\underline{x}^j))$$

Now as $P(tf_W+f_H)=0$ it follows from (2 9) that

$$H_{t+1}(Z,\theta)\le \text{const}\sum_{j=1}^{\infty}\tilde{\pi}_*\nu(\tilde{\pi}(\underline{x}^j(n_j)))\le \text{const}\ \gamma$$

If we now let γ and then θ go to zero we get $H_{t+1}(Z)=0$ As $H_{t+1}(F)=0$ this gives $H_{t+1}(E)=p$ □

REMARK All the results of this section rely heavily on the assumption that E has a weak invariant foliation, and we suspect that without it they are not true We shall develop this remark in a forthcoming paper

4 *Some examples of self-affine sets*

In this last section we describe some examples of self-affine sets for which we can check whether or not $HD((p_H\pi)_*\mu)=1$ holds We begin with the following

PROPOSITION 15 *Let J be the convex hull of $p_H(E)$ If E has a weak invariant foliation* (∗) *and $J=\bigcup_i \tau_i(J)$ then E satisfies the Darboux property*

Proof By condition (∗), the set $p_H(E)$ is self-similar under the maps τ_i Since J is compact and non-empty, the uniqueness of self-similar sets (see [H]) implies that $J=p_H(E)$ Now for any $\underline{x}\in\Sigma$ and $n\ge 0$,

$$p_H\varphi_{x(n)}(E)=\tau_{x(n)}(J)$$

which is connected Hence E satisfies the Darboux property □

A simple family of self-affine sets can be defined as follows Take $0<p,\beta<1$ such that $\max(p,1-p)<\beta$ and let $E(p,\beta)$ be the self-affine set determined by the contractions φ_0,φ_1 $I\times\mathbb{R}\to I\times\mathbb{R}$ defined by

$$\varphi_0(x,y)=(px,\beta y),\quad \varphi_1(x,y)=(p+(1-p)x,\beta y+1-\beta)$$

It is easy to check that $E(p,\beta)$ satisfies (∗) and that the assumptions of Proposition 15 are fulfilled in particular $E(p,\beta)$ satisfies the Darboux property The functions f_H,f_W $\Sigma\to\mathbb{R}$ associated to $E(p,\beta)$ are given by

$$f_H(\underline{x})\equiv\log\beta\quad f_W(\underline{x})=\begin{cases}\log p & \text{if } x_1=0\\ \log(1-p) & \text{if } x_1=1\end{cases}$$

The measure μ is particularly simple in this case Let m be any σ-invariant probability measure on Σ and let $q=m(\{\underline{x}\in\Sigma\ x_1=0\})$ Then

$$h_m(\sigma)+\int(sf_W+f_H)\,dm$$

$$\le -q\log q-(1-q)\log(1-q)+sq\log p+s(1-q)\log(1-p)+\log\beta = F(s,q)$$

and we have equality if and only if m is the product measure on Σ determined by the probability vector $(q, 1-q)$ In order to find the corresponding number t and the equilibrium state μ we have to find $t, q \in [0, 1]$ such that $F(t, q) = 0$ and $F(t, d) \leq F(t, q)$ for every $d \in [0, 1]$ With elementary calculus one obtains the following

PROPOSITION 16 *The number t is uniquely determined by the equation $p^t + (1-p)^t = \beta^{-1}$ and the equilibrium state μ is the Bernoulli measure given by the probability vector $(p^t\beta, (1-p)^t\beta)$* □

Now observe that the map $p_H\pi$ $\Sigma \to \mathbb{R}$ is given by

$$p_H\pi(\underline{x}) = (1-\beta) \sum_{n=1}^{\infty} \beta^{n-1} x_n$$

This expresses $p_H\pi$ as a series of independent random variables on the probability space (Σ, μ) Therefore, the Fourier transform of the measure $(p_H\pi)_*\mu$ is the infinite product of the Fourier transforms of measures $((1-\beta)\beta^{n-1}x_n)_*\mu$ which can easily be computed by hand Using this one can prove, in the same way as in [E], the following

THEOREM 17 *For all $p \in (0, 1)$ there exists $\beta(p) \in (\max(p, 1-p), 1)$ and a set $Z(p)$ of full measure in $[\beta(p), 1)$ with the property that for any $\beta \in Z(p)$ the measure $(p_H\pi)_*\mu_{p\,\beta}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ Furthermore $\lim_{\varepsilon \to 0} HD(([\beta(p), 1]\backslash Z(p)) \cap (1-\varepsilon, 1)) = 0$* □

Since the Hausdorff dimension of any measure that is absolutely continuous to Lebesgue measure on $\mathbb{R}$ is equal to 1 we can combine the above with Theorem 13 to get

COROLLARY 18 *For every $(p, \beta) \in \bigcup_{p \in (0\,1)} \{p\} \times Z(p)$ we have $HD(E(p, \beta)) = D_B(E(p, \beta))$ In particular, the set of parameter values (p, β) for which $HD(E(p, \beta)) = D_B(E(p, \beta))$ has positive Lebesgue measure in $(0, 1)^2$*

The above class is a generalization of some examples considered in [**PU**] Define a mapping h $I \to I$ by

$$h(x) = \begin{cases} p^{-1}x & 0 \leq x < p \\ (1-p)^{-1}x - p/(1-p) & p \leq x \leq 1 \end{cases}$$

and let f $I \to \mathbb{R}$ be the map $f(x) = (1-\beta) \sum_{n=0}^{\infty} \beta^n r_n(x)$, where $r_n = I_{[p\,1]} \circ h^n$ It is easy to check that $E(p, \beta)$ coincides up to a countable set with the graph of f Fixing $p = \frac{1}{2}$ puts us in precisely the class of functions considered in § 6 of [**PU**] Some examples are given there of sets $E(\frac{1}{2}, \beta)$ for which $HD((p_H\pi)_*\mu) < 1$ and $HD(E(\frac{1}{2}, \beta)) < D_B(E(\frac{1}{2}, \beta))$ They correspond to β being the reciprocal of a Pisot number

Finally we shall briefly describe a subclass of self-affine sets (that are graphs of continuous functions) introduced in [**K**] for which the conjecture stated in the introduction can be proved completely We say that f $I \to I$ is a self-affine function if there exists $1 > H > 0$ and an integer $r \geq 4$ such that for every $n \geqq 1$, $0 \leq i < r^n$ and $0 \leq h < r^{-n}$ we have

$$f(ir^{-n} + h) - f(ir^{-n}) = T_{n\,i} r^{-nH} f(r^n h)$$

where $T_{n,i}$ equals either 1 or -1 It is easy to see that a bounded self-affine function is continuous if and only if the above condition holds for any $0 \le h \le r^{-n}$ A useful characterisation of the class **K** of self-affine functions for which $f(0)=0, f(1)=1$ can be found in [K] Observe that for $f \in \mathbf{K}$, the graph of f coincides with the self-affine set determined by the contractions

$$\varphi_i(x, y) = ((i+x)r^{-1}, T_{1,i}r^{-H}y + f(r^{-1}i)), \quad 0 \le i < r$$

It is easy to check that $D_B(\text{graph}(f)) = 2 - H$, and it has been proved in [U] that $HD(\text{graph}(f)) = 2 - H$ if and only if $(p_H\pi)_*\mu$ (the probability distribution of $f\ I \to \mathbb{R}$) is absolutely continuous with respect to Lebesgue measure on $[0, 1]$

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